Graduate Texts in Mathematics

Gaisi Takeuti
Wilson M. Zaring

Introduction to Axiomatic Set Theory

Second Edition

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Gaisi Takeuti
Wilson M. Zaring
Department of Mathematics
University of Illinois
Urbana, IL 61801
U.S.A.

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Preface

In 1963, the first author introduced a course in set theory at the University of Illinois whose main objectives were to cover Gödel's work on the consistency of the Axiom of Choice (AC) and the Generalized Continuum Hypothesis (GCH), and Cohen's work on the independence of the AC and the GCH. Notes taken in 1963 by the second author were taught by him in 1966, revised extensively, and are presented here as an introduction to axiomatic set theory.

Texts in set theory frequently develop the subject rapidly moving from key result to key result and suppressing many details. Advocates of the fast development claim at least two advantages. First, key results are highlighted, and second, the student who wishes to master the subject is compelled to develop the detail on his own. However, an instructor using a "fast development" text must devote much class time to assisting his students in their efforts to bridge gaps in the text.

We have chosen instead a development that is quite detailed and complete. For our slow development we claim the following advantages. The text is one from which a student can learn with little supervision and instruction. This enables the instructor to use class time for the presentation of alternative developments and supplementary material. Indeed, by presenting the student with a suitably detailed development, we enable him to move more rapidly to the research frontier and concentrate his efforts on original problems rather than expending that effort redoing results that are well known.

Our main objective in this text is to acquaint the reader with Zermelo–Fraenkel set theory and bring him to a study of interesting results in one semester. Among the results that we consider interesting are the following: Sierpinski's proof that the GCH implies the AC, Rubin's proof that the

Aleph Hypothesis (AH) implies the AC, Gödel's consistency results and Cohen's forcing techniques. We end the text with a section on Cohen's proof of the independence of the Axiom of Constructibility.

In a sequel to this text entitled *Axiomatic Set Theory*, we will discuss, in a very general framework, relative constructibility, general forcing, and their relationship.

We are indebted to so many people for assistance in the preparation of this text that we would not attempt to list them all. We do, however, wish to express our appreciation to Professors Kenneth Appel, W. W. Boone, Carl Jockusch, Thomas McLaughlin, and Nobuo Zama for their valuable suggestions and advice. We also wish to thank Professor H. L. Africk, Professor Kenneth Bowen, Paul E. Cohen, Eric Frankl, Charles Kahane, Donald Pelletier, George Sacerdote, Eric Schindler, and Kenneth Slonneger, all students or former students of the authors, for their assistance at various stages in the preparation of the manuscript.

A special note of appreciation goes to Professor Hisao Tanaka, who made numerous suggestions for improving the text and to Dr. Klaus Gloede, who, through the cooperation of Springer-Verlag, provided us with valuable editorial advice and assistance.

We are also grateful to Mrs. Carolyn Bloemker for her care and patience in typing the final manuscript.

Urbana
January 1971

Gaisi Takeuti
Wilson M. Zaring

Preface to the Second Edition

Since our first edition appeared in 1971 much progress has been made in set theory. The problem that we faced with this revision was that of selecting new material to include that would make our text current, while at the same time retaining its status as an introductory text. We have chosen to make two major changes. We have modified the material on forcing to present a more contemporary approach. The approach used in the first edition was dated when that edition went to press. We knew that but thought it of interest to include a section on forcing that was close to Cohen's original approach. Those who wished to learn the Boolean valued approach could find that presentation in our second volume GTM 8. But now we feel that we can no longer justify devoting time and space to an approach that is only of historical interest.

As a second major modification, and one intended to update our text, we have added two chapters on Silver machines. The material presented here is based on Silver's lectures given in 1977 at the Logic Colloquium in Wrocław, Poland.

In order to produce a text of convenient size and reasonable cost we have had to delete some of the material presented in the first edition. Two chapters have been deleted *in toto*, the chapter on the Arithmetization of Model Theory, and the chapter on Languages, Structures, and Models. The material in Chapters 10 and 11 has been streamlined by introducing the Axiom of Choice earlier and deleting Sierpinski's proof that GCH implies AC, and Rubin's proof that AH, the aleph hypothesis, implies AC. Without these results we no longer need to distinguish between GCH and AH and so we adopt the custom in common use of calling the aleph hypothesis the generalized continuum hypothesis.

There are two other changes that deserve mention. We have altered the language of our theory by introducing different symbols for bound and free variables. This simplifies certain statements by avoiding the need to add conditions for instances of universal statements. The second change was intended to bring some perspective to our study by helping the reader understand the relative importance of the results presented here. We have used "Theorem" only for major results. Results of lesser importance have been labeled "Proposition."

We are indebted to so many people for suggestions for this revision that we dare not attempt to recognize them all lest some be omitted. But two names must be mentioned, Josef Tichy and Juichi Shinoda. Juichi Shinoda provided valuable assistance with the final version of the material on Silver machines. He also read the page proofs for the chapters on Silver machines and forcing, and suggested changes that were incorporated. Josef Tichy did an incredibly thorough proof reading of the first edition and compiled a list of misprints and errors. We have used this list extensively in the hope of producing an error free revision even though we know that that hope cannot be realized.

Finally we wish to convey our appreciation to Ms. Carolyn Bloemker for her usual professional job in typing the manuscript for new portions of this revision.

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Gaisi Takeuti
Wilson M. Zaring

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CHAPTER 1

Introduction

In 1895 and 1897 Georg Cantor (1845–1918) published, in a two-part paper, his master works on ordinal and cardinal numbers.[1] Cantor's theory of ordinal and cardinal numbers was the culmination of three decades of research on number "aggregates." Beginning with his paper on the denumerability of infinite sets,[2] published in 1874, Cantor had built a new theory of the infinite. In this theory a collection of objects, even an infinite collection, is conceived of as a single entity.

The notion of an infinite set as a complete entity was not universally accepted. Critics argued that logic is an extrapolation from experience that is necessarily finitistic. To extend the logic of the finite to the infinite entailed risks too grave to countenance. This prediction of logical disaster seemed vindicated when at the turn of the century paradoxes were discovered in the very foundations of the new discipline. Dedekind stopped publication of his *Was sind und was sollen die Zahlen*? Frege conceded that the foundation of his *Grundgesetze der Arithmetik* was destroyed.

Nevertheless set theory gained sufficient support to survive the crisis of the paradoxes. In 1908, speaking at the International Congress at Rome, the great Henri Poincaré (1854–1912) urged that a remedy be sought.[3] As a reward he promised "the joy of the physician called to treat a beautiful

[1] Beiträge zur Begründung der transfiniten Mengenlehre (Erster Artikel). *Math. Ann.* **46**, 481–512 (1895); (Zweiter Artikel) *Math. Ann.* **49**, 207–246 (1897). For an English translation see Cantor, Georg. *Contributions to the Founding of the Theory of Transfinite Numbers*. New York: Dover Publications, Inc.

[2] Uber eine Eigenschaft des Inbegriffes aller reellen algebraischen Zahlen. *J. Reine Angew. Math.* **77**, 258–262 (1874). In this paper Cantor proves that the set of all algebraic numbers is denumerable and that the set of all real numbers is not denumerable.

[3] Atti del IV Congresso Internazionale dei Matematici Roma 1909, Vol. 1, p. 182.

pathologic case." By that time Zermelo and Russell were already at work seeking fundamental principles on which a consistent theory could be built. The first axiomatization of set theory was given by Zermelo in 1908.[4]

From this one might assume that the sole purpose for axiomatizing is to avoid the paradoxes. There are however reasons to believe that axiomatic set theory would have evolved even in the absence of paradoxes. Certainly the work of Dedekind and of Frege in the foundations of arithmetic was not motivated by fear of paradoxes but rather by a desire to see what foundational principles were required. In his *Begriffsschrift* Frege states:

"..., we divide all truths that require justification into two kinds, those for which the proof can be carried out purely by means of logic and those for which it must be supported by facts of experience.... Now, when I came to consider the question to which of these two kinds the judgements of arithmetic belong, I first had to ascertain how far one could proceed in arithmetic by means of inferences alone, ..."[5]

Very early in the history of set theory it was discovered that the Axiom of Choice, the Continuum Hypothesis, and the Generalized Continuum Hypothesis are of special interest and importance. The Continuum Hypothesis is Cantor's conjectured solution to the problem of how many points there are on a line in Euclidean space.[6] A formal statement of the Continuum Hypothesis and its generalization will be given later.

The Axiom of Choice, in one formulation, asserts that given any collection of pairwise disjoint nonempty sets, there exists a set that has exactly one element in common with each set of the given collection. The discovery that the Axiom of Choice has important implications for all major areas of mathematics provided compelling reasons for its acceptance. Its status as an axiom, and also that of the Generalized Continuum Hypothesis, was however not clarified until Kurt Gödel in 1938, proved both to be consistent with the axioms of general set theory and Paul Cohen, in 1963, proved that they are each independent of the axioms of general set theory. Our major objective in this text will be a study of the contributions of Gödel and Cohen. In order to do this we must first develop a satisfactory theory of sets.

For Cantor a set was "any collection into a whole M of definite and separate objects m of our intuition or our thought."[7] This naive acceptance of any collection as a set leads us into the classical paradoxes, as for example

[4] Untersuchungen über die Grundlagen der Mengenlehre I. *Math. Ann.* **65**, 261–281 (1908). For an English translation see van Heijenoort, Jean. *From Frege to Gödel.* Cambridge: Harvard University Press, 1967.

[5] van Heijenoort, Jean. *From Frege to Gödel.* Cambridge: Harvard University Press, 1967. p. 5.

[6] See, What is Cantor's Continuum Problem? by Kurt Gödel in *Amer. Math. Monthly*, **54**, 515–525 (1947). A revised and expanded version of this paper is also found in Benacerraf, Paul and Putnam, Hilary. *Philosophy of Mathematics, Selected Readings.* Englewood Cliffs: Prentice-Hall, Inc., 1964.

[7] Cantor, Georg. *Contributions to the Founding of the Theory of Transfinite Numbers.* New York: Dover Publications, inc.

Russell's paradox: If the collection of all sets that are not elements of themselves is a set then this set has the property that it is an element of itself if and only if it is not an element of itself.

In view of Russell's paradox, and other difficulties to be discussed later, we have two alternatives in developing a theory of sets. Either we must abandon the idea that our theory is to encompass arbitrary collections in the sense of Cantor, or we must distinguish between at least two types of collections, arbitrary collections that we call classes and certain special collections that we call sets. Classes, or arbitrary collections, are however so useful and our intuitive feelings about classes are so strong that we dare not abandon them. A satisfactory theory of sets must provide a means of speaking safely about classes. There are several ways of developing such a theory.

Bertrand Russell (1872–1970) and Alfred North Whitehead (1861–1947) in their *Principia Mathematica* (1910) resolved the known difficulties with a theory of types. They established a hierarchy of types of collections. A collection x can be a member of a collection y only if y is one level higher in the hierarchy than x. In this system there are variables for each type level in the hierarchy and hence there are infinitely many primitive notions.

Two other systems, Gödel–Bernays (GB) set theory and Zermelo–Fraenkel (ZF) set theory, evolved from the work of Bernays (1937–1954), Fraenkel (1922), Gödel (1940), von Neumann (1925–1929), Skolem (1922), and Zermelo (1908). Our listing is alphabetical. We will not attempt to identify the specific contribution of each man. Following each name we have indicated the year or period of years of major contribution.

In Gödel–Bernays set theory the classical paradoxes are avoided by recognizing two types of classes, sets and proper classes. Sets are classes that are permitted to be members of other classes. Proper classes have sets as elements but are not themselves permitted to be elements of other classes. In this system we have three primitive notions; set, class and membership. In the formal language we have set variables, class variables, and a binary predicate symbol "$\in$".

In Zermelo–Fraenkel set theory we have only two primitive notions; set and membership. Class is introduced as a defined term. In the formal language we have only set variables and a binary predicate symbol "$\in$". Thus in ZF quantification is permitted only on set variables while in GB quantification is permitted on both set and class variables. As a result there are theorems in GB that are not theorems in ZF. It can however be proved that GB is a conservative extension of ZF in the sense that every well-formed formula (wff) of ZF is provable in ZF if and only if it is provable in GB.

Gödel's[8] work was done in Gödel–Bernays set theory. We, however, prefer Zermelo–Fraenkel theory in which Cohen[9] worked.

[8] Gödel, Kurt. *The Consistency of the Continuum Hypothesis.* Princeton: Princeton University Press, 1940.

[9] Cohen, Paul J. The Independence of the Continuum Hypothesis. Proc. Nat. Acad. Sci. U.S. **50**, 1143–1148 (1963).

CHAPTER 2
Language and Logic

The language of our theory consists of:

Free variables: $a_0, a_1, \ldots,$

Bound variables: $x_0, x_1, \ldots,$

A predicate symbol: $\in$,

Logical symbols: $\neg$, $\vee$, $\wedge$, $\rightarrow$, $\leftrightarrow$, $\forall$, $\exists$,

And auxiliary symbols: (,), [,].

The logical symbols, in the order listed, are for negation, disjunction, conjunction, implication, equivalence, universal quantification, and existential quantification.

We will not restrict ourselves to a minimal list of logical symbols, nor will we in general distinguish between primitive and defined logical symbols. When, in a given context, it is convenient to have a list of primitive symbols, we will assume whatever list best suits our immediate need.

We will use

$$a, b, c,$$

as metavariables whose domain is the collection of free variables

$$a_0, a_1, \ldots;$$

and we will use

$$x, y, z,$$

as metavariables whose domain is the collection of bound variables

$$x_0, x_1, \ldots.$$

When we need many metavariables we will use subscripts and rely upon the context to make clear whether, for example, x_0 is a particular bound variable of the formal language or a metavariable ranging over all bound variables of the formal language.

We will use

$$\varphi, \psi, \eta$$

as metavariables that range over all well-formed formulas (wffs).

Our rules for wffs are the following:

(1) If a and b are free variables, then $[a \in b]$ is a wff. Such formulas are called atomic.
(2) If φ and ψ are wffs, then $\neg\varphi$, $[\varphi \vee \psi]$, $[\varphi \wedge \psi]$, $[\varphi \to \psi]$, and $[\varphi \leftrightarrow \psi]$, are wffs.
(3) If φ is a wff and x is a bound variable, then $(\forall x)\varphi(x)$ and $(\exists x)\varphi(x)$ are wffs, where $\varphi(x)$ is the formula obtained from the wff φ by replacing each occurrence of some free variable a by the bound variable x. We call $(\forall x)\varphi(x)$ and $(\exists x)\varphi(x)$ respectively, the formula obtained from φ by universally, or existentially, quantifying on the variable a.

To simplify the appearance of wffs we will occasionally suppress certain grouping symbols. Our only requirement is that enough symbols be retained to assure the meaning:

Example. We will write $a_0 \in a_1$ for $[a_0 \in a_1]$ and instead of $[[a_0 \in a_0] \to [a_0 \in a_1]]$ we will write simply $a_0 \in a_0 \to a_0 \in a_1$.

Example. From the wff $a_0 \in a_1$ we obtain the wff $(\exists x)[x \in a_1]$ by existentially quantifying on a_0. We obtain the wff $(\forall y)[a_0 \in y]$ by universally quantifying on a_1. And we obtain $(\forall z)[a_0 \in a_1]$ by universally quantifying on a_2, or any other variable that does not occur in $a_0 \in a_1$.

A formula is well formed if and only if its being so is deducible from rules (1)–(3) above. It is easily proved that there is an effective procedure for determining whether a given expression. i.e., sequence of symbols, is a wff.

From the language just described we obtain Zermelo–Frankel set theory by adjoining logical axioms, rules of inference, and nonlogical axioms. The nonlogical axioms for ZF will be introduced in context and collected on pages 132–3. The logical axioms and the rules of inference for our theory are the following.

Logical Axioms.

(1) $\varphi \to [\psi \to \varphi]$.
(2) $[\varphi \to [\psi \to \eta]] \to [[\varphi \to \psi] \to [\varphi \to \eta]]$.
(3) $[\neg\varphi \to \neg\psi] \to [\psi \to \varphi]$.

(4) $(\forall x)[\varphi \to \psi(x)] \to [\varphi \to (\forall x)\psi(x)]$ where the free variable a on which we are quantifying does not occur in φ.
(5) $(\forall x)\varphi(x) \to \varphi(a)$ where $\varphi(a)$ is the formula obtained by replacing each occurrence of the bound variable x in $\varphi(x)$ by the free variable a.

Rules of Inference.

(1) From φ and $\varphi \to \psi$ to infer ψ.
(2) From φ to infer $(\forall x)\varphi(x)$ where $\varphi(x)$ is obtained from φ by replacing each occurrence of some free variable by x.

We will assume, without proof, those results from logic that we need, except one theorem. That theorem is proved on pages 114–6 and its proof presupposes the logical axioms and rules of inference set forth here.

We will use the turnstile, $\vdash$, to indicate that a wff is a theorem. That is, $\vdash \varphi$ is the metastatement that the wff φ is deducible, by the rules of inference, from the logical axioms above and the nonlogical axioms yet to be stated. To indicate that φ is deducible using only the logical axioms, we will write $\vdash_{\mathbf{LA}} \varphi$. We say that two wffs φ and ψ are logically equivalent if and only if $\vdash_{\mathbf{LA}} \varphi \leftrightarrow \psi$.

CHAPTER 3

Equality

Definition 3.1. $a = b \overset{\Delta}{\leftrightarrow} (\forall x)[x \in a \leftrightarrow x \in b]$.

Proposition 3.2.

(1) $a = a$.

(2) $a = b \rightarrow b = a$.

(3) $a = b \wedge b = c \rightarrow a = c$.

PROOF.

(1) $(\forall x)[x \in a \leftrightarrow x \in a]$.

(2) $(\forall x)[x \in a \leftrightarrow x \in b] \rightarrow (\forall x)[x \in b \leftrightarrow x \in a]$.

(3) $(\forall x)[x \in a \leftrightarrow x \in b] \wedge (\forall x)[x \in b \leftrightarrow x \in c] \rightarrow (\forall x)[x \in a \leftrightarrow x \in c]$.

$\square$

Remark. Our intuitive idea of equality is of course identity. A basic property that we expect of equality is that paraphrased as "equals may be substituted for equals," that is, if $a = b$ then anything that can be asserted of a can also be asserted of b. In particular if a certain wff holds for a it must also hold for b and *vice versa*:

$$a = b \rightarrow [\varphi(a) \leftrightarrow \varphi(b)].$$

Here $\varphi(b)$ is the formula obtained from φ by replacing each occurrence of some free variable by b, and $\varphi(a)$ is the formula obtained from φ by replacing each occurrence of the same free variable by a.

We need not postulate such a substitution principle for, as we will now show, it can be deduced from Definition 3.1 and the following weaker principle.

Axiom 1 (Axiom of Extensionality).

$$a = b \wedge a \in c \rightarrow b \in c.$$

Proposition 3.3. $a = b \rightarrow [a \in c \leftrightarrow b \in c]$.

PROOF. Axiom 1 and Proposition 3.2(2). ☐

Theorem 3.4. $a = b \rightarrow [\varphi(a) \leftrightarrow \varphi(b)]$.

PROOF (By induction on n the number of logical symbols in φ). If $n = 0$, then $\varphi(a)$ is of the form $c \in d$, $c \in a$, $a \in c$, or $a \in a$. Clearly

$$a = b \rightarrow [c \in d \leftrightarrow c \in d].$$

From the definition of equality

$$a = b \rightarrow [c \in a \leftrightarrow c \in b].$$

From Proposition 3.3

$$a = b \rightarrow [a \in c \leftrightarrow b \in c].$$

Again from the definition of equality and Proposition 3.3 respectively

$$a = b \rightarrow [a \in a \leftrightarrow a \in b],$$

$$a = b \rightarrow [a \in b \leftrightarrow b \in b].$$

Therefore

$$a = b \rightarrow [a \in a \leftrightarrow b \in b].$$

As our induction hypothesis we assume the result true for each wff having fewer than n logical symbols. If $n > 0$ and $\varphi(a)$ has exactly n logical symbols, then $\varphi(a)$ must be of the form

$$(1)\quad \neg\psi(a), \qquad (2)\quad \psi(a) \wedge \eta(a), \quad \text{or} \quad (3)\quad (\forall x)\psi(a, x).$$

In Cases (1) and (2) $\psi(a)$ and $\eta(a)$ have fewer than n logical symbols and hence from the induction hypothesis

$$a = b \rightarrow [\psi(a) \leftrightarrow \psi(b)],$$

$$a = b \rightarrow [\eta(a) \leftrightarrow \eta(b)].$$

From properties of negation and conjunction it then follows that

$$a = b \rightarrow [\neg\psi(a) \leftrightarrow \neg\psi(b)]$$

$$a = b \rightarrow [\psi(a) \wedge \eta(a) \leftrightarrow \psi(b) \wedge \eta(b)].$$

Thus if $\varphi(a)$ is $\neg\psi(a)$ or $\psi(a) \wedge \eta(a)$,

$$a = b \rightarrow [\varphi(a) \leftrightarrow \varphi(b)].$$

If $\varphi(a)$ is $(\forall x)\psi(a, x)$ we first choose a free variable c that is distinct from a and b and which does not occur in $\psi(a, x)$. Since $\psi(a, c)$ has fewer than n logical symbols it follows from the induction hypothesis that

$$a = b \rightarrow [\psi(a, c) \leftrightarrow \psi(b, c)].$$

Generalizing on c in this formula, using Logical Axiom 4, and the following result from logic

$$(\forall x)[\psi(a, x) \leftrightarrow \psi(b, x)] \rightarrow [(\forall x)\psi(a, x) \leftrightarrow (\forall x)\psi(b, x)]$$

we arrive at the conclusion that

$$a = b \rightarrow [(\forall x)\psi(a_1 x) \leftrightarrow (\forall x)\psi(b_1 x)]. \qquad \square$$

Remark. Extensionality assures us that a set is completely determined by its elements. From a casual acquaintance with this axiom one might assume that extensionality is a substitution principle having more to do with logic than set theory. This suggests that if equality were taken as a primitive notion then perhaps this axiom could be dispensed with. Dana Scott[1] however, has proved that this cannot be done without weakening the system. Thus, even if we were to take equality as a primitive logical notion it would still be necessary to add an extensionality axiom.[2]

[1] *Essays on the Foundations of Mathematics.* Amsterdam: North-Holland Publishing Company 1962, pp. 115–131.

[2] See Quine, Willard Van Orman. *Set Theory and its Logic.* Cambridge: Harvard University Press, 1969, 30f.

CHAPTER 4

Classes

We pointed out in the Introduction that one objective of axiomatic set theory is to avoid the classical paradoxes. One such paradox, the Russell paradox, arose from the naïve acceptance of the idea that given any property there exists a set whose elements are those objects having the given property, i.e., given a wff φ containing one free variable, there exists a set that contains all objects for which φ holds and contains no object for which φ does not hold. More formally there exists a set a such that

$$(\forall x)[x \in a \leftrightarrow \varphi(x)].$$

This principle, called the Axiom of Abstraction, was accepted by Frege in his *Grundgesetze der Arithmetik* (1893). In a letter[1] to Frege (1902) Bertrand Russell pointed out that the principle leads to the following paradox.

Consider the wff $b \notin b$. If there exists a set a such that

$$(\forall x)[x \in a \leftrightarrow x \notin x]$$

then in particular

$$a \in a \leftrightarrow a \notin a.$$

The idea of the collection of all objects having a specified property is so basic that we could hardly abandon it. But if it is to be retained how shall the paradox be resolved? The Zermelo–Fraenkel approach is the following.

For each wff $\varphi(a, a_1, \ldots, a_n)$ we will introduce a class symbol

$$\{x \mid \varphi(x, a_1, \ldots, a_n)\}$$

[1] van Heijenoort, Jean. *From Frege to Gödel.* Cambridge: Harvard University Press, 1967, pp. 124–125.

which is read "the class of all x such that $\varphi(x, a_1, \ldots, a_n)$." Our principal interpretation is that the class symbol $\{x \mid \varphi(x)\}$ denotes the class of individuals that have the property φ. We will show that class is an extension of the notion of set in that every set is a class but not every class is a set.

We will extend the $\in$-relation to class symbols in such a way that an object is an element of a class $\{x \mid \varphi(x)\}$ if and only if that object is a set and it has the defining property for the class. The Russell paradox is then resolved by showing that $\{x \mid x \notin x\}$ is a proper class, i.e., a class that is not a set. It is then disqualified for membership in any class, including itself, on the grounds that it is not a set.

Were we to adjoin the symbols

$$\{x \mid \varphi(x)\}$$

to our object language it would be necessary to extend our rules for wffs and add axioms governing the new symbols. We choose instead to introduce classes as defined terms. It is, of course, essential that we provide an effective procedure for reducing to primitive symbols any formula that contains a defined term. We begin by defining the contexts in which class symbols are permitted to appear. Our only concern will be their appearance in wffs in the *wider sense* as defined by the following rules.

Definition 4.1. (1) If a and b are free variables, then $a \in b$ is a wff in the wider sense.

(2) If φ and ψ are wffs in the wider sense and a and b are free variables then $a \in \{x \mid \psi(x)\}$, $\{x \mid \varphi(x)\} \in b$, and $\{x \mid \varphi(x)\} \in \{x \mid \psi(x)\}$ are wffs in the wider sense.

(3) If φ and ψ are wffs in the wider sense then $\neg\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \rightarrow \psi$, and $\varphi \leftrightarrow \psi$ are wffs in the wider sense.

(4) If φ is a wff in the wider sense and x is a bound variable then $(\exists\, x)\varphi(x)$ and $(\forall\, x)\varphi(x)$ are wffs in the wider sense.

A formula is a wff in the wider sense iff its being so is deducible from (1)–(4).

It is our intention that every wff in the wider sense be an abbreviation for a wff in the original sense. It is also our intention that a set belong to a class iff it has the defining property of that class, i.e.,

$$a \in \{x \mid \varphi(x)\} \qquad \text{iff} \quad \varphi(a).$$

Definition 4.2. If φ and ψ are wffs in the wider sense then

(1) $a \in \{x \mid \varphi(x)\} \overset{\triangle}{\leftrightarrow} \varphi(a)$.

(2) $\{x \mid \varphi(x)\} \in a \overset{\triangle}{\leftrightarrow} (\exists\, y)[y \in a \wedge (\forall\, z)[z \in y \leftrightarrow \varphi(z)]]$.

(3) $\{x \mid \varphi(x)\} \in \{x \mid \psi(x)\} \overset{\triangle}{\leftrightarrow} (\exists\, y)[y \in \{x \mid \psi(x)\} \wedge (\forall\, z)[z \in y \leftrightarrow \varphi(z)]]$.

Remark. From Definition 4.2 it is easily proved that each wff in the wider sense φ is reducible to a wff φ^* that is determined uniquely by the following rules.

Definition 4.3. If φ and ψ are wffs in the wider sense then

(1) $[a \in b]^* \overset{\triangle}{\leftrightarrow} a \in b$.
(2) $[a \in \{x \mid \varphi(x)\}]^* \overset{\triangle}{\leftrightarrow} \varphi^*(a) \overset{\triangle}{\leftrightarrow} [\varphi(a)]^*$.
(3) $[\{x \mid \varphi(x)\} \in a]^* \overset{\triangle}{\leftrightarrow} (\exists y)[y \in a \wedge (\forall z)[z \in y \leftrightarrow \varphi^*(z)]]$.
(4) $[\{x \mid \varphi(x)\} \in \{x \mid \psi(x)\}]^* \overset{\triangle}{\leftrightarrow} (\exists y)[\psi^*(y) \wedge (\forall z)[z \in y \leftrightarrow \varphi^*(z)]]$.
(5) $[\neg \varphi]^* \overset{\triangle}{\leftrightarrow} \neg \varphi^*$.
(6) $[\varphi \wedge \psi]^* \overset{\triangle}{\leftrightarrow} \varphi^* \wedge \psi^*$.
(7) $[(\forall x)\varphi(x)]^* \overset{\triangle}{\leftrightarrow} (\forall x)\varphi^*(x)$.

Proposition 4.4. *Each* wff *in the wider sense φ, is reducible to one and only one* wff *φ^* determined from φ by the rules* (1)–(7) *of Definition* 4.3.

PROOF (By induction on n the number of logical symbols plus class symbols, in φ). If $n = 0$, i.e., if φ has no logical symbols or class symbols, then φ must be of the form $a \in b$. By (1) of Definition 4.3, φ^* is $a \in b$.

As our induction hypothesis we assume that each wff in the wider sense having fewer than n logical and class symbols is reducible to one and only one wff that is determined by the rules (1)–(7) of Definition 4.3. If φ is a wff in the wider sense having exactly n logical and class symbols and if $n > 0$ then φ must be of one of the following forms:

(1) $a \in \{x \mid \psi(x)\}$,
(2) $\{x \mid \psi(x)\} \in a$,
(3) $\{x \mid \psi(x)\} \in \{x \mid \eta(x)\}$,
(4) $\neg \psi$,
(5) $\psi \wedge \eta$,
(6) $(\forall x)\psi$.

In each case ψ and η have fewer than n logical and class symbols and hence there are unique wffs ψ^* and η^* determined by ψ and η respectively and the rules (1)–(7) of Definition 4.3. Then by rules (2)–(7) φ determines a unique wff φ^*. □

Remark. From Proposition 4.4 every wff in the wider sense φ is an abbreviation for a wff φ^*. The proof tacitly assumes the existence of an effective procedure for determining whether or not a given formula is a wff in the wider sense. That such a procedure exists we leave as an exercise for the reader. From such a procedure it is immediate that there is an effective procedure for determining φ^* from φ.

Proposition 4.4 also assures us that in Definitions 4.1 and 4.2 we have not extended the notion of class but have only extended the notation for classes for if $\varphi(x)$ is a wff in the wider sense then

$$\{x \mid \varphi(x)\}.$$

and

$$\{x \mid \varphi^*(x)\}$$

are the same class. This is immediate from Proposition 4.4 and equality for classes which we now define. We wish this definition to encompass not only equality between class and class but also between set and class. For this, and other purposes, we introduce the notion of a *term.*

By a term we mean a free variable or a class symbol. We shall use capital Roman letters

$$A, B, C, \ldots$$

as metavariables on terms.

Definition 4.5. If A and B are terms then

$$A = B \overset{\triangle}{\leftrightarrow} (\forall\, x)[x \in A \leftrightarrow x \in B].$$

Proposition 4.6. $A \in B \leftrightarrow (\exists\, x)[x = A \wedge x \in B]$.

Proof. Definitions 4.2 and 4.5. □

Proposition 4.7. *If A, B, and C are terms then*

(1) $A = A$,

(2) $A = B \rightarrow B = A$,

(3) $A = B \wedge B = C \rightarrow A = C$.

The proof is similar to that of Proposition 3.2 and is left to the reader.

Proposition 4.8. *If A and B are terms and φ is a* wff *in the wider sense, then*

$$A = B \rightarrow [\varphi(A) \leftrightarrow \varphi(B)].$$

The proof is by induction. It is similar to the proof of Theorem 3.4 and is left to the reader.

Proposition 4.9. $a = \{x \mid x \in a\}$.

Proof. $(\forall\, x)[x \in a \leftrightarrow x \in a]$. □

Remark. Proposition 4.9 establishes that every set is a class. We now wish to establish that not all classes are sets. We introduce the predicates $\mathscr{M}(A)$ and $\mathscr{P}\!r(A)$ for "A is a set" and "A is a proper class" respectively.

Definition 4.10. $\mathscr{M}(A) \overset{\triangle}{\leftrightarrow} (\exists x)[x = A]$.

$$\mathscr{P}r(A) \overset{\triangle}{\leftrightarrow} \neg \mathscr{M}(A).$$

Proposition 4.11. $\mathscr{M}(a)$.

PROOF. $a = a$ □

Proposition 4.12. $A \in \{x \mid \varphi(x)\} \leftrightarrow \mathscr{M}(A) \wedge \varphi(A)$.

PROOF. Definitions 4.2 and 4.10 and Propositions 4.6 and 4.8. □

Definition 4.13. $\mathrm{Ru} \triangleq \{x \mid x \notin x\}$.

Proposition 4.14. $\mathscr{P}r(\mathrm{Ru})$.

PROOF. From Proposition 4.12

$$\mathscr{M}(\mathrm{Ru}) \to [\mathrm{Ru} \in \mathrm{Ru} \leftrightarrow \mathrm{Ru} \notin \mathrm{Ru}].$$

Therefore Ru is a proper class. □

Remark. Since the Russell class, Ru, is a proper class the Russell paradox is resolved. It should be noted that the Russell class is the first nonset we have encountered. Others will appear in the sequel.

We now have examples to show that the class of individuals for which a given wff φ holds may be a set or a proper class. Those sets, $\{x \mid \varphi(x)\}$, for which $\varphi(x)$ contains no free variable, we call *definable* sets.

As a notational convenience for the work ahead we add the following definitions.

Definition 4.15.

(1) $(\forall x_1, \ldots, x_n)\varphi(x_1, \ldots, x_n) \overset{\triangle}{\leftrightarrow} (\forall x_1) \cdots (\forall x_n)\varphi(x_1, \ldots, x_n)$.

(2) $(\exists x_1, \ldots, x_n)\varphi(x_1, \ldots, x_n) \overset{\triangle}{\leftrightarrow} (\exists x_1) \cdots (\exists x_n)\varphi(x_1, \ldots, x_n)$.

(3) $(\forall x_1, \ldots, x_n \in A)\varphi(x_1, \ldots, x_n) \overset{\triangle}{\leftrightarrow}$
$(\forall x_1, \ldots, x_n)[x_1 \in A \wedge \cdots \wedge x_n \in A \to \varphi(x_1, \ldots, x_n)]$.

(4) $(\exists x_1, \ldots, x_n \in A)\varphi(x_1, \ldots, x_n) \overset{\triangle}{\leftrightarrow}$
$(\exists x_1, \ldots, x_n)[x_1 \in A \wedge \cdots \wedge x_n \in A \wedge \varphi(x_1, \ldots, x_n)]$.

(5) $a_1, \ldots, a_n \in A \overset{\triangle}{\leftrightarrow} a_1 \in A \wedge \cdots \wedge a_n \in A$.

Definition 4.16. If τ is a term and φ is a wff, then

$$\{\tau(x_1, \ldots, x_n) \mid \varphi(x_1, \ldots, x_n)\} \triangleq \{y \mid (\exists x_1, \ldots, x_n)[y = \tau(x_1, \ldots, x_n) \wedge \varphi(x_1, \ldots, x_n)]\}.$$

CHAPTER 5

The Elementary Properties of Classes

In this chapter we will introduce certain properties of classes with which the reader is probably familiar. The immediate consequences of the definitions are for the most part elementary and easily proved; consequently they will be left to the reader as exercises.

We begin with the notion of unordered pair, $\{a, b\}$, and ordered pair $\langle a, b\rangle$.

Definition 5.1. $\{a, b\} \triangleq \{x \mid x = a \vee x = b\}$.
$\{a\} \triangleq \{a, a\}$.

Remark. The symbol $\{a, b\}$ we read as "the pair a, b," and the symbol $\{a\}$ we read as "singleton a." We postulate that pairs are sets.

Axiom 2 (Axiom of Pairing). $\mathscr{M}(\{a, b\})$.

Definition 5.2. $\langle a, b\rangle \triangleq \{x \mid x = \{a\} \vee x = \{a, b\}\}$.

Remark. We read $\langle a, b\rangle$ as "the ordered pair a, b."

Exercises

Prove the following.

(1) $c \in \{a, b\} \leftrightarrow c = a \vee c = b$.

(2) $c \in \{a\} \leftrightarrow c = a$.

(3) $c \in \langle a, b\rangle \leftrightarrow c = \{a\} \vee c = \{a, b\}$.

(4) $\{a\} = \{b\} \leftrightarrow a = b$.

(5) $\{a\} = \{b, c\} \leftrightarrow a = b = c.$

(6) $\langle a, b\rangle = \langle c, d\rangle \leftrightarrow a = c \wedge b = d.$

(7) $\mathscr{M}(\langle a, b\rangle).$

(8) $(\forall x)[a \in x \rightarrow b \in x] \rightarrow a = b.$

(9) $A \in B \rightarrow \mathscr{M}(A).$

Remark. The notions of unordered pair and ordered pair have natural generalizations to unordered n-tuple, $\{a_1, a_2, \ldots, a_n\}$ and ordered n-tuple, $\langle a_1, a_2, \ldots, a_n\rangle$.

Definition 5.3. $\{a_1, a_2, \ldots, a_n\} \triangleq \{x \mid x = a_1 \vee x = a_2 \vee \cdots \vee x = a_n\}.$

Definition 5.4. $\langle a_1, a_2, \ldots, a_n\rangle \triangleq \langle\langle a_1, \ldots, a_{n-1}\rangle, a_n\rangle, n \geqq 3.$

Remark. Since ordered pairs are sets it follows by induction that ordered n-tuples are also sets. From the fact that unordered pairs are sets we might also hope to prove by induction that unordered n-tuples are sets. For such a proof however we need certain properties of set union.

Definition 5.5. $\cup(A) \triangleq \{x \mid (\exists y)[x \in y \wedge y \in A]\}.$

Axiom 3 (Axiom of Unions). $\mathscr{M}(\cup(a)).$

Definition 5.6. $A \cup B \triangleq \{x \mid x \in A \vee x \in B\}.$

$$A \cap B \triangleq \{x \mid x \in A \wedge x \in B\}.$$

Remark. The symbol $\cup(A)$ denotes the union of the members of A; we will read this symbol simply as "union A." We read $A \cup B$ as "A union B" and we read $A \cap B$ as "A intersect B."

Proposition 5.7. $a \cup b = \cup\{a, b\}.$

PROOF.
$$\begin{aligned} a \cup b &= \{x \mid x \in a \vee x = b\} \\ &= \{x \mid (\exists y)[x \in y \wedge y \in \{a, b\}]\} \\ &= \cup(\{a, b\}). \end{aligned}$$ □

Corollary 5.8. $\mathscr{M}(a \cup b).$

PROOF. Proposition 5.7, the Axiom of Unions, and the Axiom of Pairing. □

Exercises

Prove the following.

(1) $b \in \{a_1, a_2, \ldots, a_n\} \leftrightarrow b = a_1 \vee b = a_2 \vee \cdots \vee b = a_n$.

(2) $\{a_1, a_2, \ldots, a_{n+1}\} = \{a_1, a_2, \ldots, a_n\} \cup \{a_{n+1}\}, n \geqq 1$.

(3) $\mathscr{M}(\{a_1, a_2, \ldots, a_n\})$.

(4) $\mathscr{M}(\langle a_1, a_2, \ldots, a_n\rangle)$.

(5) $a \in b \cup \{b\} \leftrightarrow a \in b \vee a = b$.

(6) $A \cup B = B \cup A$.

(7) $A \cap B = B \cap A$.

(8) $(A \cup B) \cup C = A \cup (B \cup C)$.

(9) $(A \cap B) \cap C = A \cap (B \cap C)$.

(10) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

(11) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Remark. We next introduce the notions of subclass $A \subseteq B$, and power set, $\mathscr{P}(a)$.

Definition 5.9. $A \subseteq B \overset{\Delta}{\leftrightarrow} (\forall x)[x \in A \rightarrow x \in B]$.

$$A \subset B \overset{\Delta}{\leftrightarrow} A \subseteq B \wedge A \neq B.$$

Definition 5.10. $\mathscr{P}(a) \triangleq \{x \mid x \subseteq a\}$.

Remark. We read $A \subseteq B$ as "A is a subclass of B"; $A \subset B$ we read as "A is a proper subclass of B"; and we read $\mathscr{P}(a)$ as "the power set of a."

Axiom 4 (Axiom of Powers). $\mathscr{M}(\mathscr{P}(a))$.

Exercises

Reduce the following wffs in the wider sense to wffs.

(1) $\{a, b\} \in \cup(c)$,

(2) $\mathscr{M}(\{a, b\})$,

(3) $\mathscr{M}(\cup(a))$,

(4) $\mathscr{M}(\mathscr{P}(a))$.

Prove the following.

(5) $A \subseteq B \wedge B \subseteq C \rightarrow A \subseteq C$.

(6) $A \subseteq B \rightarrow C \cap A \subseteq C \cap B$.

(7) $A \subseteq B \rightarrow C \cup A \subseteq C \cup B$.

(8) $A \subseteq B \leftrightarrow B = (A \cup B)$.

(9) $A \subseteq B \leftrightarrow A = (A \cap B)$.

(10) $A \subseteq A$.

(11) $A \subseteq A \cup B$.

(12) $A \cap B \subseteq A \wedge A \cap B \subseteq B$.

(13) $A \subset B \leftrightarrow (\exists x)[x \in A \wedge x \notin B]$.

(14) $A \subset B \rightarrow [\cup(A) \subseteq \cup(B)]$.

(15) $A = B \rightarrow [\cup(A) = \cup(B)]$.

(16) $a \in \mathscr{P}(a)$.

(17) $\cup(\mathscr{P}(a)) = a$.

(18) $a \subset b \leftrightarrow \mathscr{P}(a) \subset \mathscr{P}(b)$.

(19) $a = b \leftrightarrow \mathscr{P}(a) = \mathscr{P}(b)$.

Remark. The Axiom of Abstraction asserts that the class of all individuals that have a given property φ, is a set. Using class variables we can state this simply as

$$\mathscr{M}(A).$$

But, since the Axiom of Abstraction leads to Russell's paradox, we must reject it.

Zermelo proposed to replace the Axiom of Abstraction by an axiom that asserts that the class of all individuals in a given set a that have a specified property φ, is a set.

The Axiom Schema of Separation.

$$\mathscr{M}(a \cap A).$$

It is easily shown that this axiom does not lead to the Russell paradox.

Zermelo's set theory, as presented in 1908, consisted essentially of our Axioms 1–4, the Axiom Schema of Separation, an axiom of infinity, (See Axiom 7, page 43), and an axiom of choice. The theory we are developing differs from Zermelo's theory of 1908 in two respects. First, for the consistency proofs of Chapter 15, we must exclude the axiom of choice from the list of axioms of our basic theory. Second, Zermelo's Axiom of Separation is replaced by a much stronger axiom due to Fraenkel.

In 1922, Fraenkel proposed a modification of Zermelo's theory in which the Axiom Schema of Separation is replaced by an axiom that asserts that functions map sets onto sets.[1]

The condition that a wff $\varphi(a, b)$ should define a function, i.e., that

$$\{\langle x, y\rangle \mid \varphi(x, y)\}$$

should be a single valued relation is simply that

$$(\forall x, y, z)[\varphi(x, y) \wedge \varphi(x, z) \rightarrow y = z].$$

If this is the case and if

$$A = \{x \mid (\exists y)\varphi(x, y)\} \qquad \text{and} \qquad B = \{y \mid (\exists x)\varphi(x, y)\}$$

then the function in question maps A onto B and by *Fraenkel's* axiom maps $a \cap A$ onto a subset of B. That is

$$\mathscr{M}(\{y \mid (\exists x \in a)\varphi(x, y)\}).$$

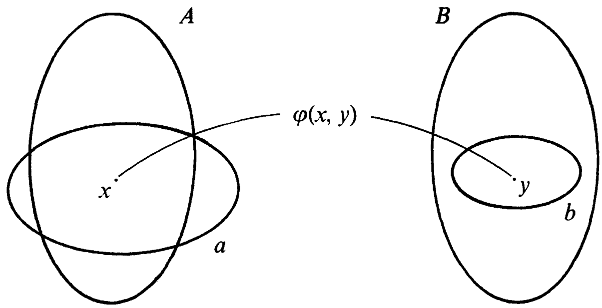

Figure 1

Axiom 5 (Axiom Schema of Replacement).

$$[(\forall x)(\forall y)(\forall z)[\varphi(x, y) \wedge \varphi(x, z) \rightarrow y = z] \rightarrow \mathscr{M}(\{y \mid (\exists x \in a)\varphi(x, y)\})].$$

Remark. From Fraenkel's axiom we can easily deduce Zermelo's. The two are however not equivalent. Indeed Richard Montague has proved that ZF is not a finite extension of Zermelo set theory.[2]

Proposition 5.11 (Zermelo's Schema of Separation).

$$\mathscr{M}(a \cap A)$$

PROOF. Applying Axiom 5 to the wff $b \in A \wedge b = c$ where b and c do not occur in A, we have that

$$(\forall x, y, z)[x \in A \wedge x = y] \wedge [x \in A \wedge x = z] \rightarrow y = z.$$

[1] This same idea was formulated, independently, by Thoralf Skolem, also in 1922.

[2] *Essays on the Foundations of Mathematics.* Amsterdam: North-Holland Publishing Company.

Therefore

$$\mathscr{M}(\{y \mid (\exists\, x \in a)[x \in A \;\wedge\; x = y]\})$$

i.e.,

$$\mathscr{M}(a \cap A). \qquad \square$$

Definition 5.12. $A - B \triangleq \{x \mid x \in A \;\wedge\; x \notin B\}$.

Remark. The class $A - B$ is called the complement of B relative to A but we will read the symbol $A - B$ simply as "A minus B."

Hereafter we will write $\{x \in a \mid \varphi(x)\}$ for $\{x \mid x \in a \;\wedge\; \varphi(x)\}$.

Proposition 5.13. $\mathscr{M}(a - A)$.

PROOF. $a - A = \{x \in a \mid x \notin A\}$. $\square$

Definition 5.14. $0 \triangleq \{x \mid x \neq x\}$.

Proposition 5.15. $a - a = 0$.

PROOF.
$$\begin{aligned} a - a &= \{x \in a \mid x \notin a\} \\ &= \{x \mid x \neq x\} \\ &= 0. \end{aligned} \qquad \square$$

Corollary 5.16. $\mathscr{M}(0)$.

PROOF. Propositions 5.15 and 5.13. $\square$

Remark. We read 0 as "the empty set."

Proposition 5.17.

(1) $(\forall\, x)[x \notin 0]$.

(2) $a \neq 0 \leftrightarrow (\exists\, x)[x \in a]$.

PROOF.

(1) $(\forall\, x)[x = x]$. Therefore $(\forall\, x)[x \notin 0]$.

(2) $a \neq 0 \leftrightarrow (\exists\, x)[x \in 0 \;\wedge\; x \notin a] \;\vee\; (\exists\, x)[x \in a \;\wedge\; x \notin 0]$.

Since $(\forall\, x)[x \notin 0]$ we conclude that $a \neq 0 \leftrightarrow (\exists\, x)[x \in a]$. $\square$

Remark. To exclude the possibility that a set can be an element of itself and also to exclude the possibility of having "$\in$-*loops*," i.e., $a_1 \in a_2 \in \cdots \in a_n \in a_1$, Zermelo introduced his Axiom of Regularity, also known as the Axiom of Foundation, which asserts that every nonempty set a contains an element x with the property that no element of x is also an element of a. A stronger form of this axiom asserts the same property of nonempty classes. Later we will prove that the weak and strong forms are in fact equivalent.

Axiom 6 (Axiom of Regularity, weak form).

$$a \neq 0 \rightarrow (\exists\, x \in a)[x \cap a = 0].$$

Axiom 6′ (Axiom of Regularity, strong form).

$$A \neq 0 \rightarrow (\exists\, x \in A)[x \cap A = 0].$$

Proposition 5.18. $\neg[a_1 \in a_2 \in \cdots \in a_n \in a_1]$.

PROOF. Let $a = \{a_1, a_2, \ldots, a_n\}$. Suppose that $a_1 \in a_2 \in \cdots \in a_n \in a_1$. Then $(\forall\, x)[x \in a \rightarrow x \cap a \neq 0]$. This contradicts Regularity.

Corollary 5.19. $a \notin a$.

PROOF. Proposition 5.18 with $n = 1$. □

Definition 5.20. $V \triangleq \{x \mid x = x\}$.

Proposition 5.21. $\mathscr{P}r(V)$.

PROOF. Since $V = V$ it follows that if V is a set, then $V \in V$. □

Remark. From the strong form of Regularity we can deduce the following induction principle.

Proposition 5.22. $(\forall\, x)[x \subseteq A \rightarrow x \in A] \rightarrow A = V$.

PROOF. Assume that $(\forall\, x)[x \subseteq A \rightarrow x \in A]$. If $B = V - A$ and if $B \neq 0$ then by (strong) Regularity there exists a set a such that

$$a \in B \wedge a \cap B = 0$$

that is

$$(\forall\, y)[y \in a \rightarrow y \notin B].$$

But since $B = V - A$,

$$(\forall\, y)[y \notin B \rightarrow y \in A].$$

Thus $a \subseteq A$ and hence, by our hypothesis, $a \in A$. But this contradicts the fact that $a \in B$. Therefore $B = 0$ and $A = V$. (See Exercise (1) below.) □

Remark. Proposition 5.22 assures us that if every set a has a certain property, $\varphi(a)$, whenever each element of a has that property then every set does indeed have the property. Consider the following example. If each element of a set a has no infinite descending $\in$-chain then clearly a has no infinite descending $\in$-chain. Therefore there are no infinite descending $\in$-chains.

EXERCISES

Prove the following.

(1) $0 \subseteq A \wedge A \subseteq V$.

(2) $(\forall x)[x \notin A] \rightarrow A = 0$.

(3) $A \subseteq a \rightarrow \mathscr{M}(A)$.

(4) $\mathscr{M}(A) \rightarrow \mathscr{M}(A \cap B)$.

(5) $\mathrm{Ru} = V$.

(6) $A \notin A$.

(7) $A - B = 0 \leftrightarrow A \subseteq B$.

(8) $A - B = A \cap \{x | x \notin B\}$.

(9) $A - (B \cup C) = (A - B) \cap (A - C)$.

(10) $A - (B \cap C) = (A - B) \cup (A - C)$.

(11) $A - (B - A) = A$.

(12) $A \cap (B - C) = (A \cap B) - (A \cap C) = (A \cap B) - C$.

(13) $A \cup (B - C) = (A \cup B) - (C - A) = (A \cup B) - ((B \cap C) - A)$.

(14) $A - B \subseteq A$.

(15) $A \subseteq B \rightarrow [C - B \subseteq C - A]$.

CHAPTER 6

Functions and Relations

Definition 6.1. $A \times B \triangleq \{x \mid (\exists y \in A)(\exists z \in B)[x = \langle y, z\rangle]\}$.

Remark. We read the symbol $A \times B$ as "A cross B."

Proposition 6.2. $\mathscr{M}(a \times b)$.

PROOF.

$$\begin{aligned} c \in a \times b &\to (\exists x, y)[x \in a \wedge y \in b \wedge c = \langle x, y\rangle]. \\ &\to (\exists x, y)[\{x\} \subseteq a \cup b \wedge \{x, y\} \subseteq a \cup b \wedge c = \langle x, y\rangle]. \\ &\to (\exists x, y)[\{x\}, \{x, y\} \in \mathscr{P}(a \cup b) \wedge c = \langle x, y\rangle]. \\ &\to (\exists x, y)[\langle x, y\rangle \in \mathscr{P}(\mathscr{P}(a \cup b)) \wedge c = \langle x, y\rangle]. \\ &\to c \in \mathscr{P}(\mathscr{P}(a \cup b)). \end{aligned}$$

Therefore $a \times b \subseteq \mathscr{P}(\mathscr{P}(a \cup b))$; hence $\mathscr{M}(a \times b)$. $\square$

Definition 6.3. $A^1 \triangleq A$.

(1) $A^{n+1} \triangleq A^n \times A$.

(2) $A^{-1} \triangleq \{\langle x, y\rangle \mid \langle y, x\rangle \in A\}$.

Remark. We read A^{-1} as "A converse." If A contains elements that are not ordered pairs, for example, if $A = \{\langle 0, 1\rangle, 0\}$ then $(A^{-1})^{-1} \neq A$; indeed for the example at hand $A^{-1} = \{\langle 1, 0\rangle\}$ and $(A^{-1})^{-1} = \{\langle 0, 1\rangle\}$.

Definition 6.4.

(1) $\mathscr{R}el(A) \overset{\triangle}{\leftrightarrow} A \subseteq V^2$.

(2) $\mathscr{U}n(A) \overset{\triangle}{\leftrightarrow} (\forall x, y, z)[\langle x, y\rangle, \langle x, z\rangle \in A \to y = z]$.

(3) $\mathscr{U}n_2(A) \overset{\triangle}{\leftrightarrow} \mathscr{U}n(A) \wedge \mathscr{U}n(A^{-1})$.
(4) $\mathscr{F}nc(A) \overset{\triangle}{\leftrightarrow} \mathscr{R}el(A) \wedge \mathscr{U}n(A)$.
(5) $\mathscr{F}nc_2(A) \overset{\triangle}{\leftrightarrow} \mathscr{R}el(A) \wedge \mathscr{U}n_2(A)$.

Remark. We read

$\mathscr{R}el(A)$ as "A is a relation,"
$\mathscr{U}n(A)$ as "A is single valued,"
$\mathscr{U}n_2(A)$ as "A is one-to-one,"
$\mathscr{F}nc(A)$ as "A is a function,"

and

$\mathscr{F}nc_2(A)$ as "A is a one-to-one function."

Definition 6.5.

(1) $\mathscr{D}(A) \triangleq \{x \mid (\exists y)[\langle x, y\rangle \in A]\}$.
(2) $\mathscr{W}(A) \triangleq \{y \mid (\exists x)[\langle x, y\rangle \in A]\}$.

Remark. We read $\mathscr{D}(A)$ and $\mathscr{W}(A)$ as "domain of A" and "range of A" respectively.

It should be noted that a class does not have to be a relation in order to have a domain and a range. Indeed every class has both. The domain of A is simply the class of first entries of those ordered pairs that are in A and the range of A is the class of second entries of those ordered pairs that are in A.

Definition 6.6.

(1) $A \upharpoonright B \triangleq A \cap (B \times V)$.
(2) $A``B \triangleq \mathscr{W}(A \upharpoonright B)$.
(3) $A \circ B \triangleq \{\langle x, y\rangle \mid (\exists z)[\langle x, z\rangle \in B \wedge \langle z, y\rangle \in A]\}$.

Remark. We read

$A \upharpoonright B$ as "the restriction of A to B,"
$A``B$ as "the image of B under A,"

and

$A \circ B$ as "the composite of A with B."

Note that $A \upharpoonright B$ is the class of ordered pairs in A having first entry in B and $A``B$ is the class of second entries of those ordered pairs in A that have first entry in B.

Exercises

Prove the following.

(1) $(A^{-1})^{-1} \subseteq A$.

(2) $(A^{-1})^{-1} = A \leftrightarrow \mathscr{R}el(A)$.

(3) $A \times B \subseteq V^2$.

(4) $V^2 \subset V$.

(5) $\mathscr{R}el(A) \wedge \mathscr{R}el(B) \rightarrow \mathscr{R}el(A \cup B)$.

(6) $(\forall x)[x \in A \rightarrow \mathscr{R}el(x)] \rightarrow \mathscr{R}el(\cup(A))$.

(7) $\mathscr{U}n_2(A) \rightarrow (\forall w, x, y, z)[\langle w, x\rangle, \langle z, y\rangle \in A \rightarrow [w = z \leftrightarrow x = y]]$.

(8) $\mathscr{F}nc_2(A) \rightarrow \mathscr{F}nc(A) \wedge \mathscr{F}nc(A^{-1})$.

(9) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

(10) $A \subseteq B \rightarrow \mathscr{D}(A) \subseteq \mathscr{D}(B)$.

(11) $A \subseteq B \rightarrow \mathscr{W}(A) \subseteq \mathscr{W}(B)$.

(12) $\mathscr{R}el(A \upharpoonright B)$.

(13) $\langle a, b\rangle \in A \upharpoonright B \leftrightarrow \langle a, b\rangle \in A \wedge a \in B$.

(14) $\mathscr{D}(A \upharpoonright B) = B \cap \mathscr{D}(A)$.

(15) $A \upharpoonright B \subseteq A$.

(16) $[A = A \upharpoonright \mathscr{D}(A)] \leftrightarrow \mathscr{R}el(A)$.

(17) $B \subseteq C \rightarrow [(A \upharpoonright C) \upharpoonright B = A \upharpoonright B]$.

(18) $\mathscr{U}n(A) \rightarrow \mathscr{U}n(A \upharpoonright B)$.

(19) $b \in A``B \leftrightarrow (\exists x)[\langle x, b\rangle \in A \wedge x \in B]$.

(20) $[A \subseteq B \wedge C \subseteq D] \rightarrow A``C \subseteq B``D$.

(21) If $A = \{\langle\langle x, y\rangle, \langle y, x\rangle\rangle \mid x \in V \wedge y \in V\}$, then $\mathscr{U}n(A) \wedge A``B = B^{-1}$.

(22) If $A = \{\langle\langle x, y\rangle, x\rangle \mid x \in V \wedge y \in V\}$, then $\mathscr{U}n(A) \wedge A``B = \mathscr{D}(B)$.

(23) If $A = \{\langle\langle x, y\rangle, y\rangle \mid x \in V \wedge y \in V\}$, then $\mathscr{U}n(A) \wedge A``B = \mathscr{W}(B)$.

(24) $\mathscr{R}el(A \circ B)$.

(25) $\mathscr{U}n(A) \wedge \mathscr{U}n(B) \rightarrow \mathscr{U}n(A \circ B)$.

(26) $\mathscr{U}n(A) \wedge \mathscr{U}n(B) \rightarrow \mathscr{F}nc(A \circ B)$.

(27) $\mathscr{U}n_2(A) \wedge \mathscr{U}n_2(B) \rightarrow \mathscr{U}n_2(A \circ B)$.

(28) $\mathscr{U}n_2(A) \wedge \mathscr{U}n_2(B) \rightarrow \mathscr{F}nc_2(A \circ B)$.

(29) $\mathscr{F}nc(A) \wedge \mathscr{F}nc(B) \rightarrow \mathscr{F}nc(A \circ B)$.

(30) $\mathscr{F}nc_2(A) \wedge \mathscr{F}nc_2(B) \rightarrow \mathscr{F}nc_2(A \circ B)$.

(31) $A^{-1} = (A \cap V^2)^{-1}$.

Proposition 6.7. $\mathcal{U}n(A) \to \mathcal{M}(A``a)$.

Proof. From Definition 6.4(2)

$$\mathcal{U}n(A) \leftrightarrow (\forall x)(\forall y)(\forall z)[\langle x, y\rangle \in A \wedge \langle x, z\rangle \in A \to y = z].$$

Then from the Axiom Schema of Replacement it follows that

$$\{y \mid (\exists x \in a)[\langle x, y\rangle \in A]\}$$

is a set, that is, $A``a$ is a set. □

Remark. Proposition 6.7 assures us that single valued relations, i.e., functions, map sets onto sets.

Corollary 6.8.

(1) $\mathcal{M}(a^{-1})$.

(2) $\mathcal{M}(\mathcal{D}(a))$.

(3) $\mathcal{M}(\mathcal{W}(a))$.

Proof. (1) If $A = \{\langle\langle x, y\rangle, \langle y, x\rangle\rangle \mid x, y \in V\}$, then A is single valued and hence by Proposition 6.7, $A``a$ is a set. But $A``a = a^{-1}$; therefore a^{-1} is a set.

(2) If $A = \{\langle\langle x, y\rangle, x\rangle \mid x, y \in V\}$, then A is single valued and $A``a = \mathcal{D}(a)$.

(3) If $A = \{\langle\langle x, y\rangle, y\rangle \mid x, y \in V\}$, then A is single valued and $A``a = \mathcal{W}(a)$. □

Corollary 6.9.

(1) $\mathcal{M}(A \times B) \leftrightarrow \mathcal{M}(B \times A)$.

(2) $\mathcal{P}r(A) \wedge B \neq 0 \to \mathcal{P}r(A \times B) \wedge \mathcal{P}r(B \times A)$.

Proof. (1) Suppose $C = \{\langle\langle x, y\rangle, \langle y, x\rangle\rangle \mid x, y \in V\}$. Then C is single valued, $C``(A \times B) = (B \times A)$ and $C``(B \times A) = (A \times B)$.

(2) If $B \neq 0$, then $(\exists y)[y \in B]$. Let $C = \{\langle\langle x, y\rangle, x\rangle \mid x \in A\}$. Then C is single valued and $\mathcal{D}(C) \subseteq A \times B$. Assuming that $A \times B$ is a set it follows that $\mathcal{D}(C)$ is a set. But $A = C``\mathcal{D}(C)$ and hence, by Proposition 6.7, A is a set. From this contradiction we conclude that $A \times B$ is a proper class and hence by (1) so is $B \times A$. □

Definition 6.10. $(\exists!\, x)\varphi(x) \overset{\Delta}{\leftrightarrow} (\exists x)\varphi(x) \wedge (\forall x, y)[\varphi(x) \wedge \varphi(y) \to x = y]$.

Remark. We read $(\exists!\, x)\varphi(x)$ as "there exists a unique x such that $\varphi(x)$."

Definition 6.11. $A`b \triangleq \{x \mid (\exists y)[x \in y \wedge \langle b, y\rangle \in A] \wedge (\exists!\, y)[\langle b, y\rangle \in A]\}$.

Remark. We read $A`b$ as "the value of A at b."

Proposition 6.12.

(1) $\langle b, c\rangle \in A \wedge (\exists!\, y)[\langle b, y\rangle \in A] \rightarrow A`b = c$.

(2) $\neg(\exists!\, y)[\langle b, y\rangle \in A] \rightarrow A`b = 0$.

PROOF. (1) From Definition 6.11, $\langle b, c\rangle \in A \wedge (\exists!\, y)[\langle b, y\rangle \in A]$ implies

$$a \in A`b \leftrightarrow a \in c$$

i.e., $A`b = c$.

(2) From Definition 6.11, $\neg(\exists!\, y)[\langle b, y\rangle \in A]$ implies

$$(\forall x)[x \notin A`b]$$

i.e., $A`b = 0$. □

Remark. From Proposition 6.12 we see that Definition 6.11 is an extension of the notion of function value. If A is a function and if b is in $\mathscr{D}(A)$ then $A`b$ is the value of $A$ at $b$ in the usual sense. If $b$ is not in $\mathscr{D}(A)$ then $A`b = 0$. If A is not a function $A`b$ is still defined. Indeed if $b$ is not in $\mathscr{D}(A)$ then $A`b = 0$. If b is in $\mathscr{D}(A)$ but there are two different ordered pairs in A with first entry b then again $A`b = 0$. If $b$ is in $\mathscr{D}(A)$ and $\langle b, c\rangle$ is the only ordered pair in $A$ with first entry $b$ then $A`b = c$.

Corollary 6.13. $\mathscr{M}(A`b)$.

PROOF. Proposition 6.12 assures us that $(\exists\, y)[A`b = y]$. □

Definition 6.14.

(1) $\{A`x \mid x \in B\} \triangleq \{y \mid (\exists\, x \in B)[y = A`x]\}$.

(2) $\bigcup_{x \in B} A`x \triangleq \bigcup \{A`x \mid x \in B\}$.

Remark. We read $\{A`x \mid x \in B\}$ as "the class of all $A`x$ such that $x \in B$," and we read

$$\bigcup_{x \in B} A`x$$

as "the union of all $A`x$ for $x \in B$."

Definition 6.15.

(1) $A \mathscr{F}n\, B \overset{\Delta}{\leftrightarrow} \mathscr{F}nc(A) \wedge \mathscr{D}(A) = B$.

(2) $A \mathscr{F}n_2\, B \overset{\Delta}{\leftrightarrow} \mathscr{F}nc_2(A) \wedge \mathscr{D}(A) = B$.

(3) $F : A \longrightarrow B \overset{\Delta}{\leftrightarrow} F \mathscr{F}n\, A \wedge \mathscr{W}(F) \subseteq B$.

(4) $F : A \xrightarrow[\text{onto}]{} B \overset{\Delta}{\leftrightarrow} F \mathscr{F}n\, A \wedge \mathscr{W}(F) = B$.

(5) $F : A \xrightarrow{1-1} B \overset{\Delta}{\leftrightarrow} F \mathscr{F}n_2\, A \wedge \mathscr{W}(F) \subseteq B$.

(6) $F : A \xrightarrow[\text{onto}]{1-1} B \overset{\Delta}{\leftrightarrow} F \mathscr{F}n_2\, A \wedge \mathscr{W}(F) = B$.

Remark. We read

$A \mathscr{F}\!n\, B$ as "A is a function on B,"

$A \mathscr{F}\!n_2\, B$ as "A is a one-to-one function on B,"

$F: A \to B$ as "F maps A into B,"

$F: A \xrightarrow[\text{onto}]{} B$ as "F maps A onto B,"

$F: A \xrightarrow{1-1} B$ as "F maps A one-to-one into B,"

and

$F: A \xrightarrow[\text{onto}]{1-1} B$ as "F maps A one-to-one onto B."

Theorem 6.16.

(1) $A \mathscr{F}\!n\, a \to \mathscr{M}(A)$.

(2) $A \mathscr{F}\!n_2\, a \to \mathscr{M}(A)$.

PROOF. (1) If A is a function on a, then $A \subseteq a \times A``a$ and $A$ is single valued. But if $A$ is single valued, then by Proposition 6.7, $A``a$ is a set, and so by Proposition 6.2, $a \times A``a$ is a set. Since $A \subseteq a \times A``a$, it then follows that A is a set.

(2) If A is a one-to-one function on a, then A is a function on a and hence A is a set by (1). □

Proposition 6.17. $\mathscr{U}\!n(A) \to \mathscr{M}(A \upharpoonright a)$.

PROOF. If A is single valued, then certainly $A \upharpoonright a$ is single valued. Since, by Definition 6.6(1), $A \upharpoonright a$ is a relation, it follows that $A \upharpoonright a$ is a function on $\mathscr{D}(A \upharpoonright a)$. Furthermore, since $\mathscr{D}(A \upharpoonright a) \subseteq a$ it follows that $\mathscr{D}(A \upharpoonright a)$ is a set. Then by Proposition 6.16(1), $A \upharpoonright a$ is a set.

EXERCISES

Prove the following.

(1) $\mathscr{P}\!r(A) \to \mathscr{P}\!r(A^2)$.

(2) $\mathscr{P}\!r(V^2)$.

(3) $\mathscr{U}\!n(B) \wedge (\exists x)[A - \mathscr{W}(B \upharpoonright x) = 0] \to \mathscr{M}(A)$.

(4) $A \mathscr{F}\!n\, C \wedge B \mathscr{F}\!n\, D \to [A = B \leftrightarrow [C = D \wedge (\forall x)[x \in C \to A`x = B`x]]]$.

(5) $\mathscr{U}\!n(A) \wedge \mathscr{U}\!n(B) \wedge a \in \mathscr{D}(A \circ B) \to (A \circ B)`a = A`B`a$.

(6) $A_1 \mathscr{F}\!n\, B_1 \wedge A_2 \mathscr{F}\!n\, B_2 \wedge \mathscr{W}(A_2) \subseteq B_1 \to A_1 \circ A_2 \mathscr{F}\!n\, B_2$.

(7) $A_1 \mathscr{F}\!n_2\, B_1 \wedge A_2 \mathscr{F}\!n_2\, B_2 \wedge \mathscr{W}(A_2) = B_1 \to A_1 \circ A_2 \mathscr{F}\!n_2\, B_2 \wedge \mathscr{W}(A_1 \circ A_2) = \mathscr{W}(A_1)$.

(8) $A \mathscr{F}n_2 B \wedge \mathscr{W}(A) = C \rightarrow A^{-1} \mathscr{F}n_2 C \wedge \mathscr{W}(A^{-1}) = B$.

(9) $\mathscr{U}n(A) \wedge \mathscr{M}(B) \rightarrow \mathscr{M}(A``B)$.

(10) $\mathscr{U}n(A) \rightarrow A``B = \{A`x \mid x \in B \cap \mathscr{D}(A)\}$.

(11) $A \mathscr{F}nB \vee A \mathscr{F}n_2 B \rightarrow A = A \upharpoonright B$.

(12) $(\exists x)(\exists y)[x \neq y \wedge \langle b, x\rangle \in A \wedge \langle b, y\rangle \in A] \rightarrow A`b = 0$.

(13) $\neg(\exists x)[\langle b, x\rangle \in A] \rightarrow A`b = 0$.

Remark. In later chapters we will study structures consisting of a class A on which is defined a relation R, i.e., $R \subseteq A^2$. Since for any class B, $B \cap A^2 \subseteq A^2$ we see that every class B determines a relation on A in a very natural way. We therefore choose to begin our discussion with a very general theory of ordered pairs of classes $[A, R]$ that we will call relational systems.

Definition 6.18. $a\, R\, b \overset{\Delta}{\leftrightarrow} \langle a, b\rangle \in R$.

Remark. We read $a\, R\, b$ simply as "$a\, R\, b$."

In the material ahead we will be interested in several types of relational systems, $[A, R]$. We will be interested in systems in which R orders A and systems in which R partially orders A, in the following sense.

Definition 6.19.

(1) $R \operatorname{Or} A \overset{\Delta}{\leftrightarrow} (\forall x, y \in A)[x\, R\, y \leftrightarrow \neg[x = y \vee y\, R\, x]] \wedge (\forall x, y, z \in A)[x\, R\, y \wedge y\, R\, z \rightarrow x\, R\, z]$.

(2) $R \operatorname{Po} A \overset{\Delta}{\leftrightarrow} (\forall x \in A)[x\, R\, x] \wedge (\forall x, y \in A)[x\, R\, y \wedge y\, R\, x \rightarrow x = y] \wedge (\forall x, y, z \in A)[x\, R\, y \wedge y\, R\, z \rightarrow x\, R\, z]$.

Remark. There are several properties of relational systems whose proofs depend upon the classes of R-predecessors:

$$\{x \mid x\, R\, a\}$$

Proposition 6.20. $(R^{-1})``\{a\} = \{x \mid x\, R\, a\}$.

PROOF.
$$\begin{aligned}(R^{-1})``\{a\} &= \{x \mid \langle a, x\rangle \in R^{-1}\}\\ &= \{x \mid \langle x, a\rangle R\}\\ &= \{x \mid x\, R\, a\}.\end{aligned}$$ □

Remark. From Proposition 6.20 we see that $A \cap (R^{-1})``\{a\} = 0$ means that no element of A precedes a in the sense of R. If, in addition, $a \in A$ then a is an R-minimal element of A. We wish to consider relations with respect to which each subclass of a given class has an R-minimal element. Such a relation we call a *founded* relation. Since we cannot quantify on class symbols we must formulate our definition in terms of subsets and impose additional conditions that will enable us to deduce the property for subclasses. Later we will show that these additional conditions are not essential.

Definition 6.21.

$$R \operatorname{Fr} A \overset{\Delta}{\leftrightarrow} (\forall x)[x \subseteq A \wedge x \neq 0 \rightarrow (\exists y \in x)[x \cap (R^{-1})``\{y\} = 0]].$$

Remark. We read $F \operatorname{Fr} A$ as "F is a founded relation on A."

Definition 6.22. $E \triangleq \{\langle x, y\rangle \mid x \in y\}$.

Remark. From the Axiom of Regularity we see that the $\in$-relation E is a founded relation on V. As in the case of the $\in$-relation, founded relations have no relational loops and, as we will prove later, no infinite descending relational chains.

Proposition 6.23.

$$R \operatorname{Fr} A \wedge a_1 \in A \wedge \cdots \wedge a_n \in A \rightarrow \neg[a_1 R a_2 \wedge a_2 R a_3 \wedge \cdots \wedge a_n R a_1].$$

The proof is left to the reader.

Remark. There are two types of founded relations that are of special interest, the well-founded relations and the well-ordering relations.

Definition 6.24.

(1) $R \operatorname{Wfr} A \overset{\Delta}{\leftrightarrow} R \operatorname{Fr} A \wedge (\forall x \in A)[\mathscr{M}(A \cap (R^{-1})``\{x\})]$.

(2) $R \operatorname{We} A \overset{\Delta}{\leftrightarrow} R \operatorname{Fr} A \wedge (\forall x \in A)(\forall y \in A)[x R y \vee x = y \vee y R x]$.

(3) $R \operatorname{Wfwe} A \overset{\Delta}{\leftrightarrow} R \operatorname{Wfr} A \wedge R \operatorname{We} A$.

Remark. Note that R is a founded relation on A iff each nonempty subset of A has an R-minimal element. Furthermore, R is a well-founded relation on A iff each nonempty subset of A has an R-minimal element and each R-initial segment of A is a set. By an R-initial segment of A we mean the class of all elements in A that R-precede a given element of A, i.e., $A \cap (R^{-1})``\{a\}$ for $a \in A$. For example, each E-initial segment of V is a set, indeed

$$(E^{-1})``\{a\} = \{x \mid x \in a\} = a.$$

Then $(\forall x)[a \cap x = a \cap (E^{-1})``\{x\}]$ and hence from the Axiom of Regularity

$$a \neq 0 \rightarrow (\exists x \in a)[a \cap (E^{-1})``\{x\} = 0]$$

that is, E is well founded on V.

There do exist founded relations that are not well founded. Let A be the class of all finite sets and for $a, b \in A$ define $a R b$ to mean that a has fewer elements than b. Given any nonempty collection of finite sets there is a set in the collection that has the least number of elements. Thus R is founded on A. However the R-initial segment of A that contains all finite sets that R-precede a given doubleton set contains all singleton sets hence is a proper class. Thus R is not well founded on A.

R is a well ordering of A iff R determines an R-minimal element for each nonempty subset of A and the elements in A are pairwise R-comparable. If there were elements $a, b \in A$ that were not R-comparable, i.e., neither $a\,R\,b$ nor $b\,R\,a$, then both a and b would be R-minimal elements of $\{a, b\}$. Conversely if a and b are R-comparable then a and b cannot both be R-minimal elements of the same set. Thus if R well orders A then R determines a unique R-minimal element for each nonempty subset of A. That R is a transitive relation satisfying trichotomy we leave to the reader:

Proposition 6.25. R We $A \to R$ Or A.

EXERCISES

(1) $R \text{ Fr } A \wedge B \subseteq A \to R \text{ Fr } B$.

(2) $R \text{ Fr } a \to R \text{ Wfr } a$.

(3) $R \text{ Wfr } A \wedge B \subseteq A \to R \text{ Wfr } B$.

(4) $R \text{ We } A \wedge B \subseteq A \to R \text{ We } B$.

(5) $R \text{ We } A \to (\forall x, y \in A)[x\,R\,y \to \neg[x = y \vee y\,R\,x]]$.

(6) $R \text{ We } A \to (\forall x, y, z \in A)[x\,R\,y \wedge y\,R\,z \to x\,R\,z]$.

Remark. If a relation R well orders a class A, does it follow that R determines an R-minimal element for every nonempty subclass of A? If R is a well-founded well ordering of A, i.e., R Wfwe A then the answer is, Yes:

Proposition 6.26.

$$R \text{ Wfwe } A \wedge B \subseteq A \wedge B \neq 0 \to (\exists x \in B)[B \cap (R^{-1})``\{x\} = 0].$$

PROOF. If B is not empty, then B contains an element b. If $B \cap (R^{-1})``\{b\} = 0$, then $b$ is the set we seek. But suppose that $B \cap (R^{-1})``\{b\} \neq 0$. If in addition $B \subseteq A$ and R is a well-founded well ordering of A, then R is also a well-founded well ordering of B. Since $b \in B$ it follows that $B \cap (R^{-1})``\{b\}$ is a set. In fact, it is a subset of B and hence it has an R-minimal element. that is, there exists an a such that

$$a \in B \cap (R^{-1})``\{b\} \wedge B \cap (R^{-1})``\{b\} \cap (R^{-1})``\{a\} = 0.$$

From this it follows that $a \in B$ and $a\,R\,b$. Then, since R is a transitive relation

$$B \cap (R^{-1})``\{a\} \subseteq B \cap (R^{-1})``\{b\} \cap (R^{-1})``\{a\}$$

and since $B \cap (R^{-1})``\{b\} \cap (R^{-1})``\{a\} = 0$, it follows that $B \cap (R^{-1})``\{a\} = 0$. Consequently

$$a \in B \wedge B \cap (R^{-1})``\{a\} = 0$$

that is

$$(\exists x \in B)[B \cap (R^{-1})``\{x\} = 0]. \qquad \square$$

Proposition 6.27.

R Wfwe $A \wedge B \subseteq A \wedge (\forall x \in A)[A \cap (R^{-1})``\{x\} \subseteq B \rightarrow x \in B] \rightarrow A = B.$

Proof. If $A - B \neq 0$ then by Proposition 6.26

$$(\exists x \in A - B)[(A - B) \cap (R^{-1})``\{x\} = 0]$$

then

$$A \cap (R^{-1})``\{x\} \subseteq B.$$

Since $x \in A$ it follows from hypothesis that $x \in B$. But this contradicts the fact that $x \in A - B$.

Therefore $A - B = 0$ that is $A \subseteq B$. Since by hypothesis $B \subseteq A$ we conclude that $A = B$. $\square$

Remark. Proposition 6.27 is an induction principle. To prove that $(\forall x \in A)\varphi(x)$, we consider

$$B = \{x \in A \mid \varphi(x)\}.$$

If for any R that is a well-founded well ordering on A we can prove

$$(\forall x \in A)[A \cap (R^{-1})``\{x\} \subseteq B \rightarrow x \in B]$$

it then follows that $A = B$, i.e. $(\forall x \in A)\varphi(x)$.

Later it will be shown that Propositions 6.26 and 6.27 are over hypothesized. We will prove that the hypothesis R Wfwe A can be replaced by R Fr A. See Propositions 9.21 and 9.22 pages 80–1.

Clearly two relational systems $[A, R_1]$ and $[A, R_2]$ are essentially the same if R_1 and R_2 have the same relational part in common with A^2, i.e., if $A^2 \cap R_1 = A^2 \cap R_2$. Even if R_1 and R_2 do not have the same relational part there is a sense in which the two relational systems are equivalent. They may be equivalent in the sense that there exists an isomorphism between the two relational systems.

Definition 6.28.

$$H \operatorname{Isom}_{R_1, R_2}(A_1, A_2) \overset{\Delta}{\leftrightarrow} H : A_1 \xrightarrow[\text{onto}]{1-1} A_2 \wedge$$
$$(\forall x, y \in A_1)[xR_1y \leftrightarrow H`xR_2H`y].$$

Remark. We read $H \operatorname{Isom}_{R_1, R_2}(A_1, A_2)$ as "H is an R_1, R_2" isomorphism of A_1 onto A_2.

Definition 6.29. $I \triangleq \{\langle x, x\rangle \mid x \in V\}$.

Proposition 6.30.

(1) $(I \restriction A)\mathrm{Isom}_{R,R}(A, A)$.

(2) $H\ \mathrm{Isom}_{R_1,R_2}(A_1, A_2) \rightarrow H^{-1}\ \mathrm{Isom}_{R_2,R_1}(A_2, A_1)$.

(3) $H_1\ \mathrm{Isom}_{R_1,R_2}(A_1, A_2) \wedge H_2\ \mathrm{Isom}_{R_2,R_3}(A_2, A_3) \rightarrow$
$H_2 \circ H_1\ \mathrm{Isom}_{R_1,R_3}(A_1, A_3)$.

The proof is left to the reader.

Proposition 6.31. *If* $H\ \mathrm{Isom}_{R_1,R_2}(A_1, A_2) \wedge B \subseteq A_1 \wedge a \in A_1$, *then*

(1) $B \cap (R_1^{-1})``\{a\} = 0 \leftrightarrow H``B \cap (R_2^{-1})``\{H`a\} = 0$,

(2) $H``(A_1 \cap (R_1^{-1})``\{a\}) = A_2 \cap (R_2^{-1})``\{H`a\}$.

Proof

(1)
$$\begin{aligned} b \in B \cap (R_1^{-1})``\{a\} &\rightarrow b \in B \wedge b\, R_1\, a \\ &\rightarrow H`b \in H``B \wedge H`b\, R_2\, H`a \\ &\rightarrow H`b \in H``B \cap (R_2^{-1})``\{H`a\}. \end{aligned}$$

$$\begin{aligned} b \in H``B \cap (R_2^{-1})``\{H`a\} &\rightarrow (\exists\, y \in B)[b = H`y \wedge b\, R_2\, H`a] \\ &\rightarrow (\exists\, y \in B)[H`y\, R_2\, H`a] \\ &\rightarrow (\exists\, y \in B)[y\, R_1\, a] \\ &\rightarrow (\exists\, y)[y \in B \cap (R_1^{-1})``\{a\}]. \end{aligned}$$

(2)
$$\begin{aligned} H``(A_1 \cap (R_1^{-1})``\{a\}) &= \{z \mid (\exists\, y \in A_1)[y\, R_1\, a \wedge z = H`y]\} \\ &= \{z \mid (\exists\, y \in A_1)[z = H`y \wedge H`y\, R_2\, H`a]\} \\ &= A_2 \cap (R_2^{-1})``\{H`a\}. \end{aligned}$$ □

Remark. Proposition 6.30 assures us that isomorphism between relational systems is an equivalence relation. From Proposition 6.31 we see that such isomorphisms preserve minimal elements and preserve initial segments. From this it is easy to prove the following. Details are left to the reader.

Proposition 6.32. *If* $H\ \mathrm{Isom}_{R_1,R_2}(A_1, A_2)$, *then*

(1) $R_1\ \mathrm{Fr}\ A_1 \leftrightarrow R_2\ \mathrm{Fr}\ A_2$,

(2) $R_1\ \mathrm{Wfr}\ A_1 \leftrightarrow R_2\ \mathrm{Wfr}\ A_2$,

(3) $R_1\ \mathrm{We}\ A_1 \leftrightarrow R_2\ \mathrm{We}\ A_2$.

Remark. From Proposition 6.32 we see that if in a given equivalence class of isomorphic relational systems, there is a relational system that is founded then every relational system in that equivalence class is founded. Similarly if there is a relational system that is well founded then all systems in that class are well founded; if one system is a well ordering all are well orderings.

Each equivalence class represents a particular type of ordering. Suppose that we are given a particular type of ordering, $[A_1, R_1]$ with $R_1 \subseteq A_1^2$ and a class A_2, can we define an ordering on A_2 of the same type, that is, can we define a relation $R_2 \subseteq A_2^2$ such that the ordering $[A_1, R_1]$ is order isomorphic to the ordering $[A_2, R_2]$?

From the definition of order isomorphism we see that it is necessary that there exist a one-to-one correspondence between A_1 and A_2. This is also sufficient.

Proposition 6.33. *If*

$$H: A_1 \xrightarrow[\text{onto}]{1-1} A_2 \wedge R_2 = \{\langle H`x, H`y\rangle \mid x \in A_1 \wedge y \in A_1 \wedge \langle x, y\rangle \in R_1\}$$

then

$$H \operatorname{Isom}_{R_1, R_2}(A_1, A_2).$$

The proof is left to the reader.

Remark. The relation R_2 in Proposition 6.33 is said to be induced on A_2 by the one-to-one function H and the relation R_1, on A_1. The proposition assures us that if a one-to-one correspondence exists between two classes then any type of ordering that can be defined on one class can also be defined on the other class. While this is a very useful result it leaves unanswered the question of what types of relations are definable on a given class A. Are there founded relations definable on A? Are there well-founded relations on A? Can A be well ordered? The first two questions are easily answered because the $\in$-relation is well founded on A.

The last question is the most interesting. From the work of Paul Cohen we know that the question of whether or not every set can be well ordered, is undecidable in ZF. We will have more to say on this subject later.

EXERCISE

If R_2 We A_2 then

$$H \operatorname{Isom}_{R_1, R_2}(A_1, A_2) \rightarrow H: A_1 \xrightarrow[\text{onto}]{1-1} A_2 \wedge$$

$$(\forall x \in A_1)[H`x \in A_2 - H``(R_1^{-1})``\{x\} \wedge (A_2 - H``(R_1^{-1})``\{x\}) \cap (R_2^{-1})``\{H`x\} = 0].$$

CHAPTER 7

Ordinal Numbers

The theory of ordinal numbers is essentially a theory of well-ordered sets. For Cantor an ordinal number was "the general concept which results from (a well-ordered aggregate) M if we abstract from the nature of its elements while retaining their order of precedence ...." It was Gottlob Frege (1848–1925) and Bertrand Russell (1872–1970), working independently, who removed Cantor's numbers from the realm of psychology. In 1903 Russell defined an ordinal number to be an equivalence class of well-ordered sets under order isomorphism. Russell's definition has a certain intuitive appeal. By his definition the ordinal number one is the class of all well-ordered singleton sets, the ordinal number two is the class of all well-ordered doubleton sets, etc. But this definition has a serious defect from the point of view of ZF set theory because the class of all singleton sets is a proper class, as is the class of all doubleton sets, etc. For our purposes we would like ordinal numbers to be sets and to acheive this we take a different approach from that of Russell.

Our approach is that of von Neumann. We choose to define ordinal numbers to be particular members of equivalence classes rather than the equivalence classes themselves. The particular sets that we choose to be our ordinal numbers are sets that are well ordered by the $\in$-relation and which are transitive in the following sense.

Definition 7.1. $\mathrm{Tr}(A) \overset{\Delta}{\leftrightarrow} (\forall x \in A)[x \subseteqq A]$.

Remark. We read $\mathrm{Tr}(A)$ as "A is transitive."

Proposition 7.2. $\mathrm{Tr}(A) \wedge B \in A \rightarrow B \subset A$.

Proof. If $B \in A$, then B is a set. If in addition A is transitive, then by Definition 7.1, $B \subseteq A$. But $B = A$ implies that $A \in A$ which contradicts Corollary 5.19. Consequently $B \subset A$. □

Remark. In spite of our claim that we are going to define ordinal numbers to be sets we begin by defining ordinal classes.

Definition 7.3. $\mathrm{Ord}(A) \overset{\Delta}{\leftrightarrow} \mathrm{Tr}(A) \wedge (\forall x, y \in A)[x \in y \vee x = y \vee y \in x]$.

Remark. We read $\mathrm{Ord}(A)$ as "A is an ordinal."

Since, by the Axiom of Regularity, the $\in$-relation E is founded, indeed well founded, on every class, it follows that E well orders every ordinal.

Proposition 7.4. $\mathrm{Ord}(A) \rightarrow E \text{ We } A$.

Proof. Obvious from Definitions 7.3, 6.24(2), 6.21, and Axiom 6′. □

Remark. Proposition 7.4 assures us that every subset of an ordinal class A has an E-minimal element. We can in fact prove a stronger result, namely that every subclass of an ordinal A has an E-minimal element:

Proposition 7.5. $\mathrm{Ord}(A) \wedge B \subseteq A \wedge B \neq 0 \rightarrow (\exists x \in B)[B \cap x = 0]$.

Proof. From Proposition 7.4, E well orders A. Since E is also well founded on A, i.e., E is founded and E-initial segments of A are sets, it follows from Proposition 6.26 that B has an E-minimal element, i.e.,

$$(\exists x \in B)[B \cap (E^{-1})``\{x\} = 0].$$

But $(E^{-1})``\{x\} = x$. □

Proposition 7.6. $\mathrm{Ord}(A) \wedge a \in A \rightarrow \mathrm{Ord}(a)$.

Proof. Since A is transitive $a \in A$ implies $a \subseteq A$. Consequently if $b, c \in a$, then $b, c \in A$ and hence, since A is an ordinal,

$$b \in c \vee b = c \vee c \in b$$

that is,

$$(\forall x, y \in a)[x \in y \vee x = y \vee y \in x].$$

It then remains to prove that a is transitive, i.e., that $b \in a$ implies $b \subseteq a$. Toward this end we note that if $c \in b$, then since A is transitive, $c, a \in A$ and hence

$$c \in a \vee c = a \vee a \in c.$$

But $c = a \wedge c \in b \wedge b \in a$ and $a \in c \wedge c \in b \wedge b \in a$ each contradict Proposition 5.18, and so we conclude that $c \in a$. Thus, having shown that

$$c \in b \rightarrow c \in a$$

we have proved that

$$b \in a \rightarrow b \subseteqq a.$$

Consequently we have shown that a is transitive and hence a is an ordinal. $\square$

Remark. We now wish to prove that the $\in$-relation also well orders the class of ordinal sets. From this and Proposition 7.6 it will then follow that the class of ordinal sets is an ordinal class.

Proposition 7.7. $\mathrm{Ord}(A) \wedge \mathrm{Tr}(B) \rightarrow [B \subset A \leftrightarrow B \in A]$.

PROOF. Since A is an ordinal, A is transitive and so by Proposition 7.2

$$B \in A \rightarrow B \subset A.$$

Conversely if $B \subset A$, then $A - B \neq 0$. From Proposition 7.5, $A - B$ has an E-minimal element b, that is,

$$(b \in A - B) \wedge (A - B) \cap b = 0.$$

Clearly $b \in A$. To prove that $B \in A$ we will prove that $B = b$. Toward this end we note that since $b \in A$ and A is transitive $b \subset A$. But $(A - B) \cap b = 0$, and so $b \subseteqq B$.

To prove that $B \subseteqq b$ we observe that if $c \in B$ then, since $B \subset A$, $c \in A$. But A is an ordinal class and $b \in A$. Therefore

$$c \in b \vee c = b \vee b \in c.$$

From the transitivity of B we see that $[b \in c \vee b = c] \wedge c \in B$ implies that $b \in B$. But this contradicts the fact that $b \in A - B$. We conclude that $c \in b$, that is, $B \subseteqq b$.

Then $b = B \wedge b \in A$; hence $B \in A$. $\square$

Corollary 7.8. $\mathrm{Ord}(A) \wedge \mathrm{Ord}(B) \rightarrow [B \subset A \leftrightarrow B \in A]$.

PROOF. $\mathrm{Ord}(B) \rightarrow \mathrm{Tr}(B)$. $\square$

Remark. Among other things Propositions 7.6 and 7.7 assure us that a transitive subclass of an ordinal is an ordinal.

Proposition 7.9. $\mathrm{Ord}(A) \wedge \mathrm{Ord}(B) \rightarrow \mathrm{Ord}(A \cap B)$.

PROOF. We first note that $A \cap B \subseteq A$. Furthermore, since A and B are transitive

$$\begin{aligned} a \in A \cap B &\to a \in A \wedge a \in B \\ &\to a \subset A \wedge a \subset B \\ &\to a \subset A \cap B. \end{aligned}$$

Therefore $A \cap B$ is a transitive subclass of the ordinal A and hence by Propositions 7.6 and 7.7, $A \cap B$ is an ordinal class. □

Proposition 7.10. $\mathrm{Ord}(A) \wedge \mathrm{Ord}(B) \to [A \in B \vee A = B \vee B \in A]$.

PROOF. We first observe that $A \cap B \subseteq A \wedge A \cap B \subseteq B$. If $A \cap B \subset A \wedge A \cap B \subset B$, then $A \cap B \in A \wedge A \cap B \in B$ (Propositions 7.9 and 7.7) hence $A \cap B \in A \cap B$. But this contradicts Proposition 5.18. Therefore $A \cap B = A$ or $A \cap B = B$, i.e., $A \subseteq B$ or $B \subseteq A$. Hence, by Corollary 7.8

$$A \in B \vee A = B \vee B \in A.$$ □

Definition 7.11. $\mathrm{On} \triangleq \{x \mid \mathrm{Ord}(x)\}$.

Proposition 7.12. $\mathrm{Ord}(\mathrm{On})$.

PROOF. From Proposition 7.6,

$$a \in \mathrm{On} \to a \subseteq \mathrm{On}$$

i.e., On is transitive. From Propositions 7.6 and 7.10

$$(\forall x, y \in \mathrm{On})[x \in y \vee x = y \vee y \in x].$$

Therefore, by Definition 7.3, On is an ordinal. □

Proposition 7.13. $\mathscr{P}\!\imath(\mathrm{On})$.

PROOF. Were On a set it would follow that $\mathrm{On} \in \mathrm{On}$. But this contradicts Proposition 5.18. □

Corollary 7.14. $\mathrm{Ord}(A) \to A \in \mathrm{On} \vee A = \mathrm{On}$.

PROOF. From Propositions 7.10 and 7.12, $A \in \mathrm{On} \vee A = \mathrm{On} \vee \mathrm{On} \in A$. But since On is a proper class we cannot have $\mathrm{On} \in A$. □

Corollary 7.15. $\mathrm{Ord}(A) \to A \subseteq \mathrm{On}$.

PROOF. Corollary 7.14, Proposition 7.12 and Corollary 7.8. □

Remark. The elements of On are the ordinal numbers in our system. We have proved that every ordinal class is an ordinal number except one, On. The ordinal numbers play such an important role in the theory ahead that we find it convenient to use the symbols

$$\alpha, \beta, \gamma, \ldots$$

as variables on ordinal numbers. We will not distinguish between free and bound variables except that we will not use the same symbol for both a free and a bound variable in the same formula. We will rely upon the reader to make a proper interpretation of formulas involving ordinal variables, subject to the following definition.

Definition 7.16.

(1) $\varphi(\alpha) \overset{\Delta}{\leftrightarrow} [\mathrm{Ord}(x) \to \varphi(x)]$

(2) $(\forall \alpha)\varphi(\alpha) \overset{\Delta}{\leftrightarrow} (\forall x)[\mathrm{Ord}(x) \to \varphi(x)]$

(3) $(\exists \alpha)\varphi(\alpha) \overset{\Delta}{\leftrightarrow} (\exists x)[\mathrm{Ord}(x) \wedge \varphi(x)]$.

Remark. Definition 7.16 is deliberately ambiguous and intended to shift attention away from certain formal details that should no longer require attention. We, thereby, hope to be able to focus more intently on the information that our formulas convey. But if called upon to explain in what sense Definition 7.16 is a definition, that is, to explain for example, what formula $(\forall \alpha)\varphi(\alpha)$ is an abbreviation for, we would do so by standardizing our list of ordinal variables,

free variables: $\alpha_0, \alpha_1, \ldots,$
bound variables: $\beta_0, \beta_1, \ldots,$

and specifying that

$\varphi(\beta_n)$ is $[\mathrm{Ord}(a_n) \to \varphi(a_n)]$
$(\forall \alpha_n)\varphi(\alpha_n)$ is $(\forall x_n)[\mathrm{Ord}(x_n) \to \varphi(x_n)]$
$(\exists \alpha_n)\varphi(\alpha_n)$ is $(\exists x_n)[\mathrm{Ord}(x_n) \wedge \varphi(x_n)]$.

Having now made it clear that matters can be set straight, we will not bother to do so here or in similar definitions to follow.

Theorem 7.17 (The Principle of Transfinite Induction). *If* (1) $A \subseteq \mathrm{On}$ *and* (2) $(\forall \alpha)[\alpha \subseteq A \to \alpha \in A]$, *then* $A = \mathrm{On}$.

PROOF. To prove that $A = \mathrm{On}$, given that $A \subseteq \mathrm{On}$, it is sufficient to prove that $\mathrm{On} \subseteq A$. Suppose that On is not a subclass of A. Then $\mathrm{On} - A \neq 0$ and hence by Propositions 7.12 and 7.5. $(\exists \alpha \in \mathrm{On} - A)[(\mathrm{On} - A) \cap \alpha = 0]$. Since $\alpha \subset \mathrm{On}$ it follows that $\alpha \subseteq A$. Then by (2), $\alpha \in A$. But this contradicts the fact that $\alpha \in \mathrm{On} - A$. Therefore $\mathrm{On} - A = 0$, i.e., $\mathrm{On} \subseteq A$. Then from (1), $A = \mathrm{On}$. ☐

Definition 7.18.

(1) $\alpha < \beta \overset{\Delta}{\leftrightarrow} \alpha \in \beta$.

(2) $\alpha \leqq \beta \overset{\Delta}{\leftrightarrow} \alpha < \beta \vee \alpha = \beta$.

(3) $\max(\alpha, \beta) \triangleq \alpha \cup \beta$.

EXERCISES

Prove the following.

(1) $(\forall\, \alpha)[\alpha \subset \mathrm{On}]$.

(2) $(\forall\, \alpha)(\forall\, \beta)[\alpha < \beta \vee \alpha = \beta \vee \beta < \alpha]$.

(3) $\alpha \leqq \beta \wedge \beta < \gamma \rightarrow \alpha < \gamma$.

(4) $\alpha < \beta \wedge \beta \leqq \gamma \rightarrow \alpha < \gamma$.

(5) $\alpha \leqq \beta \leftrightarrow \alpha \subseteqq \beta$.

(6) $(\forall\, \alpha)(\exists\, \beta)[\beta = \alpha \cup \{\alpha\}]$.

(7) $\mathrm{Tr}(A) \leftrightarrow (\forall\, x)(\forall\, y)[x \in y \wedge y \in A \rightarrow x \in A]$.

(8) $\mathrm{Ord}(A) \wedge a \in A \rightarrow a = a \cap A$.

(9) $A \subseteqq \mathrm{On} \wedge A \neq 0 \rightarrow (\exists\, \alpha \in A)(\forall\, \beta \in A)[\alpha \leqq \beta]$.

(10) $\mathrm{Tr}(A) \rightarrow [\cup(A) \subseteqq A]$.

(11) $\cup(\mathrm{On}) = \mathrm{On}$.

(12) $\mathrm{Ord}(\max(\alpha, \beta))$.

(13) $\alpha \leqq \max(\alpha, \beta) \wedge \beta \leqq \max(\alpha, \beta)$.

(14) $\alpha = \max(\alpha, \beta) \vee \beta = \max(\alpha, \beta)$.

Proposition 7.19. $A \subseteqq \mathrm{On} \rightarrow \mathrm{Ord}(\cup(A))$.

PROOF. If $A \subseteqq \mathrm{On}$, then since by Proposition 7.6, elements of ordinals are ordinals, it follows that

$$\cup(A) \subseteqq \mathrm{On}.$$

Furthermore if $a \in \cup(A)$ then, from the definition of union, there exists a set b such that

$$a \in b \wedge b \in A.$$

But since b is an ordinal, and hence is transitive, we have

$$a \subseteqq b \wedge b \in A.$$

Consequently

$$a \subseteqq \cup(A).$$

Thus $\cup(A)$ is a transitive subclass of the ordinal On and hence $\cup(A)$ is an ordinal. $\square$

Proposition 7.20. $A \subseteqq \mathrm{On} \wedge \alpha \in A \rightarrow \alpha \leqq \cup(A)$.

Proof. By Proposition 7.19 $\cup(A)$ is an ordinal. Furthermore

$$\begin{aligned} \alpha \in A &\to \alpha \subseteqq \cup(A) \\ &\to \alpha \leqq \cup(A). \end{aligned}$$ □

Proposition 7.21. $A \subseteqq \mathrm{On} \wedge (\forall \beta \in A)[\beta \leqq \alpha] \to \cup(A) \leqq \alpha$.

Proof. If $\beta \in \cup(A)$, then $(\exists \gamma)[\beta < \gamma \wedge \gamma \in A]$. Therefore $\beta < \gamma \wedge \gamma \leqq \alpha$. Hence $\beta < \alpha$, that is, $\cup(A) \subseteqq \alpha$. □

Remark. Proposition 7.20 assures us that $\cup(A)$ is an upper bound for the class of ordinals A. Proposition 7.21 assures us that $\cup(A)$ is also the smallest upper bound for A. Furthermore if A has a maximal ordinal, that is if $(\exists \alpha \in A)(\forall \beta \in A)[\beta \leqq \alpha]$ then $\alpha = \cup(A)$ and hence $\cup(A) \in A$. If A has no maximal element then $\cup(A) \notin A$. In particular if A is an ordinal number, i.e. $(\exists \alpha)[\alpha = A]$ and if $\cup(\alpha) \notin \alpha$ then $\cup(\alpha) = \alpha$. Such an ordinal is called a *limit ordinal.*

Definition 7.22. $\alpha + 1 \triangleq \alpha \cup \{\alpha\}$.

Example. $0 + 1 = 0 \cup \{0\} = \{0\} \triangleq 1.$

$1 + 1 = 1 \cup \{1\} = \{0\} \cup \{1\} = \{0, 1\} \triangleq 2.$

Proposition 7.23. $(\forall \alpha)[\alpha \in \alpha + 1 \wedge \alpha \subseteqq \alpha + 1]$.

Proof. We need only observe that $\alpha \in (\alpha \cup \{\alpha\}) \wedge \alpha \subseteqq (\alpha \cup \{\alpha\})$. □

Proposition 7.24. $(\forall \alpha)[\alpha + 1 \in \mathrm{On}]$.

Proof. Since $(\forall \alpha)[\alpha \in \mathrm{On} \wedge \alpha \subseteqq \mathrm{On}]$ it follows that $\alpha + 1 \subseteqq \mathrm{On}$. Furthermore if $a \in \alpha + 1$, then $a \in \alpha$ or $a = \alpha$ and hence $a \subseteqq \alpha$. But $\alpha \subseteqq \alpha + 1$, hence $a \subseteqq \alpha + 1$. Thus $\alpha + 1$ is a transitive subset of an ordinal class hence $\alpha + 1$ is an ordinal, that is, $\alpha + 1 \in \mathrm{On}$. □

Proposition 7.25. $\neg[\alpha < \beta < \alpha + 1]$.

Proof. If $\alpha < \beta \wedge \beta < \alpha + 1$ then $\alpha \in \beta \wedge [\beta \in \alpha \vee \beta = \alpha]$. But this contradicts Proposition 5.18. □

Proposition 7.26. $a \subset \mathrm{On} \to (\forall \alpha \in a)[\alpha < \cup(a) + 1]$.

Proof. From Propositions 7.19 and 7.20, $\cup(a)$ is an ordinal and $\alpha \leqq \cup(a)$. From Proposition 7.23 we then conclude that $\alpha < \cup(a) + 1$. □

Remark. Proposition 7.26 assures us that given any ordinal number there is an ordinal number that is larger, indeed given any set of ordinal numbers there is an ordinal number that is larger than each element of the given set. The naïve acceptance of On as a set would then lead to the paradox of the largest ordinal, that is, the existence of an ordinal number that is larger than every ordinal including itself. This paradox was discovered by Cantor in 1895. It first appeared in print in 1897 having been rediscovered by Burali-Forti, whose names it now bears.

Definition 7.27. $K_{\mathrm{I}} \triangleq \{\alpha \mid \alpha = 0 \vee (\exists \beta)[\alpha = \beta + 1]\}$.

$K_{\mathrm{II}} \triangleq \mathrm{On} - K_{\mathrm{I}}$.

Definition 7.28. $\omega \triangleq \{\alpha \mid \alpha \cup \{\alpha\} \subseteqq K_{\mathrm{I}}\}$.

Exercises

Prove the following.

(1) $\alpha \in K_{\mathrm{II}} \leftrightarrow \alpha \neq 0 \wedge \alpha = \cup(\alpha)$.

(2) $\omega \neq 0$.

(3) $(\forall \alpha)[\alpha + 1 \in K_{\mathrm{I}}]$.

(4) $\omega \subseteqq K_{\mathrm{I}}$.

(5) $\beta \in \omega \leftrightarrow (\forall \gamma)[\gamma \leqq \beta \rightarrow \gamma \in K_{\mathrm{I}}]$.

(6) $\beta < \alpha \wedge \alpha \in K_{\mathrm{II}} \rightarrow (\exists \gamma)[\beta < \gamma < \alpha]$.

(7) $\alpha \subseteqq K_{\mathrm{I}} \rightarrow \alpha \subseteqq \omega$.

(8) $\alpha < \beta \rightarrow \alpha + 1 \leqq \beta$.

(9) $\alpha < \beta \rightarrow \alpha + 1 < \beta + 1$.

Remark. From Exercise 1 we see that K_{II} is the class of all limit ordinals. We will refer to the elements of K_{II} as limit ordinals and to the elements of K_{I} as nonlimit ordinals. The elements in ω are the natural numbers or non-negative integers as we will now show by proving that they satisfy the Peano postulates. As a notational convenience we will use

$$i, j, k, \ldots, m, n$$

as variables on ω.

Definition 7.29.

(1) $\varphi(i) \overset{\triangle}{\leftrightarrow} [a \in \omega \rightarrow \varphi(a)]$.

(2) $(\forall i)\varphi(i) \overset{\triangle}{\leftrightarrow} (\forall x)[x \in \omega \rightarrow \varphi(x)]$.

(3) $(\exists i)\varphi(i) \overset{\triangle}{\leftrightarrow} (\exists x)[x \in \omega \wedge \varphi(x)]$.

Proposition 7.30 (Peano's Postulates).

(1) $0 \in \omega$.

(2) $(\forall i)[i + 1 \in \omega]$.

(3) $(\forall i)[i + 1 \neq 0]$.

(4) $(\forall i)(\forall j)[i + 1 = j + 1 \leftrightarrow i = j]$.

(5) $0 \in A \wedge (\forall i)[i \in A \rightarrow i + 1 \in A] \rightarrow \omega \subseteq A$.

PROOF. (1) From Definition 7.27 we have that $0 \in K_{\mathrm{I}}$. Therefore $0 \cup \{0\} \subseteq K_{\mathrm{I}}$ hence $0 \in \omega$.

(2) Since $i + 1 \in K_{\mathrm{I}}$ and since $i \in \omega \rightarrow i + 1 \subseteq K_{\mathrm{I}}$ it follows that $(i + 1) \cup \{i + 1\} \subseteq K_{\mathrm{I}}$, i.e., $i + 1 \in \omega$.

(3) Clearly $i \in i + 1$. Therefore $i + 1 \neq 0$.

(4) If $i = j$, then $(i + 1) = (j + 1)$. Conversely if $i + 1 = j + 1$ then since $i \in i + 1$ we have $i \in j \vee i = j$ and $j \in i \vee j = i$. Since $i \in j \wedge j \in i$ and $i \in j \wedge j = i$ each contradict Proposition 5.18 we conclude that $i = j$.

(5) If $\omega - A \neq 0$ then there is a smallest element i in $\omega - A$. This smallest element is not 0 because, by hypothesis $0 \in A$ and hence $0 \notin \omega - A$. Furthermore $i \in \omega$ and $\omega \subseteq K_{\mathrm{I}}$, therefore $i \in K_{\mathrm{I}}$. It then follows, from the definition of K_{I} and the fact that $i \neq 0$, that $(\exists \beta)[i = \beta + 1]$. Furthermore $i \subseteq i + 1$ and $i + 1 \subseteq K_{\mathrm{I}}$. Therefore $\beta + 1 \subseteq K_{\mathrm{I}}$, and hence $\beta \in \omega$. Since $\beta < i$ and i is the smallest element in $\omega - A$ we must have $\beta \notin \omega - A$. Consequently, $\beta \in A$. But by hypothesis $\beta \in \omega \wedge \beta \in A$ implies $\beta + 1 \in A$, that is, $i \in A$. This is a contradiction that forces us to conclude that $\omega \subseteq A$. □

Corollary 7.31 (The Principle of Finite Induction). *If* $A \subseteq \omega \wedge 0 \in A \wedge (\forall i)[i \in A \rightarrow i + 1 \in A]$ *then* $A = \omega$.

PROOF. Obvious from Proposition 7.30. □

Proposition 7.32. $\mathrm{Ord}(\omega)$.

PROOF. Since $\omega \subseteq K_{\mathrm{I}}$ and $K_{\mathrm{I}} \subseteq \mathrm{On}$ it follows that $\omega \subseteq \mathrm{On}$. Furthermore if $a \in b$ and $b \in \omega$, then b is an ordinal and $b + 1 \subseteq K_{\mathrm{I}}$. Then $a \subseteq b, b \subseteq b + 1$ and $b + 1 \subseteq K_{\mathrm{I}}$. Therefore $a \in K_{\mathrm{I}}$ and $a \subseteq K_{\mathrm{I}}$, consequently $a \cup \{a\} \subseteq K_{\mathrm{I}}$. Thus $a \in \omega$ and hence ω is transitive. Since ω is a transitive subclass of an ordinal, ω is an ordinal. □

Remark. From Proposition 7.32 we see that either $\omega \in \mathrm{On}$ or $\omega = \mathrm{On}$. But which of these alternatives is true? The question is whether or not ω is a set. This cannot be resolved by the axioms stated thus far. We choose to resolve the issue by postulating that ω is a set.

Axiom 7 (Axiom of Infinity). $\mathscr{M}(\omega)$.

Proposition 7.33. $\omega \in K_{\mathrm{II}}$.

PROOF. From Axiom 7 and Proposition 7.32 we have that $\omega \in \mathrm{On}$. Since $\omega \subseteq K_{\mathrm{I}}$ it follows that if $\omega \in K_{\mathrm{I}}$ then $\omega + 1 \subseteq K_{\mathrm{I}}$ and hence $\omega \in \omega$. Therefore $\omega \in K_{\mathrm{II}}$. □

Proposition 7.34. $F \mathscr{F}\!n\, \omega \rightarrow (\exists n)[F`(n+1) \notin F`n]$.

Proof. From the Axiom of Infinity, F is a function whose domain is a set. Therefore the range of F, $\mathscr{W}(F)$, is a set. Then from the Axiom of Regularity there exists a set a in $\mathscr{W}(F)$ such that

$$\mathscr{W}(F) \cap a = 0.$$

But since $a \in \mathscr{W}(F)$ there exists an integer n, such that $a = F`n$. Since $\mathscr{W}(F) \cap a = 0$ and since $F`(n+1) \in \mathscr{W}(F)$ we have $F`(n+1) \notin F`n$. □

Remark. Proposition 7.34 assures us that there are no infinite descending $\in$-chains. Given a nonempty set a_0, $\exists\, a_1 \in a_0$. If $a_1 \neq 0$, $\exists\, a_2 \in a_1$, etc. However, by Theorem 7.34, we see that after a finite number of steps we must arrive at a set $a_n \in a_{n-1}$ with $a_n = 0$.

While every descending $\in$-chain must be of finite length it does not follow that for a given set a there is a bound on the length of the $\in$-chains descending from a. Consider, for example, ω:

$$(\forall\, n)[0 \in 1 \in 2 \in \cdots \in n \in \omega].$$

Exercises

Prove the following.

(1) $\alpha \in K_{\mathrm{II}} \rightarrow \omega \leqq \alpha$.

(2) $A \subseteq \omega \wedge A \neq 0 \rightarrow (\exists\, k \in A)(\forall\, i \in A)[k \leqq i]$.

(3) $F: \omega \rightarrow A \wedge R \operatorname{Fr} A \rightarrow (\exists\, n)[F`(n+1) \notin (R^{-1})``\{F`n\}]$.

Definition 7.35. $\cap(A) \triangleq \{x \mid (\forall\, y \in A)[x \in y]\}$.

Remark. Note that $\cap(A)$ is the class of all those objects that are elements of every set in A. We call this class the intersection of the elements of A, or simply the intersection of A. Consequently, we read $\cap(A)$ as "intersection A."

Exercises

Prove the following.

(1) $\cap(\{a, b\}) = a \cap b$.

(2) $\cap(0) = V$.

(3) $\operatorname{Tr}(A) \rightarrow \cap(A) \subseteq A$

(4) $a \in A \rightarrow \cap(A) \in a$.

(5) $A \neq 0 \rightarrow \mathscr{M}(\cap(A))$.

(6) $A \subseteq \operatorname{On} \wedge A \neq 0 \rightarrow \operatorname{Ord}(\cap(A))$.

(7) $A \subseteq \operatorname{On} \wedge A \neq 0 \rightarrow (\forall\, \beta \in A)[\cap(A) \leqq \beta]$.

Definition 7.36.

(1) $\sup(A) \triangleq \cup(A \cap \text{On})$.

(2) $\sup_{<\beta}(A) \triangleq \cup(A \cap \beta)$.

Remark. We read these symbols as "the supremum of A" and "the supremum of A below β" respectively.

EXERCISES

Prove the following.

(1) $\sup(\omega) = \omega$.

(2) $\sup(\alpha + 1) = \alpha$.

(3) $\alpha \in K_{\text{II}} \to \sup(\alpha) = \alpha$.

(4) $\alpha \in K_{\text{I}} \to \alpha = \sup(\alpha) + 1$.

(5) $\sup_{<\beta+1}(\alpha) = \beta$ if $\alpha > \beta$.

(6) $\sup(A) \in \text{On} \vee \sup(A) = \text{On}$.

(7) $\sup_{<\beta}(A) \in \text{On}$.

Definition 7.37.

(1) $$\begin{aligned} \inf(A) &\triangleq \cap(A \cap \text{On}) && \text{if } A \cap \text{On} \neq 0, \\ &\triangleq 0 && \text{if } A \cap \text{On} = 0. \end{aligned}$$

(2) $\inf_{>\beta}(A) \triangleq \inf(A - \beta)$.

Remark. We read these symbols as "the infimum of A" and "the infimum of A above β" respectively.

EXERCISES

Prove the following.

(1) $\inf(\alpha) = 0$.

(2) $$\begin{aligned} \inf_{>\beta}(\alpha) &= 0 && \text{if } \alpha \leqq \beta \\ &= \beta + 1 && \text{if } \alpha > \beta. \end{aligned}$$

(3) $\inf(A) \in \text{On} \wedge \inf_{>\beta}(A) \in \text{On}$.

(4) $A \neq 0 \to \inf(A) \in A$.

(5) $\text{Ord}(A) \wedge A - \alpha \neq 0 \to \alpha = \inf(A - \alpha)$.

Definition 7.38. $\mu_\alpha(\varphi(\alpha)) \triangleq \inf(\{\alpha \mid \varphi(\alpha)\})$.

Remark. From Exercise (4) above we see that if A is a nonempty class of ordinals, then $\inf(A)$ is the minimal element in A. In view of this, we choose to ignore the exceptional case when $\{\alpha \mid \varphi(\alpha)\}$ is empty and read $\mu_\alpha(\varphi(\alpha))$ as "the smallest α such that $\varphi(\alpha)$."

Exercises

Prove the following.

(1) $(\exists\, \alpha)\varphi(\alpha) \to \varphi(\mu_\alpha(\varphi(\alpha)))$.

(2) $\sup(A) = \mu_\alpha((\forall\, \beta \in A)[\beta \leqq \alpha])$.

Remark. In the material ahead we will be interested in relational systems $[a, r]$ in which r well orders the set a. In such a case we call the relational system $[a, r]$ a well-ordered set. When the relation r is understood, or unimportant, we will speak simply of the well-ordered set a.

Since isomorphism between relational systems is an equivalence relation, it follows that every well-ordered set $[a, r]$ determines an equivalence class of well-ordered sets. In particular, since every ordinal α is well ordered by the $\in$-relation, each ordinal α determines an equivalence class of well-ordered sets. What we propose to prove is that the ordinal numbers can be taken as canonical well-ordered sets because (1) each ordinal α, as a well-ordered set, belongs to exactly one equivalence class of well-ordered sets, and (2) every equivalence class of well-ordered sets contains exactly one ordinal.

To prove that there is exactly one ordinal in every equivalence class we must prove that if r well orders a, there is one and only one ordinal α for which

$$(\exists f)[f \operatorname{Isom}_{E,r}(\alpha, a)].$$

We first prove that there is at most one such ordinal.

Proposition 7.39. $\operatorname{Ord}(A) \wedge \operatorname{Ord}(B) \wedge F \operatorname{Isom}_{E,E}(A, B) \to A = B$.

Proof. It is sufficient to prove that F is the identity function I restricted to A. This we prove by transfinite induction. If $\beta \in A \wedge (\forall\, \alpha < \beta)[F`\alpha = \alpha]$ then $F``\beta = \beta$. Furthermore since $\beta$ is the E-minimal element in $A - \beta$ and order isomorphisms map minimal elements onto minimal elements it follows that $F`\beta$ is the E-minimal element in $B - F``\beta = B - \beta$, i.e., $F`\beta = \beta$. □

Proposition 7.40.

$$\operatorname{Ord}(A) \wedge \operatorname{Ord}(B) \wedge F_1 \operatorname{Isom}_{E,R}(A, C) \wedge F_2 \operatorname{Isom}_{E,R}(B, C) \to A = B.$$

Proof. Proposition 7.39 and the fact that $F_2 \circ F_1^{-1} \operatorname{Isom}_{E,E}(A, B)$. □

Remark. Proposition 7.40 assures us that every well-ordered class is order isomorphic to at most one ordinal well ordered by the $\in$-relation. We will prove that while not all well-ordered classes are order isomorphic to an ordinal every well-ordered set is.

Since order isomorphisms must map minimal elements into minimal elements and initial segments into intial segments we must show that if R We a then there exists an ordinal number α and a function F mapping α onto a in such a way that $\forall \beta < \alpha$, $F`\beta$ is the R-minimal element in $a - F``\beta$. But this means that $F$ must be so defined that its value at $\beta$ depends on $\beta$, the values $F$ assumes at all ordinals smaller than $\beta$, and the additional requirement that $F`\beta$ be the R-minimal element in $a - F``\beta$. Does there exist a function fulfilling all of these requirements? We will prove that such a function does exist. Indeed we will prove that there exists exactly one function F defined on On in such a way that its value at β depends upon β, and upon the values F assumes at all ordinals smaller than β, hence upon $F \restriction \beta$, and also depends upon any previously given condition G. Such a function F is said to be defined by transfinite recursion.

We need the following lemma.

Lemma 1. $f \mathscr{F}n\, \beta \wedge (\forall \alpha < \beta)[f`\alpha = G`(f \restriction \alpha)] \wedge g \mathscr{F}n\, \gamma$
$$\wedge (\forall \alpha < \gamma)[g`\alpha = G`(g \restriction \alpha)] \wedge \beta \leqq \gamma \rightarrow (\forall \alpha < \beta)[f`\alpha = g`\alpha].$$

Proof (By induction). If $(\forall \gamma < \alpha)[f`\gamma = g`\gamma]$ then $f \restriction \alpha = g \restriction \alpha$ and hence $f`\alpha = G`(f \restriction \alpha) = G`(g \restriction \alpha) = g`\alpha$. □

Theorem 7.41 (Principle of Transfinite Recursion). *If*
$$K = \{f \mid (\exists \beta)[f \mathscr{F}n\, \beta \wedge (\forall \alpha < \beta)[f`\alpha = G`(f \restriction \alpha)]]\}$$
and if $F = \cup(K)$ *then*

(1) $F \mathscr{F}n$ On,

(2) $(\forall \alpha)[F`\alpha = G`(F \restriction \alpha)]$,

(3) $F_1 \mathscr{F}n$ On $\wedge (\forall \alpha)[F_1`\alpha = G`(F_1 \restriction \alpha)] \rightarrow F_1 = F$.

Proof. (1) Since each element of K is a relation it follows that F is a relation. Furthermore
$$\langle a, b\rangle \in F \wedge \langle a, c\rangle \in F \rightarrow (\exists f \in K)(\exists g \in K)[\langle a, b\rangle \in f \wedge \langle a, c\rangle \in g].$$
From Lemma 1 we have that $f`a = g`a$, i.e., $b = c$. Thus F is single valued, hence is a function.

If $b \in a \wedge a \in \mathscr{D}(F)$ then from the definition of F and K it follows that $(\exists f)(\exists \beta)[f \mathscr{F}n\, \beta \wedge a \in \beta]$. But ordinals are transitive and since $b \in a$, $a \in \beta$, and $\beta \subseteq \mathscr{D}(F)$ it follows that $b \in \mathscr{D}(F)$. Thus $\mathscr{D}(F)$ is a transitive subclass of On and hence is an ordinal. Therefore $\mathscr{D}(\mathscr{F}) =$ On or $(\exists \gamma)[\gamma = \mathscr{D}(F)]$. If $\gamma = \mathscr{D}(F)$ and if
$$g = F \cup \{\langle \gamma, G`(F \restriction \gamma)\rangle\}$$
then $g \mathscr{F}n(\gamma + 1) \wedge (\forall \alpha < \gamma + 1)[g`\alpha = G`(g \restriction \alpha)]$. Thus $g \in K$ and $\gamma \in \mathscr{D}(F)$. But $\gamma = \mathscr{D}(F)$. This is a contradiction; hence $\mathscr{D}(F) =$ On and $F \mathscr{F}n$ On.

(2) From (1) and the definition of F, it follows that $(\forall \alpha)(\exists f)[f \subseteq F \wedge F`\alpha = f`\alpha \wedge f`\alpha = G`(f \restriction \alpha)]$. Then $F \restriction \alpha = f \restriction \alpha$ and hence $F`\alpha = f`\alpha = G`(f \restriction \alpha) = G`(F \restriction \alpha)$.

(3) Since $\mathscr{D}(F_1) = \mathscr{D}(F)$ it is sufficient to prove that $(\forall \alpha)[F_1`\alpha = F`\alpha]$. This we do by transfinite induction. If $(\forall \alpha)[\alpha < \gamma \rightarrow F_1`\alpha = F`\alpha]$ then $F_1 \restriction \gamma = F \restriction \gamma$. Therefore $F`_1\gamma = G`(F_1 \restriction \gamma) = G`(F \restriction \gamma) = F`\gamma$. □

Corollary 7.42. $(\exists ! f)[f \mathscr{F}\!n\, \alpha \wedge (\forall \beta < \alpha)[f`\beta = G`(f \restriction \beta)]]$.

The proof is left to the reader.

Remark. Theorem 7.41 is a theorem schema and hence a metatheorem. With quantification in the metalanguage it could be stated as

$$(\forall G)(\exists ! F)[F \mathscr{F}\!n\, \mathrm{On} \wedge (\forall \alpha)[F`\alpha = G`(F \restriction \alpha)]].$$

While this statement more readily conveys the content of the theorem as it will be used it is nevertheless of interest to note that Theorem 7.41 as stated is stronger. It not only asserts the existence of the function F but prescribes a method for exhibiting F when G is given.

For certain types of problems it is sufficient to know that functions can be defined recursively on the natural numbers in the following sense. Given any function h and any set a there is a function f defined on ω in such a way that

$$f`0 = a$$
$$f`(n + 1) = h`f`n.$$

This type of recursion can be extended to functions on On by requiring that at a limit ordinal α, the value of f is the supremum of its values at the ordinals preceding α, i.e.,

$$f`\alpha = \bigcup \{f`\gamma \mid \gamma < \alpha\}.$$

To simplify our notation somewhat, we introduce the following.

Definition 7.43.

(1) $\bigcup_{x \in B} A`x \triangleq \bigcup \{A`x \mid x \in B\}$.

(2) $\bigcap_{x \in B} A`x \triangleq \bigcap \{A`x \mid x \in B\}$.

Remark. We read these symbols as "the union of $A`x$ for $x \in B$" and "the intersection of $A`x$ for $x \in B$" respectively.

We now wish to prove that given any class H and any set a, there exists a function F defined on On in such a way that

$$F`0 = a$$

and for each ordinal β

$$F`(\beta + 1) = H`F`\beta$$
$$F`\beta = \bigcup_{\gamma < \beta} F`\gamma \quad \text{if } \beta \in K_{\text{II}}.$$

Proposition 7.44. *If* $G = \{\langle x, y\rangle | [x = 0 \wedge y = a] \vee [x \neq 0 \wedge \sup(\mathscr{D}(x)) \neq \mathscr{D}(x) \wedge y = H`x` \sup(\mathscr{D}(x))] \vee [x \neq 0 \wedge \sup(\mathscr{D}(x)) = \mathscr{D}(x) \wedge y = \cup \mathscr{W}(x)]\}$ *and* $F \mathscr{F}\!n$ On $\wedge (\forall \alpha)[F`\alpha = G`(F \upharpoonright \alpha)]$ *then*

(1) $F`0 = a$,

(2) $F`(\beta + 1) = H`F`\beta$,

(3) $F`\beta = \bigcup_{\gamma < \beta} F`\gamma$, $\beta \in K_{\text{II}}$,

(4) *F is unique.*

PROOF. (1) $F`0 = G`(F \upharpoonright 0) = G`0 = a$.

(2) $F`(\beta + 1) = G`(F \upharpoonright (\beta + 1))$. Since $\mathscr{D}(F \upharpoonright (\beta + 1)) = \beta + 1$ and $\sup(\beta + 1) = \beta \neq \beta + 1$ we have that

$$G`(F \upharpoonright (\beta + 1)) = H`(F \upharpoonright (\beta + 1))` \sup(\beta + 1) = H`F`\beta,$$

i.e., $F`(\beta + 1) = H`F`\beta$.

(3) $F`\beta = G`(F \upharpoonright \beta)$. Since $\mathscr{D}(F \upharpoonright \beta) = \beta$ and $\beta \in K_{\text{II}}$ we have that $\sup(\beta) = \beta$. Hence $G`(F \upharpoonright \beta) = \cup(\mathscr{W}(F \upharpoonright \beta)) = \bigcup_{\gamma < \beta} F`\gamma$, i.e., $F`\beta = \bigcup_{\gamma < \beta} F`\gamma$.

(4) It is easily prove by induction that F is unique. ☐

Remark. Again we point out that Proposition 7.44 is a stronger result than the one we proposed to prove because it not only assures the existence of the function F but shows how to produce F given H and a.

In the statement of Proposition 7.44 we have chosen to say simply that "F is unique." The reader should have no trouble determining the wff that should appear here.

Corollary 7.45 (Principle of Finite Recursion).

$$(\exists! f)[f \mathscr{F}\!n\, \omega \wedge f`0 = a \wedge (\forall k)[f`(k + 1) = H`f`k]].$$

PROOF. If in Proposition 7.44 we restrict F to ω then $F \upharpoonright \omega$ is a function on ω and hence is a set. Therefore $(\exists f)[f = F \upharpoonright \omega]$. Then $f \mathscr{F}\!n\, \omega$, $f`0 = a$, and $(\forall k)[f`(k + 1) = H`f`k]$.

It is easily proved, by induction, that f is unique. ☐

Remark. In the study of order isomorphisms we are especially interested in those order-preserving functions that map ordinals onto ordinals. Any function whose domain is an ordinal and whose range is a class of ordinal numbers we will call an *ordinal function*. If in addition an ordinal function is order preserving we say that it is strictly monotone.

Definition 7.46.

$$\mathrm{Orf}(G) \overset{\Delta}{\leftrightarrow} G\ \mathscr{F}\!n\ \mathscr{D}(G) \wedge \mathrm{Ord}(\mathscr{D}(G)) \wedge \mathscr{W}(G) \subseteq \mathrm{On}.$$

$$\mathrm{Smo}(G) \overset{\Delta}{\leftrightarrow} \mathrm{Orf}(G) \wedge (\forall \alpha \in \mathscr{D}(G))(\forall \beta \in \mathscr{D}(G))[\alpha < \beta \rightarrow G`\alpha < G`\beta].$$

Exercises

Prove the following.

(1) $\mathrm{Smo}(G) \rightarrow (\forall \alpha \in \mathscr{D}(G))[\alpha \leqq G`\alpha]$.

(2) If in Proposition 7.44, $a \in \mathrm{On}$ and H is a strictly monotone ordinal function on On then F is a strictly monotone ordinal function on On.

(3) $F\ \mathrm{Isom}_{E,E}(A, B) \wedge \mathrm{Ord}(A) \wedge B \subseteq \mathrm{On} \rightarrow \mathrm{Smo}(F)$.

(4) State and prove a generalization of Theorem 7.41 in which On is replaced by a well-ordered class A.

Remark. The principle of transfinite recursion assures us that we can define a function F on On in such a way that its value at α is dependent on its values at all ordinals less than α and on any given condition G. If R well orders A and if F is to be an order-preserving isomorphism from some ordinal onto A then $F`\alpha$ must be the R-minimal element in $A - F``\alpha$. Suppose that we could define $G$ in such a way that $G`(F \upharpoonright \alpha)$ is the R-minimal element in $A - F``\alpha$. Then clearly F would be an order-preserving map from ordinals into A. It would then only remain to be proved that A is exhausted, i.e., $\mathscr{W}(F) = A$. In fact we will discover that there are two cases of interest. If R is a well-founded relation then $\mathscr{W}(F) = A$; if R is not well founded then $\mathscr{W}(F)$ will be an R-initial segment of A.

Note that $\mathscr{W}(F)$ is R transitive, that is, $a\ R\ b \wedge b \in \mathscr{W}(F) \rightarrow a \in \mathscr{W}(F)$.

Proposition 7.47. *If*

$$R\ \mathrm{We}\ A \wedge R\ \mathrm{Wfr}\ A \wedge B \subseteq A \wedge (\forall x \in A)(\forall y \in B)[x\ R\ y \rightarrow x \in B]$$

then

$$A = B \vee (\exists x \in A)[B = A \cap (R^{-1})``\{x\}].$$

Proof. If $A \neq B$ then, $A - B \neq 0$ since $B \subseteq A$. Thus $A - B$ has an R-minimal element, a (Proposition 6.26). Then $a \in A$ and $a \notin B$. Since by hypothesis

$$b \in A \wedge b\ R\ c \wedge c \in B \rightarrow b \in B$$

it follows that

$$(\forall x \in B)[x\ R\ a]$$

for otherwise $a \in B$. Therefore $B \subseteq [A \cap (R^{-1})``\{a\}]$.

Furthermore suppose that $b \in [A \cap (R^{-1})``\{a\}]$. Then $b \in A$ and $b\,R\,a$. But $a$ is the R-minimal element in $A - B$. Therefore $b \notin A - B$, i.e. $b \in B$. Thus $[A \cap (R^{-1})``\{a\}] \subseteq B$ and hence $B = [A \cap (R^{-1})``\{a\}]$. □

Remark. In Proposition 7.47 the requirement that R be well founded is only used to establish that $A - B$ has an R-minimal element. Later we will show that it is sufficient for R to be founded. Consequently this result follows if R is a well-ordering relation that is not well founded.

Proposition 7.48. *If $F\ \mathscr{F}\!n$ On $\wedge$ $(\forall \alpha)[F`\alpha \in [A - F``\alpha]]$, then*

(1) $\mathscr{W}(\mathrm{F}) \subseteq A$,

(2) $\mathscr{U}\!n_2(F)$,

(3) $\mathscr{P}\!r(A)$.

Proof. (1) Since $(\forall \alpha)[F`\alpha \in (A - F``\alpha)]$ it follows that $(\forall \alpha)[F`\alpha \in A]$ and hence $\mathscr{W}(F) \subseteq A$.

(2) If $\alpha < \beta$, then $F`\alpha \in F``\beta$. Since by hypothesis $F`\beta \in (A - F``\beta)$, it follows that $F`\beta \notin F``\beta$. Therefore $F`\alpha \neq F`\beta$. We have just proven that if $\alpha \neq \beta$, then $F`\alpha \neq F`\beta$. Consequently, if $F`\alpha = F`\beta$, then $\alpha = \beta$. Therefore F is one-to-one.

(3) Since $\mathscr{W}(F) \subseteq A$, it follows that if A is a set, then $\mathscr{W}(F)$ is also a set. But since F is one-to-one $\mathscr{W}(F)$ is a set iff On is a set. Therefore A is a proper class. □

Proposition 7.49. *If $F\ \mathscr{F}\!n$ On $\wedge$ $(\forall \alpha)[A - F``\alpha \neq 0 \to F`\alpha \in A - F``\alpha]$ $\wedge$ $\mathscr{M}(A)$ then*

$$(\exists \alpha)[(\forall \beta < \alpha)[A - F``\beta \neq 0] \wedge F``\alpha = A \wedge \mathscr{U}\!n_2(F \restriction \alpha)].$$

Proof. If $(\forall \alpha)[A - F``\alpha \neq 0]$ then $(\forall \alpha)[F`\alpha \in A - F``\alpha]$ and by Proposition 7.48 A is a proper class. Since by hypothesis A is a set we conclude that $(\exists \alpha)[A - F``\alpha = 0]$. There is then a smallest such α, i.e.,

$$(\exists \alpha)[A - F``\alpha = 0 \wedge (\forall \beta < \alpha)[A - F``\beta \neq 0]].$$

If $x \in F``\alpha$ then $(\exists \beta < \alpha)[x = F`\beta]$. But if $\beta < \alpha$, then $[A - F``\beta] \neq 0$ and hence $F`\beta \in [A - F``\beta]$, i.e., $x \in A$. Thus $F``\alpha \subseteq A$. But since $[A - F``\alpha] = 0$, $A \subseteq F``\alpha$. Therefore $A = F``\alpha$.

Since F is a function $F \restriction \alpha$ is single valued. Furthermore if $\gamma < \alpha$ and $\beta < \alpha$ and if $\gamma < \beta$, then

$$F`\gamma \in F``\beta \wedge F`\beta \in A - F``\beta$$

i.e., $F`\gamma \in F``\beta \wedge F`\beta \notin F``\beta$. Then $F`\beta \neq F`\gamma$. We have proved that if $\gamma < \alpha$ and $\beta < \alpha$ and if $\gamma \neq \beta$, then $F`\beta \neq F`\gamma$. Consequently, if $\gamma < \alpha$ and $\beta < \alpha$ and if $F`\gamma = F`\beta$, then $\gamma = \beta$, that is $F \restriction \alpha$ is one-to-one. □

Remark. In the proof of Proposition 7.48 we see that the requirement that $(\forall\,\alpha)[F`\alpha \in A - F``\alpha]$ assures us that the function $F$ defined by transfinite induction will be one-to-one. Conversely if $F$ is one-to-one then $\mathscr{W}(F)$ will be a proper class and $(\forall\,\alpha)[F`\alpha \in A - F``\alpha]$. Furthermore if $\mathscr{W}(F)$ is a set then F cannot be one-to-one. In this case Proposition 7.49 assures us that if F fulfills the requirements for one-to-oneness "as long as it can," i.e., until $\mathscr{W}(F)$ is exhausted then the restriction of F to some ordinal α will map α one-to-one onto $\mathscr{W}(F)$. From this we can prove that every well-ordered set is order isomorphic to an ordinal number.

Proposition 7.50. *If R is well founded on A and well orders A, if*

(1) $G = \{\langle x, y\rangle \mid y \in [A - \mathscr{W}(x)] \wedge [(A - \mathscr{W}(x)) \cap (R^{-1})``\{y\} = 0]\}$,

(2) $F \mathscr{F}\!n$ On, and

(3) $(\forall\,\alpha)[F`\alpha = G`(F \restriction \alpha)]$, *then*

$$A - F``\alpha \neq 0 \rightarrow F`\alpha \in A - F``\alpha.$$

Proof. As the first step in our proof we will show that G is single valued. For this purpose suppose that $\langle x, y_1\rangle, \langle x, y_2\rangle \in G$. Then

$$y_1 \in [A - \mathscr{W}(x)] \wedge y_2 \in [A - \mathscr{W}(x)]$$

and $[(A - \mathscr{W}(x)) \cap (R^{-1})``\{y_1\}] = 0 \wedge [(A - \mathscr{W}(x)) \cap (R^{-1})``\{y_2\}] = 0$. Therefore $y_1 \notin (R^{-1})``\{y_2\} \wedge y_2 \notin (R^{-1})``\{y_1\}$. Since R well orders A we must have $y_1 = y_2$.

Furthermore, if $[A - \mathscr{W}(x)] \neq 0$ then since R is well founded on A and well orders A, $A - \mathscr{W}(x)$ has an R-minimal element, y. But $G`x = y$, i.e., $G`x \in [A - \mathscr{W}(x)]$.

We have now shown that if $A - \mathscr{W}(x) \neq 0$, then $G`x \in A - \mathscr{W}(x)$. In particular, if x is $F \restriction \alpha$, it follows, since $\mathscr{W}(F \restriction \alpha) = F``\alpha$, that if $A - F``\alpha \neq 0$, then $G`(F \restriction \alpha) \in A - F``\alpha$. But $G`(F \restriction \alpha) = F`\alpha$. □

Proposition 7.51. *If A is a proper class that is well ordered by R, and R is well founded on A, if*

(1) $G = \{\langle x, y\rangle \mid y \in [A - \mathscr{W}(x)] \wedge [(A - \mathscr{W}(x)) \cap (R^{-1})``\{y\}] = 0\}$,

(2) $F \mathscr{F}\!n$ On, and

(3) $(\forall\,\alpha)[F`\alpha = G`(F \restriction \alpha)]$,

then

$$F \operatorname{Isom}_{E,R}(\text{On}, A).$$

Proof. If $(\exists\,\alpha)[A - F``\alpha = 0]$ then $A \subseteq F``\alpha$. Since $F``\alpha$ is a set it would then follow that $A$ is a set. Since $A$ is a proper class it follows that $(\forall\,\alpha)[A - F``\alpha \neq 0]$. From Proposition 7.50 and the defining properties of F and G it follows that $F`\alpha$ is the R-minimal element in $A - F``\alpha$, i.e.,

$$F`\alpha \in [A - F``\alpha] \quad \text{and} \quad [(A - F``\alpha) \cap (R^{-1})``\{F`\alpha\}] = 0.$$

From Proposition 7.48 it then follows that $\mathscr{W}(F) \subseteqq A$ and F is one-to-one. To prove that F is onto we note that if $y \in \mathscr{W}(F)$ then $(\exists\, \alpha)[y = F`\alpha]$. Furthermore since $F`\alpha$ is the R-minimal element in $A - F``\alpha$

$$x\, R\, y \to x \notin [A - F``\alpha]$$

then

$$\begin{aligned} x \in A \wedge x\, R\, y &\to x \in F``\alpha \\ &\to x \in \mathscr{W}(F). \end{aligned}$$

Then R, A, and $\mathscr{W}(F)$ satisfy the hypotheses of Proposition 7.47:

$$R \text{ We } A \wedge R \text{ Wfr } A \wedge \mathscr{W}(F) \subseteqq A \wedge \\ (\forall\, x \in A)(\forall\, y \in \mathscr{W}(F))[x\, R\, y \to x \in \mathscr{W}(F)].$$

Consequently, from Proposition 7.47 we conclude that $\mathscr{W}(F) = A \vee (\exists\, x \in A)[\mathscr{W}(F) = A \cap (R^{-1})``\{x\}]$. But $\mathscr{W}(F)$ cannot be an R-initial segment of A because R-initial segments of A are sets, and $\mathscr{W}(F)$ being the one-to-one image of the proper class On cannot be a set. Therefore $\mathscr{W}(F) = A$ and

$$F\colon \text{On} \xrightarrow[\text{onto}]{1-1} A.$$

Finally if $\alpha < \beta$ then $F``\alpha \subseteqq F``\beta$ and hence $[A - F``\beta] \subseteqq [A - F``\alpha]$. Since $F`\beta \in [A - F``\beta]$ it follows that $F`\beta \in [A - F``\alpha]$. But $F`\alpha$ is the R-minimal element of $A - F``\alpha$. Hence

$$F`\alpha\, R\, F`\beta \vee F`\alpha = F`\beta.$$

Since F is one-to-one, $F`\alpha \neq F`\beta$ and so

$$\alpha < \beta \to F`\alpha\, R\, F`\beta$$

i.e., $F \text{ Isom}_{E,R}(\text{On}, A)$. □

Corollary 7.52. *If A is a proper class of ordinals, if*

(1) $G = \{\langle x, y\rangle \mid y \in [A - \mathscr{W}(x)] \wedge [(A - \mathscr{W}(x)) \cap (E^{-1})``\{y\}] = 0\}$,

(2) $F \mathscr{F}\!n$ On, and

(3) $(\forall\, \alpha)[F`\alpha = G`(F \restriction \alpha)]$

then

$$F \text{ Isom}_{E,E}(\text{On}, A).$$

PROOF. E We $A \wedge E$ Wfr A. □

Proposition 7.53. R We $A \wedge \mathscr{M}(A) \to (\exists!\, \alpha)(\exists!\, f)[f \text{ Isom}_{E,R}(\alpha, A)]$.

PROOF. If $G = \{\langle x, y\rangle \mid y \in [A - \mathscr{W}(x)] \wedge [(A - \mathscr{W}(x)) \cap (R^{-1})``\{y\}] = 0\}$, $F \mathscr{F}\!n$ On, and if $(\forall\, \alpha)[F`\alpha = G`(F \restriction \alpha)]$, then by Propositions 7.49 and 7.50, $(\exists\, \alpha)[F``\alpha = A \wedge \mathscr{U}\!n_2(F \restriction \alpha)]$, i.e. $(F \restriction \alpha)\colon \alpha \xrightarrow[\text{onto}]{1-1} A$.

That $F \upharpoonright \alpha$ is order preserving is proved as in the proof of Proposition 7.51 and is left to the reader. Then $(F \upharpoonright \alpha)\mathrm{Isom}_{E,R}(\alpha, A)$. But $F \upharpoonright \alpha$ is a set, i.e. $(\exists f)[f = F \upharpoonright \alpha]$. Then $(\exists f)[f\ \mathrm{Isom}_{E,R}(\alpha, A)]$. □

The uniqueness argument is left to the reader.

Corollary 7.54. $A \subseteq \mathrm{On} \wedge \mathscr{M}(A) \to (\exists!\, \alpha)(\exists!\, f)[f\,\mathrm{Isom}_{E,E}(\alpha, A)]$.

PROOF. $A \subseteq \mathrm{On} \to E\ \mathrm{We}\ A$. □

Remark. Since E is a well-founded relation and well foundedness is preserved under order isomorphism it follows that the requirement in Proposition 7.51 that R be well founded on A cannot be removed. In its absence we can only prove that On is order isomorphic to some R-initial segment of A. That this can occur we show by an example, the so-called lexicographical ordering on On × On.

Definition 7.55. $\mathrm{Le} \triangleq \{\langle\langle\alpha, \beta\rangle, \langle\gamma, \delta\rangle\rangle \mid \alpha < \gamma \vee [\alpha = \gamma \wedge \beta < \delta]\}$.

Proposition 7.56.

(1) $\mathrm{Le}\ \mathrm{We}\ \mathrm{On}^2 \wedge$
$[B \subseteq \mathrm{On}^2 \wedge B \neq 0 \to (\exists x \in B)[B \cap (\mathrm{Le}^{-1})``\{x\} = 0]]$.

(2) $\neg \mathrm{Le}\ \mathrm{Wfr}\ \mathrm{On}^2$.

PROOF. (1) The proof is left to the reader.

(2) If $F`\alpha = \langle 0, \alpha\rangle$ then $F\colon \mathrm{On} \xrightarrow[\text{onto}]{1-1} (\mathrm{Le}^{-1})``\{\langle 1, 0\rangle\}$, consequently $(\mathrm{Le}^{-1})``\{\langle 1, 0\rangle\}$ is a proper class. □

Remark. From the lexicographical ordering we in turn define a relation R_0 that will be of value to us in later chapters. We will show that this relation R_0 not only well orders On^2 it is well founded on On^2.

Definition 7.57. $R_0 \triangleq \{\langle\langle\alpha, \beta\rangle, \langle\gamma, \delta\rangle\rangle \mid \max(\alpha, \beta) < \max(\gamma, \delta)$
$\vee [\max(\alpha, \beta) = \max(\gamma, \delta) \wedge \langle\alpha, \beta\rangle \mathrm{Le} \langle\gamma, \delta\rangle]\}$.

Proposition 7.58.

(1) $R_0\, \mathrm{We}\, \mathrm{On}^2 \wedge [B \subseteq \mathrm{On}^2 \wedge B \neq 0 \to (\exists x \in B)[B \cap (R_0^{-1})``\{x\} = 0]]$.

(2) R_0 Wfr On^2.

PROOF. (1) The proof is left to the reader.

(2) If $\gamma = \max(\alpha, \beta) + 1$ and $\langle\delta, \tau\rangle\, R_0\, \langle\alpha, \beta\rangle$, then $\max(\delta, \tau) \leq \max(\alpha, \beta) < \gamma$. Therefore $\langle\delta, \tau\rangle \in \gamma \times \gamma$ and hence $\mathrm{On}^2 \cap (R_0^{-1})``\{\langle\alpha, \beta\rangle\} \subseteq \gamma \times \gamma$. Since $\gamma \times \gamma$ is a set, $\mathrm{On}^2 \cap (R_0^{-1})``\{\langle\alpha, \beta\rangle\}$ is a set. Thus R_0 is well founded on On^2. □

Remark. We have shown that R_0 well orders On^2 and is well founded on On^2. Consequently the relational system $[R_0, \mathrm{On}^2]$ is order isomorphic to $[E, \mathrm{On}]$. By Proposition 7.51 there exists an order isomorphism, indeed a unique order isomorphism between the two systems.

Definition 7.59. $J_0 \operatorname{Isom}_{R_0, E}(\mathrm{On}^2, \mathrm{On})$.

CHAPTER 8
Ordinal Arithmetic

In Chapter 7 we defined $\alpha + 1$ to be $\alpha \cup \{\alpha\}$. We proved that $\alpha + 1$ is an ordinal, that is, $\alpha + 1$ is a transitive set that is well ordered by the $\in$-relation. As a well-ordered set $\alpha + 1$ has an initial segment α and its "terminal" segment beginning with α consists of just a single element, namely α.

If we add 1 to $\alpha + 1$ we obtain an ordinal with an initial segment α and a terminal segment, beginning with α, consisting of two elements α and $\alpha + 1$. Since this terminal segment $\{\alpha, \alpha + 1\}$ is order isomorphic to $2 \triangleq 1 + 1$ we call the sum of $\alpha + 1$ and 1, $\alpha + 2$.

In general, by $\alpha + \beta$ we mean an ordinal obtained from α by adding 1, β times. That is, $\alpha + \beta$ is an ordinal with an initial segment α and a terminal segment, beginning with α, that is order isomorphic to β. That such an ordinal number exists is clear from the fact that $(\{0\} \times \alpha) \cup (\{1\} \times \beta)$ is well ordered by the lexicographical ordering Le. With respect to Le, $\{0\} \times \alpha$ is an initial segment order isomorphic to α and $\{1\} \times \beta$ is a terminal segment order isomorphic to β.

It would then seem reasonable to define $\alpha + \beta$ as the ordinal that is order isomorphic to $(\{0\} \times \alpha) \cup (\{1\} \times \beta\}$. However for certain purposes it is preferable to define $\alpha + \beta$ recursively in the following way.

Definition 8.1.

$$\alpha + 0 \triangleq \alpha,$$
$$\alpha + (\beta + 1) \triangleq (\alpha + \beta) + 1,$$
$$\alpha + \beta \triangleq \bigcup_{\gamma < \beta} (\alpha + \gamma), \beta \in K_{\mathrm{II}}.$$

Remark. Definition 8.1 is an example of a very convenient form of definition by transfinite recursion. To define the addition of β to α, i.e., $\alpha + \beta$ we specify the result of adding 0 to α, we define the sum of α and $(\beta + 1)$ as an operation on $\alpha + \beta$ namely the operation of adding one, and we define $\alpha + \beta$ for $\beta \in K_{\text{II}}$ as the supremum of the set of sums $\alpha + \gamma$ for $\gamma < \beta$.

That this is sufficient to define $\alpha + \beta$ for all α and β is clear from Proposition 7.44. If in Proposition 7.44, $H = \{\langle \alpha, \alpha + 1\rangle \mid \alpha \in \text{On}\}$ and $a = \alpha$ then

$$F`0 = \alpha,$$

$$F`(\beta + 1) = H`F`\beta = F`\beta + 1,$$

$$F`\beta = \bigcup_{\gamma<\beta} F`\gamma, \beta \in K_{\text{II}},$$

i.e., $\alpha + \beta = F`\beta$.

The reader should have little difficulty convincing himself, or herself, that Definition 8.1 captures our intuitive notion that $\alpha + \beta$ is an ordinal with initial segment α and terminal segment that is order isomorphic to β. In addition, Definition 8.1 is designed for proofs by induction. Recall that in order to prove $(\forall\, \alpha)\varphi(\alpha)$ by induction we need only prove

(1) $(\forall\, \beta \in \alpha)\varphi(\beta) \to \varphi(\alpha)$.

Since for each α either $\alpha = 0$ or $(\exists\, \gamma)[\alpha = \gamma + 1]$ or $\alpha \in K_{\text{II}}$, we can prove (1) by proving

$$\varphi(0),$$

$$\varphi(\alpha) \to \varphi(\alpha + 1),$$

and

$$\alpha \in K_{\text{II}} \wedge (\forall\, \beta \in \alpha)\varphi(\beta) \to \varphi(\alpha).$$

Definition 8.1 lends itself well to such proofs as we will now demonstrate.

Proposition 8.2. $\alpha + \beta \in \text{On}$.

Proof (By transitive induction on β). For $\beta = 0$ we have $\alpha + 0 = \alpha \in \text{On}$. If $\alpha + \beta \in \text{On}$ then $\alpha + (\beta + 1) = (\alpha + \beta) + 1 \in \text{On}$. If $\beta \in K_{\text{II}}$ and $(\forall \gamma)[\gamma < \beta \to \alpha + \gamma \in \text{On}]$ then $\alpha + \beta = \bigcup_{\gamma<\beta}(\alpha + \gamma) \in \text{On}$. $\square$

Proposition 8.3. $0 + \alpha = \alpha + 0 = \alpha$.

Proof. By definition $\alpha + 0 = \alpha$. If $0 + \alpha = \alpha$, then $0 + (\alpha + 1) = (0 + \alpha) + 1 = \alpha + 1$. If $\alpha \in K_{\text{II}}$ and $(\forall\, \beta)[\beta < \alpha \to 0 + \beta = \beta]$ then $0 + \alpha = \bigcup_{\beta<\alpha}(0 + \beta) = \bigcup_{\beta<\alpha} \beta = \alpha$. $\square$

Remark. We frequently wish to prove a property of ordinals that holds for all ordinals greater than or equal to some ordinal $\gamma > 0$ and which may fail to hold for ordinals smaller than γ:

(1) $(\forall\, \alpha)[\alpha \geqq \gamma \to \varphi(\alpha)]$.

Such an assertion can be proved by induction using the following approach. Let $\psi(\alpha)$ be the wff

$$\alpha < \gamma \vee \varphi(\alpha).$$

Then to prove (1) it is sufficient to prove $(\forall \alpha)\psi(\alpha)$. To prove this by induction it is sufficient to prove three things:

(a) $\psi(0)$,

(b) $\psi(\alpha) \to \psi(\alpha + 1)$,

(c) $\alpha \in K_{\mathrm{II}} \wedge (\forall \beta \in \alpha)\psi(\beta) \to \psi(\alpha)$.

Since $\gamma > 0$ the proof of (a) is already established so we have nothing to do. To prove (b) we have nothing to do if $\alpha + 1 < \gamma$. For $\alpha + 1 = \gamma$ we must prove $\varphi(\gamma)$ and for $\alpha + 1 > \gamma$ we must prove $\varphi(\alpha) \to \varphi(\alpha + 1)$. To prove (c) we must prove $\varphi(\gamma)$ if $\gamma \in K_{\mathrm{II}}$ and we must prove that

$$\alpha \in K_{\mathrm{II}} \wedge (\forall \beta)[\gamma \leqq \beta < \alpha \wedge \varphi(\beta)] \to \varphi(\alpha).$$

In summary we must prove

$$\varphi(\gamma),$$

$$\alpha \geqq \gamma \wedge \varphi(\alpha) \to \varphi(\alpha + 1),$$

and

$$\alpha \in K_{\mathrm{II}} \wedge (\forall \beta)[\gamma \leqq \beta < \alpha \wedge \varphi(\beta)] \to \varphi(\alpha).$$

Let us illustrate:

Proposition 8.4. $\alpha < \beta \to \gamma + \alpha < \gamma + \beta$.

Proof (By transfinite induction on β). For $\beta = \alpha + 1$ we have $\gamma + \alpha < (\gamma + \alpha) + 1 = \gamma + (\alpha + 1)$. If $\alpha < \beta \to \gamma + \alpha < \gamma + \beta$ and if $\alpha < \beta + 1$ then $\alpha < \beta \vee \alpha = \beta$. In either case we have that $\gamma + \alpha \leqq \gamma + \beta < (\gamma + \beta) + 1 = \gamma + (\beta + 1)$. If $\beta \in K_{\mathrm{II}}$ and $(\forall \delta)[\alpha < \delta < \beta \to \gamma + \alpha < \gamma + \delta]$ then $\gamma + \alpha < \bigcup_{\delta<\beta} (\gamma + \delta) = \gamma + \beta$. □

Corollary 8.5. $\gamma + \alpha = \gamma + \beta \leftrightarrow \alpha = \beta$.

Proof. If $\alpha = \beta$, then by Theorem 3.4

$$x \in \gamma + \alpha \leftrightarrow x \in \gamma + \beta.$$

Consequently $\gamma + \alpha = \gamma + \beta$. That is

(1) $\alpha = \beta \to \gamma + \alpha = \gamma + \beta$.

From Proposition 8.4

(2) $\alpha < \beta \to \gamma + \alpha < \gamma + \beta$

and

(3) $\beta < \alpha \to \gamma + \beta < \gamma + \alpha$.

From (1), (2), and (3) it follows that

$$\gamma + \alpha = \gamma + \beta \leftrightarrow \alpha = \beta. \qquad \square$$

Remark. For the proofs of several results on ordinal arithmetic we need the following property of suprema.

Proposition 8.6. $(\forall\, \alpha \in A)(\exists\, \beta \in B)[\alpha \leqq \beta] \rightarrow \sup(A) \leqq \sup(B)$.

PROOF. If $\gamma \in \sup(A)$ then $(\exists\, \alpha)[\gamma \in \alpha \wedge \alpha \in A]$. But $\alpha \in A \rightarrow (\exists\, \beta)[\beta \in B \wedge \alpha \leqq \beta]$ i.e., $(\exists\, \beta)[\gamma \in \beta \wedge \beta \in B]$. Therefore $\gamma \in \sup(B)$ and hence $\sup(A) \leqq \sup(B)$. $\square$

Remark. That Proposition 8.6 can not be an iff result is established by the counter-example

$$\sup(\omega) = \sup(\omega + 1).$$

Proposition 8.7. $\alpha \leqq \beta \rightarrow \alpha + \gamma \leqq \beta + \gamma$.

PROOF (By transfinite induction on γ). If $\alpha \leqq \beta$, then $\alpha + 0 \leqq \beta + 0$. If $\alpha + \gamma \leqq \beta + \gamma$ then $\alpha + (\gamma + 1) \leqq \beta + (\gamma + 1)$. If $\gamma \in K_{\mathrm{II}}$ and $(\forall\, \delta < \gamma)[\alpha + \delta \leqq \beta + \delta]$ then $\alpha + \gamma = \bigcup_{\delta<\gamma} (\alpha + \delta) \leqq \bigcup_{\delta<\gamma} (\beta + \delta) = \beta + \gamma$. $\square$

Proposition 8.8. $\alpha \leqq \beta \rightarrow (\exists!\, \gamma)[\alpha + \gamma = \beta]$.

PROOF. Since $\alpha \geqq 0$ it follows from Propositions 8.7 and 8.3 that $\alpha + \beta \geqq 0 + \beta = \beta$. Thus there exists a smallest ordinal γ such that $\alpha + \gamma \geqq \beta$. If $\gamma \in K_{\mathrm{I}}$ then $\gamma = 0 \vee (\exists\, \delta)[\gamma = \delta + 1]$. If $\gamma = 0$ then $\alpha \geqq \beta \wedge \alpha \leqq \beta$. Therefore $\alpha = \beta$ and $\alpha + \gamma = \beta$. If $\gamma = \delta + 1$ then $\delta < \gamma$ and $\alpha + \delta < \beta$. Then $\alpha + \delta + 1 \leqq \beta$, i.e., $\alpha + \gamma \leqq \beta$. But $\alpha + \gamma \geqq \beta$; therefore $\alpha + \gamma = \beta$. If $\gamma \in K_{\mathrm{II}}$ then $(\forall\, \delta < \gamma)[\alpha + \delta < \beta]$. Therefore

$$\alpha + \gamma = \bigcup_{\delta<\beta} (\alpha + \delta) \leqq \beta.$$

Again since $\alpha + \gamma \geqq \beta$, we have that $\alpha + \gamma = \beta$.

From Corollary 8.5 we see that if $\alpha + \gamma = \beta$ and $\alpha + \delta = \beta$ then $\gamma = \delta$. $\square$

Proposition 8.9. $m + n \in \omega$.

PROOF (By finite induction on n). For $n = 0$ we have $m + 0 = m \in \omega$. If $m + n \in \omega$, then $m + (n + 1) = (m + n) + 1 \in \omega$. $\square$

Proposition 8.10. $n < \omega \wedge \omega \leqq \alpha \rightarrow n + \alpha = \alpha$.

PROOF (By transfinite induction on α). If $\alpha = \omega$ we have

$$n + \omega = \bigcup_{\gamma<\omega} (n + \gamma).$$

By Proposition 8.9, $\gamma < \omega \rightarrow n + \gamma < \omega$. Hence

$$\bigcup_{\gamma<\omega} (n + \gamma) \leqq \omega.$$

On the other hand, by Proposition 8.8, $(\forall \beta < \omega)(\exists \gamma \in \omega)[\beta \leqq n + \gamma]$. Then

$$\omega = \bigcup_{\beta\in\omega} \beta \leqq \bigcup_{\gamma<\omega} (n + \gamma)$$

by Proposition 8.6. Thus $n + \omega = \omega$. By Definition 8.1 and the induction hypothesis

$$n + (\alpha + 1) = (n + \alpha) + 1 = \alpha + 1.$$

Finally, if $\alpha \in K_{\mathrm{II}}$ then from the induction hypothesis

$$n + \alpha = \bigcup_{\beta<\alpha} (n + \beta) = \bigcup_{\beta<\alpha} \beta = \alpha. \qquad \square$$

Remark. From Proposition 8.10 we see that ordinal addition is not commutative:

$$1 + \omega = \omega \neq \omega + 1.$$

Furthermore $1 + \omega = 2 + \omega$ but $1 \neq 2$. Thus we do *not* have a right-hand cancellation law. From Corollary 8.5 we see that we do however have a left-hand cancellation law.

Proposition 8.4 assures us of the additivity property for inequalities for addition from the left. Proposition 8.6 however suggests that addition from the right may not preserve strict inequality. This is the case as we see from the following example.

$$1 < 2 \quad \text{but} \quad 1 + \omega = 2 + \omega.$$

Proposition 8.8 shows that subtraction, when permitted, is unique.

Finally ordinal addition is associative. The proof requires the following result.

Proposition 8.11. $\beta \in K_{\mathrm{II}} \rightarrow \alpha + \beta \in K_{\mathrm{II}}$.

PROOF. If $\beta \in K_{\mathrm{II}}$, then $\beta \neq 0$. Therefore $\alpha + \beta \neq 0$. Thus $\alpha + \beta \in K_{\mathrm{II}}$ or $(\exists \delta)[\alpha + \beta = \delta + 1]$. But if $\beta \in K_{\mathrm{II}}$ then

$$\alpha + \beta = \bigcup_{\gamma<\beta} (\alpha + \gamma).$$

Since $\delta \in \delta + 1$ it follows that if $\alpha + \beta = \delta + 1$ then

$$\delta \in \bigcup_{\gamma<\beta} (\alpha + \gamma)$$

that is $(\exists \gamma)[\gamma < \beta \wedge \delta \in \alpha + \gamma]$. But if $\delta \in \alpha + \gamma$ then $(\delta + 1) \in (\alpha + \gamma + 1)$.

Since $\beta \in K_{\text{II}}$ and $\gamma < \beta$ it follows that $\gamma + 1 < \beta$. Therefore

$$\delta + 1 \in \bigcup_{\gamma < \beta} (\alpha + \gamma)$$

i.e. $\delta + 1 \in \delta + 1$.

Since this is a contradiction we conclude that $\alpha + \beta \in K_{\text{II}}$. □

Proposition 8.12. $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$.

PROOF (By transfinite induction on γ). For $\gamma = 0$ we note that $(\alpha + \beta) + 0 = \alpha + \beta = \alpha + (\beta + 0)$. If $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$ then $(\alpha + \beta) + (\gamma + 1) = ((\alpha + \beta) + \gamma) + 1 = (\alpha + (\beta + \gamma)) + 1 = \alpha + ((\beta + \gamma) + 1) = \alpha + (\beta + (\gamma + 1))$. If $\gamma \in K_{\text{II}}$ and $(\alpha + \beta) + \delta = \alpha + (\beta + \delta)$ for $\delta < \gamma$ then

$$(\alpha + \beta) + \gamma = \bigcup_{\delta < \gamma} ((\alpha + \beta) + \delta) = \bigcup_{\delta < \gamma} (\alpha + (\beta + \delta)).$$

Furthermore since $\gamma \in K_{\text{II}}$ we have by Proposition 8.11 that $\beta + \gamma \in K_{\text{II}}$. Therefore

$$\alpha + (\beta + \gamma) = \bigcup_{\eta < \beta + \gamma} (\alpha + \eta).$$

If $\delta < \gamma$ and $\eta = \beta + \delta$, then $\eta < \beta + \gamma$ and $\alpha + (\beta + \delta) \leqq \alpha + \eta$. Conversely if $\eta < \beta + \gamma$, then $\eta < \beta$ or $(\exists\, \delta)[\eta = \beta + \delta]$. Suppose that $\eta < \beta$. Then $\alpha + \eta \leqq \alpha + (\beta + 0)$ and $0 < \gamma$. On the otherhand if $\eta = \beta + \delta$ then $\alpha + \eta \leqq \alpha + (\beta + \delta)$ and since $\eta < \beta + \gamma$ we have that $\delta < \gamma$. Thus, by Proposition 8.6

$$\bigcup_{\delta < \gamma} (\alpha + (\beta + \delta)) = \bigcup_{\eta < \beta + \gamma} (\alpha + \eta)$$

i.e.,

$$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma).$$ □

Proposition 8.13. $\alpha \geqq \omega \to (\exists!\, \beta)(\exists!\, n)[\beta \in K_{\text{II}} \wedge \alpha = \beta + n]$.

PROOF. If $A = \{\gamma \in K_{\text{II}} | \gamma \leqq \alpha\}$ and if $\beta = \cup(A)$, then $\beta \in K_{\text{II}}$ and $\beta \leqq \alpha$. Therefore, by Proposition 8.8, $(\exists\, \gamma)[\beta + \gamma = \alpha]$. If $\gamma \geqq \omega$, then $(\exists\, \delta)[\gamma = \omega + \delta]$ and $\alpha = \beta + (\omega + \delta) = (\beta + \omega) + \delta$. But $\beta + \omega \in K_{\text{II}}$ and $\beta + \omega \leqq \alpha$. Thus $\beta + \omega \in A$; but $\beta < \beta + \omega$. This contradicts the definition of β; hence $\gamma < \omega$.

If $\alpha = \beta_1 + n_1 = \beta_2 + n_2$ with $\beta_1 \leqq \beta_2$ then $(\exists\, \gamma)[\beta_1 + \gamma = \beta_2]$, i.e.,

$$\beta_1 + n_1 = \beta_1 + \gamma + n_2,$$

$$n_1 = \gamma + n_2.$$

Since $\gamma \leqq \gamma + n_2$ we must have that $\gamma < \omega$. Furthermore since $\beta_1 + \gamma = \beta_2$ and $\beta_2 \in K_{\text{II}}$ it follows that $\gamma = 0$, i.e., $\beta_1 = \beta_2$ and $n_1 = n_2$. □

Definition 8.14. $\alpha \dot{-} \beta \triangleq \cap\{\gamma | \beta + \gamma \geqq \alpha\}$.

EXERCISES

Prove the following.

(1) $\alpha + \beta \in \omega \rightarrow \alpha \in \omega \wedge \beta \in \omega$.

(2) $\alpha \leqq \beta \rightarrow \alpha + (\beta \dot{-} \alpha) = \beta$.

(3) $\omega \dot{-} n = \omega$.

(4) $[m + n = n + m] \wedge [m + n = k + n \rightarrow m = k]$.

(5) $\alpha \leqq \alpha + \beta \wedge [\beta > 0 \rightarrow \alpha < \alpha + \beta]$.

(6) $\alpha \leqq \beta + \alpha$.

(7) $\alpha + \beta \in K_{\mathrm{II}} \leftrightarrow \beta \in K_{\mathrm{II}} \vee [\beta = 0 \wedge \alpha \in K_{\mathrm{II}}]$.

(8) $\beta \in K_{\mathrm{II}} \wedge \alpha < \beta \rightarrow (\forall n)[\alpha + n < \beta]$.

(9) $\alpha + \beta$ is order isomorphic to $(\{0\} \times \alpha) \cup (\{1\} \times \beta)$ where the order on the latter set is Le, i.e., $(\exists f)[f \text{ Isom}_{E,\text{Le}}(\alpha + \beta, (\{0\} \times \alpha) \cup (\{1\} \times \beta))]$.

(10) Prove Proposition 8.8 by transfinite induction on β.

Remark. From the foregoing we see that ordinal addition on ω has all of the arithmetic properties that we expect. Addition on On is however not commutative and the right-hand cancellation law fails.

In very much the same way as we define integer multiplication as repeated addition we can also define ordinal multiplication as repeated addition. For the justification of our definition we again appeal to Proposition 7.44.

If in Proposition 7.44, $H_\alpha = \{\langle\beta, \beta + \alpha\rangle \mid \beta \in \text{On}\}$ and if $a = 0$ then

$$F_\alpha`0 = 0,$$
$$F_\alpha`(\beta + 1) = H_\alpha`F_\alpha`\beta = F_\alpha`\beta + \alpha,$$
$$F_\alpha`\beta = \bigcup_{\gamma<\beta} F_\alpha`\gamma, \ \beta \in K_{\mathrm{II}}.$$

We define the product of α and β, i.e., $\alpha\beta$, to be $F_\alpha`\beta$.

Definition 8.15.

$$\alpha \cdot 0 \triangleq 0,$$
$$\alpha(\beta + 1) \triangleq \alpha\beta + \alpha,$$
$$\alpha\beta \triangleq \bigcup_{\gamma<\beta} \alpha\gamma, \ \beta \in K_{\mathrm{II}}.$$

Proposition 8.16. $\alpha\beta \in \text{On}$.

PROOF (By transfinite induction on β). For $\beta = 0$ we have $\alpha \cdot 0 = 0 \in \text{On}$. If $\alpha\beta \in \text{On}$ then $\alpha(\beta + 1) = \alpha\beta + \alpha \in \text{On}$. If $\beta \in K_{\mathrm{II}}$ and $\alpha\gamma \in \text{On}$ for $\gamma < \beta$, then

$$\alpha\beta = \bigcup_{\delta<\beta} \alpha\delta \in \text{On}. \qquad \square$$

Proposition 8.17. $mn \in \omega$.

PROOF (By finite induction on n). For $n = 0$ we have $m \cdot 0 = 0 \in \omega$. If $mn \in \omega$ then $m(n + 1) = mn + n \in \omega$. □

Proposition 8.18.

(1) $0 \cdot \alpha = \alpha \cdot 0 = 0$.

(2) $1 \cdot \alpha = \alpha \cdot 1 = \alpha$.

PROOF (By transfinite induction). (1) By definition $\alpha \cdot 0 = 0$ for all α including $\alpha = 0$. If $0 \cdot \alpha = 0$ then $0(\alpha + 1) = 0 \cdot \alpha + 0 = 0$. If $\alpha \in K_{\text{II}}$ and $0 \cdot \gamma = 0$ for $\gamma < \alpha$, then

$$0 \cdot \alpha = \bigcup_{\gamma < \alpha} 0 \cdot \gamma = 0.$$

(2) From (1) above $\alpha \cdot 1 = \alpha(0 + 1) = \alpha \cdot 0 + \alpha = 0 + \alpha = \alpha$. By definition $1 \cdot 0 = 0$. If $1 \cdot \alpha = \alpha$ then $1(\alpha + 1) = 1 \cdot \alpha + 1 = \alpha + 1$. If $\alpha \in K_{\text{II}}$ and $1 \cdot \gamma = \gamma$ for $\gamma < \alpha$, then

$$1 \cdot \alpha = \bigcup_{\gamma < \alpha} 1 \cdot \gamma = \alpha.$$ □

Proposition 8.19. $\alpha < \beta \wedge \gamma > 0 \leftrightarrow \gamma\alpha < \gamma\beta$.

PROOF. First we will prove, by transfinite induction on β, that $\alpha < \beta$ and $\gamma > 0$ imply $\gamma\alpha < \gamma\beta$.

If $\beta = \alpha + 1$ and $\gamma > 0$, then $\gamma\alpha < \gamma\alpha + \gamma = \gamma(\alpha + 1)$. If $\alpha < \beta \wedge \gamma > 0 \rightarrow \gamma\alpha < \gamma\beta$ and if $\alpha < \beta + 1$, then $\alpha < \beta$ or $\alpha = \beta$. In either case $\gamma\alpha \leqq \gamma\beta < \gamma\beta + \gamma = \gamma(\beta + 1)$. If $\beta \in K_{\text{II}}$ and if $(\forall\, \delta)[\alpha < \delta < \beta \wedge \gamma > 0 \rightarrow \gamma\alpha < \gamma\delta]$, then

$$\gamma\alpha < \bigcup_{\delta < \beta} \gamma\delta = \gamma\beta.$$

Conversely if $\gamma\alpha < \gamma\beta$ then $\gamma > 0$. Since $\alpha = \beta$ implies $\gamma\alpha = \gamma\beta$ and $\beta < \alpha$ and $\gamma > 0$ implies $\gamma\beta < \gamma\alpha$ we conclude that if $\gamma\alpha < \gamma\beta$ then $\alpha < \beta \wedge \gamma > 0$. □

Proposition 8.20. $\gamma\alpha = \gamma\beta \wedge \gamma > 0 \rightarrow \alpha = \beta$.

PROOF. By Proposition 8.19, $\alpha \neq \beta$ and $\gamma > 0$ imply $\alpha\gamma \neq \alpha\beta$. □

Proposition 8.21. $\alpha \leqq \beta \rightarrow \alpha\gamma \leqq \beta\gamma$.

PROOF (By transfinite induction on γ). For $\gamma = 0$ we see that $\alpha \cdot 0 = 0 \leqq \beta \cdot 0$. If $\alpha\gamma \leqq \beta\gamma$, then $\alpha(\gamma + 1) = \alpha\gamma + \alpha \leqq \beta\gamma + \beta = \beta(\gamma + 1)$. If $\gamma \in K_{\text{II}}$ and $(\forall\, \delta)[\delta < \gamma \rightarrow \alpha\delta \leqq \beta\delta]$ then

$$\alpha\gamma = \bigcup_{\delta < \gamma} \alpha\delta \leqq \bigcup_{\delta < \gamma} \beta\delta = \beta\gamma.$$ □

Proposition 8.22. $\alpha\beta = 0 \leftrightarrow \alpha = 0 \vee \beta = 0$.

Proof. If $\alpha = 0$ or $\beta = 0$, then from Proposition 8.18, $0 \cdot \beta = 0 \wedge \alpha \cdot 0 = 0$. If $\alpha \neq 0$ and $\beta \neq 0$, then $\alpha \geqq 1$ and $\beta \geqq 1$ and hence $1 \leqq \alpha \leqq \alpha\beta$, i.e., $\alpha\beta \neq 0$. □

Proposition 8.23. $\beta \in K_{\mathrm{II}} \wedge \gamma < \alpha\beta \rightarrow (\exists\, \delta)[\delta < \beta \wedge \gamma < \alpha\delta]$.

Proof. Definition 8.15. □

Proposition 8.24. $\alpha \neq 0 \wedge \beta \in K_{\mathrm{II}} \rightarrow \alpha\beta \in K_{\mathrm{II}}$.

Proof. If $\alpha \neq 0 \wedge \beta \in K_{\mathrm{II}}$ then $\alpha\beta \neq 0$. Therefore $\alpha\beta \in K_{\mathrm{II}}$ or $(\exists\, \gamma)[\gamma + 1 = \alpha\beta]$. Since $\gamma \in \gamma + 1$ and since $\beta \in K_{\mathrm{II}}$ it follows that if $\gamma + 1 = \alpha\beta$ then

$$\gamma \in \bigcup_{\delta < \beta} \alpha\delta,$$

i.e., $(\exists\, \delta)[\delta < \beta \wedge \gamma < \alpha\delta]$ (see Theorem 8.23). Then $\gamma + 1 < \alpha\delta + 1 \leqq \alpha\delta + \alpha = \alpha(\delta + 1)$. But $\beta \in K_{\mathrm{II}}$ and $\delta < \beta$ implies $\delta + 1 < \beta$, i.e., $\gamma + 1 \in \alpha(\delta + 1)$ and $\delta + 1 < \beta$. Thus

$$\gamma + 1 \in \bigcup_{\delta < \beta} \alpha\delta = \gamma + 1.$$

From this contradiction we conclude that $\alpha\beta \in K_{\mathrm{II}}$. □

Proposition 8.25. $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$.

Proof (By transfinite induction on γ). For $\gamma = 0$ we see that $\alpha(\beta + 0) = \alpha\beta = \alpha\beta + \alpha \cdot 0$. If $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$, then $\alpha(\beta + (\gamma + 1)) = \alpha((\beta + \gamma) + 1) = \alpha(\beta + \gamma) + \alpha = (\alpha\beta + \alpha\gamma) + \alpha = \alpha\beta + (\alpha\gamma + \alpha) = \alpha\beta + \alpha(\gamma + 1)$. If $\gamma \in K_{\mathrm{II}}$ and $\alpha(\beta + \delta) = \alpha\beta + \alpha\delta$ for $\delta < \gamma$ then we consider two cases $\alpha = 0$ and $\alpha \neq 0$. If $\alpha = 0$ then

$$\alpha(\beta + \gamma) = 0 = \alpha\beta + \alpha\gamma.$$

If $\alpha \neq 0$ then since $\gamma \in K_{\mathrm{II}}$ it follows that $\beta + \gamma \in K_{\mathrm{II}}$ and $\alpha\gamma \in K_{\mathrm{II}}$.

$$\alpha(\beta + \gamma) = \bigcup_{\delta < \beta + \gamma} \alpha\delta,$$

$$\alpha\beta + \alpha\gamma = \bigcup_{\eta < \alpha\gamma} (\alpha\beta + \eta).$$

If $\delta < \beta + \gamma$ then $\delta < \beta \vee (\exists\, \tau)[\tau < \gamma \wedge \delta = \beta + \tau]$. Therefore $\alpha\delta < \alpha\beta$ or

$$\alpha\delta = \alpha(\beta + \tau) = \alpha\beta + \alpha\tau = \alpha\beta + \eta$$

where $\eta = \alpha\tau$. Since $\tau < \gamma$ we have that $\alpha\tau < \alpha\gamma$, i.e., $\eta < \alpha\gamma$. Thus

$$\alpha(\beta + \gamma) \leqq \alpha\beta + \alpha\gamma.$$

If $\eta < \alpha\gamma$ then $(\exists\, \delta)[\delta < \gamma \wedge \eta < \alpha\delta]$. Therefore $\beta + \delta < \beta + \gamma$ and hence

$$\alpha\beta + \eta < \alpha\beta + \alpha\delta = \alpha(\beta + \delta).$$

Thus $\alpha\beta + \alpha\gamma \leqq \alpha(\beta + \gamma)$ and hence $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$. □

Remark. Note that $(\omega + 1)2 = (\omega + 1) + (\omega + 1) = \omega + (1 + \omega) + 1 = \omega + \omega + 1 = \omega 2 + 1 \neq \omega 2 + 2$. We do not have a right-hand distributive law.

Theorem 8.26. $(\alpha\beta)\gamma = \alpha(\beta\gamma)$.

Proof (By induction on γ). For $\gamma = 0$ we have $(\alpha\beta)\cdot 0 = 0 = \alpha \cdot 0 = \alpha(\beta \cdot 0)$. If $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ then $(\alpha\beta)(\gamma + 1) = (\alpha\beta)\gamma + \alpha\beta = \alpha(\beta\gamma) + \alpha\beta = \alpha(\beta\gamma + \beta) = \alpha(\beta(\gamma + 1))$. If $\gamma \in K_{\text{II}}$ and $\alpha\beta = 0$ then $\alpha = 0$ or $\beta = 0$ and $(\alpha\beta)\gamma = 0 = \alpha(\beta\gamma)$. If $\alpha\beta \neq 0$ then $\beta\gamma \in K_{\text{II}}$ and hence

$$(\alpha\beta)\gamma = \bigcup_{\delta < \gamma} (\alpha\beta)\delta,$$

$$\alpha(\beta\gamma) = \bigcup_{\eta < \beta\gamma} \alpha\eta.$$

But $\delta < \gamma \leftrightarrow \beta\delta < \beta\gamma$. Therefore $(\alpha\beta)\gamma = \alpha(\beta\gamma)$. □

Proposition 8.27. $\beta \neq 0 \rightarrow (\exists!\, \gamma)(\exists!\, \delta)[\alpha = \beta\gamma + \delta \wedge \delta < \beta]$.

Proof. If $\alpha < \beta$, then $\alpha = \beta \cdot 0 + \alpha \wedge \alpha < \beta$. If $\beta \leqq \alpha$ and $\gamma = \sup\{\delta \mid \beta\delta \leqq \alpha\}$ then $\gamma \geqq 1$. Suppose that $\alpha < \beta\nu$. Then $\beta\delta \leqq \alpha$ implies $\delta < \nu$ and hence $\gamma \leqq \nu$. Consequently if $\delta < \gamma$ then $\beta\delta \leqq \alpha$. If $(\exists\, \tau)[\gamma = \tau + 1]$ then $\tau < \gamma$ hence $\tau \in \{\delta \mid \beta\delta \leqq \alpha\}$ therefore $(\exists\, \nu)[\nu \in \{\delta \mid \beta\delta \leqq \alpha\} \wedge \tau < \nu]$. Thus $\nu = \gamma$, i.e., $\beta\gamma \leqq \alpha$. If $\gamma \in K_{\text{II}}$ then

$$\beta\gamma = \bigcup_{\delta < \gamma} \beta\delta \leqq \bigcup_{\delta < \gamma} \alpha = \alpha.$$

Thus $\beta\gamma \leqq \alpha$ and hence $(\exists\, \delta)[\alpha = \beta\gamma + \delta]$. If $\delta \geqq \beta$, then $(\exists\, \mu)[\delta = \beta + \mu]$ then $\alpha = \beta\gamma + \beta + \mu = \beta(\gamma + 1) + \mu$ hence $\beta(\gamma + 1) \leqq \alpha$ and $\gamma + 1 \leqq \gamma$. From this we conclude that $\delta < \beta$, i.e.,

$$\alpha = \beta\gamma + \delta \wedge \delta < \beta.$$

If $\alpha = \beta\gamma_1 + \delta_1 = \beta\gamma_2 + \delta_2$ with $\delta_1 < \beta \wedge \delta_2 < \beta \wedge \gamma_1 \leqq \gamma_2$ it then follows that $(\exists\, \nu)[\gamma_2 = \gamma_1 + \nu]$ and

$$\beta\gamma_1 + \delta_1 = \beta(\gamma_1 + \nu) + \delta_2 = \beta\gamma_1 + \beta\nu + \delta_2$$
$$\delta_1 = \beta\nu + \delta_2.$$

But $\beta\nu + \delta_2 < \beta$. Therefore $\nu = 0$ and hence $\delta_1 = \delta_2 \wedge \gamma_1 = \gamma_2$. □

Corollary 8.28. $n \neq 0 \rightarrow (\exists!\, q)(\exists!\, r)[m = nq + r \wedge r < n]$.

PROOF. By Proposition 8.27 $(\exists!\,\gamma)(\exists!\,\delta)[m = n\gamma + \delta \wedge \delta < n]$. But $n\gamma + \delta \in \omega$ implies $n\gamma \in \omega$ and $\delta \in \omega$. Furthermore if $1 \leqq n$, then $\gamma \leqq n\gamma$. Therefore $\gamma \in \omega$. $\square$

Proposition 8.29. $\gamma \in K_{\text{II}} \wedge m \neq 0 \rightarrow m(\gamma + n) = \gamma + mn.$

PROOF (By induction on $\gamma + n$). If $\gamma + n = \omega$ we have $m\omega = \bigcup_{n<\omega} mn$. Since $mn < \omega$ we have

$$\bigcup_{n<\omega} mn \leqq \omega.$$

Furthermore $p < \omega \rightarrow (\exists\, q)(\exists\, r)[p = mq + r \wedge r < m]$. But $p = mq + r \leqq mq + m = m(q + 1)$. Therefore $p \in \bigcup_{n<\omega} mn$; hence

$$\bigcup_{n<\omega} mn = \omega.$$

If $m(\gamma + n) = \gamma + mn$ then $m(\gamma + n + 1) = m(\gamma + n) + m = (\gamma + mn) + m = \gamma + m(n + 1)$. If $\gamma + n \in K_{\text{II}}$, then $n = 0$ and

$$m\gamma = \bigcup_{\delta<\gamma} m\delta.$$

If $\delta < \gamma$ then $\delta < \omega \vee \omega \leqq \delta$. If $\delta < \omega$ then $m\delta < \omega < \gamma$.

If $\omega \leqq \delta$ then $(\exists\, \beta)(\exists\, n)[\beta \in K_{\text{II}} \wedge \delta = \beta + n]$. Then from the induction hypothesis $m\delta = \beta + mn$. But $\beta \leqq \delta < \gamma$ and $\gamma \in K_{\text{II}}$. Therefore $\beta + mn < \gamma$. Since $(\forall\, \delta < \gamma)[m\delta < \gamma]$ we conclude that

$$m\gamma \leqq \gamma.$$

But $1 \leqq m$ and hence $\gamma \leqq m\gamma$. Therefore $m\gamma = \gamma$. $\square$

EXERCISES

Prove the following.

(1) $(\omega + 1)(\omega + 1) = \omega \cdot \omega + \omega + 1.$

(2) $\alpha\gamma \leqq \beta\gamma \rightarrow \gamma = 0 \vee \alpha \leqq \beta.$

(3) $mn = nm \wedge (m + n)k = mk + nk.$

(4) In Proposition 8.29 can m be replaced by α with the restriction that $\alpha < \gamma$? Give a proof or a counter example.

(5) $\alpha\beta \in K_{\text{II}} \leftrightarrow \alpha\beta \neq 0 \wedge [\beta \in K_{\text{II}} \vee \alpha \in K_{\text{II}}].$

(6) $\alpha\beta$ is order isomorphic to $\beta \times \alpha$ well ordered lexicographically, i.e., $(\exists\, f)[f \text{ Isom}_{E,\,\text{Le}}(\alpha\beta, \beta \times \alpha)]$.

Remark. When restricted to ω ordinal multiplication has the properties expected. On the class of all ordinals however multiplication is not commutative.

$$2 \cdot \omega = \omega \quad \text{and} \quad \omega \cdot 2 = \omega + \omega.$$

We do not have a right-hand cancellation law:

$$1 \cdot \omega = 2 \cdot \omega \quad \text{but} \quad 1 \neq 2.$$

Having defined multiplication as repeated addition we next define exponentiation as repeated multiplication.

Definition 8.30.

$$\alpha^0 \triangleq 1.$$

$$\alpha^{\beta+1} \triangleq \alpha^\beta \cdot \alpha.$$

$$\alpha^\beta \triangleq \bigcup_{\gamma<\beta} \alpha^\gamma, \beta \in K_{\mathrm{II}} \wedge \alpha \neq 0.$$

$$\alpha^\beta \triangleq 0, \beta \in K_{\mathrm{II}} \wedge \alpha = 0.$$

Proposition 8.31.

(1) $0^0 = 1$.

(2) $0^\beta = 0, \beta \geqq 1$.

(3) $1^\beta = 1$.

PROOF. (1) From Definition 8.30, $0^0 = 1$.

(2) If $\beta \geqq 1$, then $\beta \in K_{\mathrm{II}}$ or $(\exists\, \delta)[\beta = \delta + 1]$. If $\beta \in K_{\mathrm{II}}$ then by Definition 8.30, $0^\beta = 0$. If $\beta = \delta + 1$ then $0^\beta = 0^{\delta+1} = 0^\delta \cdot 0 = 0$.

(3) (By transfinite induction). For $\beta = 0$ we have $1^0 = 1$. If $1^\beta = 1$ then $1^{\beta+1} = 1^\beta \cdot 1 = 1^\beta = 1$. If $\beta \in K_{\mathrm{II}}$ and $1^\gamma = 1$ for $\gamma < \beta$, then $1^\beta = \bigcup_{\gamma<\beta} 1^\gamma = 1$. □

Proposition 8.32. $1 \leqq \alpha \to 1 \leqq \alpha^\beta$.

PROOF (By transfinite induction on β). First we note that $\alpha^0 = 1$. If $1 \leqq \alpha^\beta$ then since $1 \leqq \alpha$ we have $1 \leqq \alpha^\beta \leqq \alpha^\beta \cdot \alpha$, i.e., $1 \leqq \alpha^{\beta+1}$. If $\beta \in K_{\mathrm{II}}$ then since $\alpha \neq 0$

$$\alpha^\beta = \bigcup_{\gamma<\beta} \alpha^\gamma.$$

Since $0 < \beta \wedge \alpha^0 = 1$ we have $1 \leqq \bigcup_{\gamma<\beta} \alpha^\gamma = \alpha^\beta$. □

Proposition 8.33. $\alpha < \beta \wedge 1 < \gamma \to \gamma^\alpha < \gamma^\beta$.

PROOF (By transfinite induction on β). If $\beta = \alpha + 1$, then $1 < \gamma$ implies $\gamma^\alpha < \gamma^{\alpha+1}$. Suppose that

$$\alpha < \beta \wedge 1 < \gamma \to \gamma^\alpha < \gamma^\beta.$$

If $\alpha < \beta + 1$, then $\alpha < \beta$ or $\alpha = \beta$. In either case

$$\gamma^\alpha \leqq \gamma^\beta < \gamma^{\beta+1}.$$

If $\beta \in K_{\mathrm{II}}$ then since $\gamma \neq 0$

$$\gamma^{\beta} = \bigcup_{\delta < \beta} \gamma^{\delta}.$$

Furthermore, if $\alpha < \beta$ then $\alpha + 1 < \beta$ and hence

$$\gamma^{\alpha} < \bigcup_{\delta < \beta} \gamma^{\delta} = \gamma^{\beta}. \qquad \square$$

Corollary 8.34. $1 < \gamma \wedge \gamma^{\alpha} < \gamma^{\beta} \rightarrow \alpha < \beta$.

PROOF. By Proposition 8.33, $\beta \leqq \alpha \wedge 1 < \gamma \rightarrow \gamma^{\beta} \leqq \gamma^{\alpha}$. $\square$

Proposition 8.35. $\alpha < \beta \rightarrow \alpha^{\gamma} \leqq \beta^{\gamma}$.

PROOF (By transfinite induction on γ). For $\gamma = 0$ we have $\alpha^0 = 1 = \beta^0$. Suppose that $\alpha < \beta$ and $\alpha^{\gamma} \leqq \beta^{\gamma}$. Then $\alpha^{\gamma+1} = \alpha^{\gamma} \cdot \alpha \leqq \beta^{\gamma} \cdot \alpha < \beta^{\gamma} \cdot \beta = \beta^{\gamma+1}$. If $\gamma \in K_{\mathrm{II}}$, $\alpha < \beta$, and if $\alpha^{\delta} \leqq \beta^{\delta}$ for $\delta < \gamma$ then

$$\alpha^{\gamma} = \bigcup_{\delta < \gamma} \alpha^{\delta} \leqq \bigcup_{\delta < \gamma} \beta^{\delta} = \beta^{\gamma}. \qquad \square$$

Corollary 8.36. $\alpha < \beta \wedge \gamma \in K_{\mathrm{I}} \wedge \gamma \neq 0 \rightarrow \alpha^{\gamma} < \beta^{\gamma}$.

PROOF. If $\gamma \in K_{\mathrm{I}} \wedge \gamma \neq 0$, then $(\exists\, \delta)[\gamma = \delta + 1]$. By Proposition 8.35, if $\alpha < \beta$, then $\alpha^{\delta} \leqq \beta^{\delta}$. But $\alpha^{\gamma} = \alpha^{\delta} \cdot \alpha \leqq \beta^{\delta} \cdot \alpha < \beta^{\delta} \cdot \beta = \beta^{\gamma}$. $\square$

Remark. That $\alpha < \beta$ and $\gamma \in K_{\mathrm{II}}$ does not imply $\alpha^{\gamma} < \beta^{\gamma}$ follows from the observation that $2 < 3$ but

$$2^{\omega} = 3^{\omega} = \omega.$$

The proof is left to the reader.

Proposition 8.37. $\alpha > 1 \rightarrow \beta \leqq \alpha^{\beta}$.

PROOF (By transfinite induction on β). For $\beta = 0$ we have $0 \leqq \alpha^0 = 1$. If $\beta \leqq \alpha^{\beta}$ then $\beta + 1 \leqq \alpha^{\beta} + 1$. But since $\beta < \beta + 1$ we have from Proposition 8.33 that $\alpha^{\beta} < \alpha^{\beta+1}$ and hence $\alpha^{\beta} + 1 \leqq \alpha^{\beta+1}$, i.e., $\beta + 1 \leqq \alpha^{\beta+1}$. If $\beta \in K_{\mathrm{II}}$ and $\gamma < \beta$ implies $\gamma \leqq \alpha^{\gamma}$, then

$$\beta \leqq \bigcup_{\gamma < \beta} \alpha^{\gamma} = \alpha^{\beta}. \qquad \square$$

Proposition 8.38. $\alpha > 1 \wedge \beta > 0 \rightarrow (\exists!\, \delta)[\alpha^{\delta} \leqq \beta < \alpha^{\delta+1}]$.

PROOF. Since by Proposition 8.37, $\beta \leqq \alpha^{\beta}$ and since $\alpha^{\beta} < \alpha^{\beta+1}$ there exists a smallest ordinal γ such that $\beta < \alpha^{\gamma}$. From Definition 8.30 it follows that $\gamma \in K_{\mathrm{I}}$. Since $\alpha^0 = 1 \wedge \beta \geqq 1$ it follows that $\gamma \neq 0$; therefore $(\exists\, \delta)[\gamma = \delta + 1]$. But $\delta < \delta + 1$ hence $\alpha^{\delta} \leqq \beta < \alpha^{\delta+1}$.

If $\alpha^\delta \leqq \beta < \alpha^{\delta+1}$ and $\alpha^\gamma \leqq \beta < \alpha^{\gamma+1}$ and if $\delta < \gamma$, then $\delta + 1 \leqq \gamma$. Hence

$$\beta < \alpha^{\delta+1} \leqq \alpha^\gamma \leqq \beta.$$

Similarly if $\gamma < \delta$, then $\beta < \beta$. Therefore $\delta = \gamma$. □

Proposition 8.39.

(1) $\alpha > 1 \wedge \beta \in K_{II} \to \alpha^\beta \in K_{II}$.

(2) $\alpha \in K_{II} \wedge \beta > 0 \to \alpha^\beta \in K_{II}$.

PROOF. (1) If $\alpha > 1$, then $\alpha^\beta \geqq 1$ and hence $\alpha^\beta \neq 0$. Therefore $\alpha^\beta \in K_{II}$ or $(\exists\, \delta)[\delta + 1 = \alpha^\beta]$. Since $\beta \in K_{II}$ and $\alpha \neq 0$

$$\alpha^\beta = \bigcup_{\gamma<\beta} \alpha^\gamma.$$

But $\delta \in \delta + 1 = \alpha^\beta$. Consequently $(\exists\, \gamma < \beta)[\delta < \alpha^\gamma]$. Since $1 < \alpha$, and $\delta < \alpha^\gamma$ it follows that $\delta + 1 \leqq \alpha^\gamma < \alpha^{\gamma+1}$. But $\gamma + 1 < \beta$ and so $\delta + 1 \in \alpha^\beta = \delta + 1$. From this contradiction we conclude that

$$\alpha^\beta \in K_{II}.$$

(2) If $\beta \in K_{II}$ then $\alpha^\beta \in K_{II}$ by (1) above. If $\beta \in K_I$ then since $\beta \neq 0$, $(\exists\, \delta)[\beta = \delta + 1]$. Then $\alpha^\beta = \alpha^{\delta+1} = \alpha^\delta \cdot \alpha$. Since $\alpha \in K_{II}$, $\alpha^\delta \neq 0$, therefore $\alpha^\delta\alpha \in K_{II}$. □

Proposition 8.40. $\beta \in K_{II} \wedge \gamma < \alpha^\beta \to (\exists\, \delta < \beta)[\gamma < \alpha^\delta]$.

PROOF. If $\beta \in K_{II}$ and $\gamma < \alpha^\beta$ then $\alpha \neq 0$ and hence

$$\alpha^\beta = \bigcup_{\delta<\beta} \alpha^\delta.$$

Then $(\exists\, \delta < \beta)[\gamma < \alpha^\delta]$. □

Proposition 8.41. $\alpha^\beta \cdot \alpha^\gamma = \alpha^{\beta+\gamma}$.

PROOF (By transfinite induction on γ). First we note that $\alpha^\beta \cdot \alpha^0 = \alpha^\beta \cdot 1 = \alpha^\beta = \alpha^{\beta+0}$. If $\alpha^\beta \cdot \alpha^\gamma = \alpha^{\beta+\gamma}$ then $\alpha^\beta \cdot \alpha^{\gamma+1} = \alpha^\beta \cdot \alpha^\gamma\alpha = \alpha^{\beta+\gamma}\alpha = \alpha^{\beta+(\gamma+1)}$. If $\gamma \in K_{II}$ then $\beta + \gamma \in K_{II}$. If $\alpha = 0$ then $\alpha^\gamma = 0 \wedge \alpha^{\beta+\gamma} = 0$. Thus $\alpha^\beta \cdot \alpha^\gamma = 0 = \alpha^{\beta+\gamma}$. If $\alpha = 1$ then $\alpha^\beta \cdot \alpha^\gamma = 1 \cdot 1 = 1 = \alpha^{\beta+\gamma}$. If $\alpha > 1$ then $\alpha^\gamma \in K_{II}$ and

$$\alpha^\beta \cdot \alpha^\gamma = \bigcup_{\delta<\alpha^\gamma} \alpha^\beta\delta,$$

$$\alpha^{\beta+\gamma} = \bigcup_{\eta<\beta+\gamma} \alpha^\eta.$$

If $\delta < \alpha^\gamma$ then by Proposition 8.40, $(\exists\, \tau < \gamma)[\delta < \alpha^\tau]$. Since by the induction hypothesis $\tau < \gamma$ implies $\alpha^\beta\alpha^\tau = \alpha^{\beta+\tau}$, and $\beta + \tau < \beta + \gamma$

$$\alpha^\beta\delta \leqq \alpha^\beta\alpha^\tau \leqq \alpha^{\beta+\tau}$$

Thus $\alpha^\beta \cdot \alpha^\gamma \leqq \alpha^{\beta+\gamma}$. Furthermore if $\eta < \beta + \gamma$ then $\eta \leqq \beta$ or $(\exists\, \tau)[\eta = \beta + \tau]$ Suppose that $\eta \leqq \beta$. Then

$$\alpha^\eta \leqq \alpha^\beta \cdot 1 \wedge 1 < \alpha^\gamma.$$

On the other hand if $\eta = \beta + \tau$, then $\tau < \gamma$. Hence $\alpha^{\beta+\tau} = \alpha^\beta \cdot \alpha^\tau$ and

$$\alpha^\eta = \alpha^{\beta+\tau} = \alpha^\beta \cdot \alpha^\tau \wedge \alpha^\tau < \alpha^\gamma.$$

Thus $\alpha^\beta \cdot \alpha^\gamma = \alpha^{\beta+\gamma}$. □

Proposition 8.42. $(\alpha^\beta)^\gamma = \alpha^{\beta\gamma}$.

PROOF (By transfinite induction on γ). For $\gamma = 0$ we have $(\alpha^\beta)^0 = 1 = \alpha^{\beta \cdot 0}$. If $(\alpha^\beta)^\gamma = \alpha^{\beta\gamma}$ then $(\alpha^\beta)^{\gamma+1} = (\alpha^\beta)^\gamma\alpha^\beta = \alpha^{\beta\gamma}\alpha^\beta = \alpha^{\beta\gamma+\beta} = \alpha^{\beta(\gamma+1)}$. If $\gamma \in K_{\mathrm{II}}$ then $\beta = 0$ or $\beta\gamma \in K_{\mathrm{II}}$. If $\beta = 0$ then $(\alpha^\beta)^\gamma = 1^\gamma = 1 = \alpha^{\beta\gamma}$. If $\beta\gamma \in K_{\mathrm{II}}$ then $\alpha = 0$ or $\alpha \neq 0$. If $\alpha = 0$ then $\alpha^\beta = 0$ and hence $(\alpha^\beta)^\gamma = 0 = \alpha^{\beta\gamma}$. If $\alpha \neq 0$ then $\alpha^\beta \neq 0$ and

$$(\alpha^\beta)^\gamma = \bigcup_{\delta<\gamma} (\alpha^\beta)^\delta,$$

$$\alpha^{\beta\gamma} = \bigcup_{\eta<\beta\gamma} \alpha^\eta.$$

If $\delta < \gamma$ then by the induction hypothesis $(\alpha^\beta)^\delta = \alpha^{\beta\delta}$. Since $\delta < \gamma$ implies $\beta\delta < \beta\gamma$ we have that $(\alpha^\beta)^\gamma \leqq \alpha^{\beta\gamma}$. If $\eta < \beta\gamma$ then $(\exists\, \delta < \gamma)[\eta < \beta\delta]$. Hence

$$\alpha^\eta \leqq \alpha^{\beta\delta} \quad \text{and} \quad \delta < \gamma.$$

Therefore $\alpha^{\beta\gamma} = (\alpha^\beta)^\gamma$. □

Proposition 8.43.

$$\alpha > 1 \wedge \gamma_n < \alpha \wedge \cdots \wedge \gamma_0 < \alpha \wedge 0 \leqq \beta_0 < \cdots < \beta_n < \beta$$
$$\to \alpha^{\beta_n}\gamma_n + \cdots + \alpha^{\beta_0}\gamma_0 < \alpha^\beta.$$

PROOF (By induction on n). If $n = 0$ then since $\gamma_0 < \alpha$ we have that $\alpha^{\beta_0}\gamma_0 < \alpha^{\beta_0+1} \leqq \alpha^\beta$. If $n > 0$ then since $\beta_{n-1} < \beta_n < \beta$ we have as our induction hypothesis

$$\alpha^{\beta_{n-1}}\gamma_{n-1} + \cdots + \alpha^{\beta_0}\gamma_0 < \alpha^{\beta_n}.$$

Therefore

$$\alpha^{\beta_n}\gamma_n + \cdots + \alpha^{\beta_0}\gamma_0 < \alpha^{\beta_n}\gamma_n + \alpha^{\beta_n} = \alpha^{\beta_n}(\gamma_n + 1).$$

Since $\gamma_n < \alpha$ we have $\gamma_n + 1 \leqq \alpha$ and hence

$$\alpha^{\beta_n}(\gamma_n + 1) \leqq \alpha^{\beta_n+1} \leqq \alpha^\beta.$$ □

Proposition 8.44. $\beta > 0 \wedge \alpha > 1 \to (\exists!\, n)(\exists!\, \beta_0)\cdots(\exists!\, \beta_n)(\exists!\, \gamma_0)\cdots(\exists!\, \gamma_n)$ $[\beta = \alpha^{\beta_n}\gamma_n + \cdots + \alpha^{\beta_0}\gamma_0 \wedge 0 \leqq \beta_0 < \beta_1 < \cdots < \beta_n \wedge 0 < \gamma_0 < \alpha \wedge \cdots \wedge 0 < \gamma_n < \alpha]$.

PROOF (By transfinite induction on β). By Proposition 8.38, there exists a δ such that

(1) $\alpha^\delta < \beta < \alpha^{\delta+1}$.

By Proposition 8.27 there exists a τ and ν such that

(2) $\beta = \alpha^\delta\tau + \nu$

and $\nu < \alpha^\delta$. From (1) it follows that $0 < \tau < \alpha$. So if $\nu = 0$ we are through. If $\nu > 0$, then by the induction hypothesis, it follows that there exist ordinals $\beta_0, \ldots, \beta_n, \gamma_0, \ldots, \gamma_n$ as prescribed, such that

$$\nu = \alpha^{\beta_n}\gamma_n + \cdots + \alpha^{\beta_0}\gamma_0.$$

Substituting this in (2) we have

$$\beta = \alpha^\delta\tau + \alpha^{\beta_n}\gamma_n + \cdots + \alpha^{\beta_0}\gamma_0$$

and the ordinals $\delta, \beta_n, \ldots, \beta_0; \tau, \gamma_n, \ldots, \gamma_0$ are as required. □

The proof of uniqueness we leave to the reader.

Proposition 8.45. $\alpha > 1 \wedge \beta_0 < \beta_1 < \cdots < \beta_n \wedge 0 < \gamma_0 < \alpha \wedge \cdots \wedge 0 < \gamma_n < \alpha \wedge \delta \geqq \omega \to (\alpha^{\beta_n}\gamma_n + \cdots + \alpha^{\beta_0}\gamma_0)\alpha^\delta = \alpha^{\beta_n+\delta}$.

PROOF. From Proposition 8.43.

$$\alpha^{\beta_n} \leqq \alpha^{\beta_n}\gamma_n + \cdots + \alpha^{\beta_0}\gamma_0 < \alpha^{\beta_n+1}.$$

Therefore

$$\alpha^{\beta_n+\delta} = \alpha^{\beta_n}\alpha^\delta \leqq (\alpha^{\beta_n}\gamma_n + \cdots + \alpha^{\beta_0}\gamma_0)\alpha^\delta \leqq \alpha^{\beta_n+1}\alpha^\delta = \alpha^{\beta_n+1+\delta} = \alpha^{\beta_n+\delta}.$$ □

Proposition 8.46. $\alpha \in K_{\mathrm{II}} \wedge \beta_0 < \beta_1 < \cdots < \beta_n \wedge 0 < m_n \wedge \delta > 0 \to (\alpha^{\beta_n}m_n + \cdots + \alpha^{\beta_0}m_0)\alpha^\delta = \alpha^{\beta_n+\delta}$.

PROOF. From Proposition 8.43.

$$\alpha^{\beta_{n-1}}m_{n-1} \quad + \cdots + \alpha^{\beta_0}m_0 < \alpha^{\beta_n}.$$

Therefore

$$\alpha^{\beta_n} \leqq \alpha^{\beta_n}m_n + \cdots + \alpha^{\beta_0}m_0 < \alpha^{\beta_n}m_n + \alpha^{\beta_n} = \alpha^{\beta_n}(m_n + 1).$$

Then

$$\alpha^{\beta_n}\alpha^\delta \leqq (\alpha^{\beta_n}m_n + \cdots + \alpha^{\beta_0}m_0)\alpha^\delta \leqq \alpha^{\beta_n}(m_n + 1)\alpha^\delta = \alpha^{\beta_n}\alpha^\delta = \alpha^{\beta_n+\delta}.$$ □

Proposition 8.47. *If* $\alpha \in K_{\mathrm{II}} \wedge \beta > 0 \wedge m > 0$ *then*

(1) $(\alpha^\beta m)^\gamma = \alpha^{\beta\gamma}m, \gamma \in K_{\mathrm{I}} \wedge \gamma \neq 0$.

(2) $(\alpha^\beta m)^\gamma = \alpha^{\beta\gamma}, \gamma \in K_{\mathrm{II}}$.

PROOF (By transfinite induction on γ). If $\gamma = 1$, then $(\alpha^\beta m)^1 = \alpha^{\beta\cdot 1}m$. If $(\alpha^\beta m)^\gamma = \alpha^{\beta\gamma}m$ then $(\alpha^\beta m)^{\gamma+1} = (\alpha^\beta m)^\gamma \alpha^\beta m = \alpha^{\beta\gamma} m \alpha^\beta m = \alpha^{\beta\gamma}\alpha^\beta m = \alpha^{\beta(\gamma+1)}m$. If $(\alpha^\beta m)^\gamma = \alpha^{\beta\gamma}$ then $(\alpha^\beta m)^{\gamma+1} = (\alpha^\beta m)^\gamma \alpha^\beta m = \alpha^{\beta\gamma}\alpha^\beta m = \alpha^{\beta(\gamma+1)}m$. If $\gamma \in K_{\mathrm{II}}$ then

$$(\alpha^\beta)^\gamma \leqq (\alpha^\beta m)^\gamma = \bigcup_{\delta<\gamma} (\alpha^\beta m)^\delta \leqq \bigcup_{\delta<\gamma} \alpha^{\beta\delta} m \leqq \bigcup_{\delta<\gamma} \alpha^{\beta(\delta+1)} = \alpha^{\beta\gamma}. \qquad \square$$

Proposition 8.48.

$$\alpha \in K_{\mathrm{II}} \wedge \beta_0 < \beta_1 < \cdots < \beta_n \rightarrow (\alpha^{\beta_n}m_n + \cdots + \alpha^{\beta_0}m_0)^\gamma \leqq \alpha^{\beta_n\gamma}(m_n + 1).$$

PROOF. Note that $\alpha^{\beta_n}m_n + \cdots + \alpha^{\beta_0}m_0 \leqq \alpha^{\beta_n}(m_n + 1)$. Therefore by Proposition 8.47 $(\alpha^{\beta_n}m_n + \cdots + \alpha^{\beta_0}m_0)^\gamma \leqq [\alpha^{\beta_n}(m_n + 1)]^\gamma \leqq \alpha^{\beta_n\gamma}(m_n + 1)$. $\square$

Proposition 8.49.

$$\alpha \in K_{\mathrm{II}} \wedge \beta_0 < \beta_1 < \cdots < \beta_n \wedge \gamma \in K_{\mathrm{II}} \rightarrow (\alpha^{\beta_n}m_n + \cdots + \alpha^{\beta_0}m_0)^\gamma = \alpha^{\beta_n\gamma}.$$

PROOF. Note that $\alpha^{\beta_n} \leqq \alpha^{\beta_n}m_n + \cdots + \alpha^{\beta_0}m_0 \leqq \alpha^{\beta_n}(m_n + 1)$. Therefore by Proposition 8.47

$$(\alpha^{\beta_n}m_n + \cdots + \alpha^{\beta_0}m_0)^\gamma = \alpha^{\beta_n\gamma}. \qquad \square$$

Corollary 8.50.

$$\alpha \in K_{\mathrm{II}} \wedge \beta_0 < \beta_1 < \cdots < \beta_n \wedge \gamma > 0 \rightarrow (\alpha^{\beta_n}m_n + \cdots + \alpha^{\beta_0}m_0)^{\alpha^\gamma} = \alpha^{\beta_n\alpha^\gamma}.$$

PROOF. By Proposition 8.39, $\gamma > 0 \wedge \alpha \in K_{\mathrm{II}} \rightarrow \alpha^\gamma \in K_{\mathrm{II}}$. The result then follows from Proposition 8.49. $\square$

CHAPTER 9
Relational Closure and the Rank Function

In this chapter we introduce two ideas important for the work to follow. The first of these is relational closure. In later chapters we will be especially interested in sets that are transitive. While there exist sets that are not transitive every set has a transitive extension. Indeed, every set has a smallest transitive extension which we call its transitive closure.

Proposition 9.1. $(\forall x)(\exists y)[x \subseteq y \wedge \mathrm{Tr}(y) \wedge (\forall z)[x \subseteq z \wedge \mathrm{Tr}(z) \rightarrow y \subseteq z]]$.

Proof. If $G`x = x \cup (\cup(x))$, then there exists a function f defined by recursion on ω such that

$$f`0 = x$$
$$f`(n + 1) = G`f`n.$$

Furthermore, if

$$y = \bigcup_{n<\omega} f`n$$

then $x = f`0 \subseteq y$. From the definition of G

$$f`n \subseteq f`(n + 1) \wedge \cup(f`n) \subseteq f`(n + 1).$$

If $a \in b \wedge b \in y$ then $(\exists n)[b \in f`n]$ and hence

$$a \in \cup(f`n) \subseteq f`(n + 1)$$

i.e., $a \in f`(n + 1)$. Then $a \in y$ and hence y is transitive.

If $x \subseteq z \wedge \mathrm{Tr}(z)$ then we prove by induction that

$$f`n \subseteq z.$$

$f`0 = x \subseteq z$. If $f`n \subseteq z$ then since z is transitive $\cup(f`n) \subseteq z$, i.e.,

$$a \in \cup(f`n) \wedge f`n \subseteq z \to a \in z.$$

Thus $f`(n+1) = f`n \cup (\cup(f`n)) \subseteq z$.

Consequently $y = \bigcup_{n<\omega} f`n \subseteq z$. □

Definition 9.2. $\operatorname{Tr} \operatorname{Cl}(a) \triangleq \cap\{y \mid a \subseteq y \wedge \operatorname{Tr}(y)\}$.

Remark. Proposition 9.1 has a natural and useful generalization to well-founded relations.

Proposition 9.3. *If* R Wfr A and $a \subseteq A$ *then there exists a set* b *such that* $[a \subseteq b \subseteq A]$ *and*

(1) $(\forall x \in A)(\forall y)[x \, R \, y \wedge y \in b \to x \in b]$.

(2) $(\forall x \in b)[x \in a \vee (\exists n)(\exists g)[g : n + 1 \to b \wedge g`0 \in a \wedge g`n = x \wedge (\forall i < n)[g`(i+1) \, R \, g`i]]]$.

(3) $(\forall w)[[a \subseteq w \subseteq A \wedge (\forall x \in A)(\forall y)[x \, R \, y \wedge y \in w \to x \in w]] \to b \subseteq w]]$.

Proof. (1) Since R is well founded on A, it follows, for each x in A, that $A \cap (R^{-1})``\{x\}$ is a set. Therefore if

$$B = \{\langle x, A \cap (R^{-1})``\{x\}\rangle \mid x \in A\}$$

then B is a function. If $y \subseteq A$ then $B``y$ is a set hence so is $\cup(B``y)$. But

$$\begin{aligned} \cup(B``y) &= \cup\{B`x \mid x \in y\} \\ &= \cup\{A \cap (R^{-1})``\{x\} \mid x \in y\} \\ &= A \cap (R^{-1})``y. \end{aligned}$$

Thus $A \cap (R^{-1})``y$ is a set.

If $G`x = x \cup (A \cap (R^{-1})``x)$ then there exists a function f defined on ω by recursion such that $f`0 = a$ and $f`(n+1) = G`f`n$. Furthermore $f`0 = a \subseteq A$. If $f`n$ is a subset of A then since

$$f`(n+1) = f`n \cup (A \cap (R^{-1})``f`n)$$

$f`(n+1)$ is a subset of A. Thus $\cup(f``\omega)$ is a subset of $A$. If $b = \cup(f``\omega)$ then $a = f`0 \subseteq b$. From the definition of G

$$f`n \subseteq f`(n+1) \wedge A \cap (R^{-1})``f`n \subseteq f`(n+1).$$

If $x \in A$, $x \, R \, y$ and $y \in b$ then $(\exists n)[x \, R \, y \wedge y \in f`n]$, i.e., $x \in (R^{-1})``f`n \subseteq f`(n+1)$. Thus $x \in b$.

(2) $(\forall x \in b)(\exists n)[x \in f`n]$. If $n = 0$ then $f`n = a$ and $x \in a$. If the result holds for each element in $f`n$ and $x \in f`(n+1)$ then since

$$f`(n+1) = f`n \cup (A \cap (R^{-1})``f`n)$$

it follows that either $x \in f`n$, hence the conclusion follows from the induction hypothesis, or $(\exists\, y \in f`n)[x\ R\ y]$ in which case the conclusion again follows from the induction hypothesis.

(3) The proof is left to the reader. $\square$

Proposition 9.4.

$$R \text{ Wfr } A \wedge B \subseteq A \wedge B \neq 0 \rightarrow (\exists\, x \in B)[B \cap (R^{-1})``\{x\} = 0].$$

PROOF. Let $a \in B$. By Proposition 9.3 there exists a set b such that

$$\{a\} \subseteq b \subseteq A \wedge (\forall\, x \in A)(\forall\, y)[x\ R\ y \wedge y \in b \rightarrow x \in b].$$

Then $b \cap B$ is a nonempty subset of A. Therefore

$$(\exists\, x \in b \cap B)[(b \cap B) \cap (R^{-1})``\{x\} = 0].$$

If $y \in B \cap (R^{-1})``\{x\}$, then $y \in B$ and $y\ R\ x$. But $x \in b$ and hence $y \in b$. Thus

$$y \in [b \cap B \cap (R^{-1})``\{x\}].$$

Therefore $B \cap (R^{-1})``\{x\} = 0$. $\square$

Remark. Note in Proposition 9.4 that the set b has the property that $(R^{-1})``b$ is a subset of b. We say that b is closed with respect to (w.r.t) the binary relation R^{-1}.

Definition 9.5.

(1) $\mathrm{Cl}(R, A) \overset{\Delta}{\leftrightarrow} R``A \subseteq A$.

(2) $\mathrm{Cl}_2(R, A) \overset{\Delta}{\leftrightarrow} R``A^2 \subseteq A$.

Remark. We read $\mathrm{Cl}(R, A)$ as "R is closed on A" and we read $\mathrm{Cl}_2(R, A)$ as "R is closed on A^2."

Proposition 9.6. *If* $\mathrm{Cl}(R_1, A) \wedge \cdots \wedge \mathrm{Cl}(R_p, A) \wedge \mathrm{Cl}_2(S_1, A) \wedge \cdots \wedge \mathrm{Cl}_2(S_q, A)$ *if* $(\forall\, x \subseteq A)[\mathscr{M}(R_1``x) \wedge \cdots \wedge \mathscr{M}(R_p``x) \wedge \mathscr{M}(S_1``x^2) \wedge \cdots \wedge \mathscr{M}(S_q``x^2)]$, *and if* $a \subseteq A$ *then there exists a set* b *such that*

$$a \subseteq b \subseteq A \wedge \mathrm{Cl}(R_1, b) \wedge \cdots \wedge \mathrm{Cl}(R_p, b) \wedge \mathrm{Cl}_2(S_1, b) \wedge \cdots \wedge \mathrm{Cl}_2(S_q, b).$$

PROOF. If $G`x = x \cup R_1``x \cup \cdots \cup R_p``x \cup S_1``x^2 \cup \cdots \cup S_q``x^2, x \subseteq A$ then there exists a function f defined on ω by recursion such that

$$f`0 = a \wedge f`(n + 1) = G`f`n$$

Let

$$b = \bigcup_{n < \omega} f`n$$

then from the definition of G

$$f`(n+1) = f`n \cup R_1``f`n \cup \cdots \cup R_p``f`n \cup S_1``(f`n)^2 \cup \cdots \cup S_q``(f`n)^2.$$

Furthermore $f`0 = a \subseteq A$. And if $f`n \subseteq A$, then since each R_i and each S_i is closed on A it follows that $f`(n+1) \subseteq A$. Therefore $b \subseteq A$. Furthermore $f`0 = a \subseteq b$.

If $y \in R_i``b$ then $(\exists\, x \in b)[\langle x, y\rangle \in R_i]$. But

$$x \in b \rightarrow (\exists\, n)[x \in f`n].$$

Thus

$$y \in R_i``f`n \subseteq f`(n+1).$$

Therefore $y \in b$. Consequently $R_i``b \subseteq b$.

If $z \in S_i``b^2$ then $(\exists\, x, y \in b)[\langle x, y, z\rangle \in S_i.]$ Furthermore $(\exists\, m, n)[x \in f`m \wedge y \in f`n]$. If $r = \max(m, n)$ then $x, y \in f`r$. Thus

$$z \in S_i``(f`r)^2 \subseteq f`(r+1).$$

Therefore $z \in b$, and hence, $S_i``b^2 \subseteq b$. □

Proposition 9.7. *If* R Wfr A, *if* $K = \{f \mid (\exists\, y \subseteq A)[\mathrm{Cl}(R^{-1}, y) \wedge f\ \mathscr{F}n\ y \wedge (\forall\, x \in y)[f`x = G`(f \restriction (R^{-1})``\{x\})]]\}$ *and if* $F = \cup(K)$, *then*

(1) $F\ \mathscr{F}n\ A$,

(2) $(\forall\, x \in A)[F`x = G`(F \restriction (R^{-1})``\{x\})]$,

(3) F *is unique.*

The proof is left to the reader.

Remark. Proposition 9.1 assures us that every set a has a smallest transitive extension. This extension of a we call the transitive closure of a. In order to define "rank" we are interested in sets that are not only transitive but supertransitive in the sense of

Definition 9.8. $\mathrm{St}(A) \overset{\Delta}{\leftrightarrow} \mathrm{Tr}(A) \wedge (\forall\, x)[x \in A \rightarrow \mathscr{P}(x) \subseteq A]$.

Remark. We read St(A) as "A is supertransitive."

Definition 9.9.

$$R_1`0 \triangleq 0,$$
$$R_1`(\alpha+1) \triangleq \mathscr{P}(R_1`\alpha),$$
$$R_1`\alpha \triangleq \bigcup_{\beta<\alpha} R_1`\beta,\ \alpha \in K_{\mathrm{II}}.$$

Proposition 9.10.

(1) $\mathscr{M}(R_1`\alpha) \wedge \mathrm{St}(R_1`\alpha)$.

(2) $\alpha < \beta \to R_1`\alpha \in R_1`\beta \wedge R_1`\alpha \subset R_1`\beta$.

PROOF. (1) (By transfinite induction on α). If $\alpha = 0$ then $R_1`\alpha = 0$ and hence $R_1`\alpha$ is a supertransitive set. If $R_1`\alpha$ is a supertransitive set then, since $R_1`(\alpha + 1) = \mathscr{P}(R_1`\alpha)$, $R_1`(\alpha + 1)$ is a set and

$$\begin{aligned} a \in c \wedge b \subseteq c \wedge c \in R_1`(\alpha + 1) &\to a \in c \wedge b \subseteq c \wedge c \subseteq R_1`\alpha \\ &\to a \in R_1`\alpha \wedge b \subseteq R_1`\alpha \\ &\to a \subseteq R_1`\alpha \wedge b \subseteq R_1`\alpha \\ &\to a \in \mathscr{P}(R_1`\alpha) \wedge b \in \mathscr{P}(R_1`\alpha). \end{aligned}$$

Thus if $c \in R_1`(\alpha + 1)$, then $c \subseteq R_1`(\alpha + 1)$ and $\mathscr{P}(c) \subseteq R_1`(\alpha + 1)$.

If $\alpha \in K_{\mathrm{II}}$, then

$$\begin{aligned} c \in R_1``\alpha &\to (\exists\, \beta < \alpha)[c \in R_1`\beta] \\ &\to (\exists\, \beta < \alpha)[c \subseteq R_1`\beta \wedge \mathscr{P}(c) \subseteq R_1`\beta] \\ &\to c \subseteq R_1`\alpha \wedge \mathscr{P}(c) \subseteq R_1`\alpha. \end{aligned}$$

Since $R_1`\alpha$ is the union of a set it is a set.

(2) Since $R_1`\beta$ is transitive it is sufficient to prove that $R_1`\alpha \in R_1`\beta$. This we do by induction on $\beta$. Since $R_1`\beta \subseteq R_1`\beta$ we have $R_1`\beta \in \mathscr{P}(R_1`\beta) = R_1`(\beta + 1)$. In particular $R_1`\alpha \in R_1`(\alpha + 1)$.

Suppose, as our induction hypothesis, that $\alpha < \beta$ implies $R_1`\alpha \in R_1`\beta$. If $\alpha < \beta + 1$, then $\alpha < \beta$ or $\alpha = \beta$. If $\alpha = \beta$ we have

$$R_1`\alpha = R_1`\beta \in R_1`(\beta + 1).$$

If $\alpha < \beta$, then from the induction hypothesis, Definition 9.9, and the fact that $R_1`(\beta + 1)$ is transitive, we have

$$R_1`\alpha \in R_1`\beta \subseteq R_1`(\beta + 1)$$

and hence $R_1`\alpha \in R_1`(\beta + 1)$.

If $\beta \in K_{\mathrm{II}}$ then

$$R_1`\beta = \bigcup_{\gamma < \beta} R_1`\gamma.$$

Since $R_1`\alpha \in R_1`(\alpha + 1)$ it follows that

$$\alpha < \beta \to R_1`\alpha \in R_1`\beta. \qquad \square$$

Definition 9.11. $\mathrm{Wf}(a) \overset{\Delta}{\leftrightarrow} (\exists\, \alpha)[a \in R_1`\alpha]$.

Remark. $\mathrm{Wf}(a)$ is read "a is well founded." With the aid of the following theorem and the Axiom of Regularity we can prove that every set is well founded.

Proposition 9.12. $(\forall x \in a)[\mathrm{Wf}(x)] \rightarrow \mathrm{Wf}(a)$.

Proof. If each element of a is well founded and $x \in a$, then $(\exists \alpha)[x \in R'_1\alpha]$. If

$$F'x = \mu_\alpha(x \in R'_1\alpha)$$

then since F is a function $F''a$ is a set and indeed a subset of On. Therefore $\cup(F''a)$ is an ordinal. If $\beta = \cup(F''a) + 1$ then $F''a \subseteq \beta$, i.e., $x \in a \rightarrow F'x < \beta$. By Proposition 9.10

$$R'_1F'x \subseteq R'_1\beta.$$

Also $x \in R'_1F'x$ and so $x \in R'_1\beta$. Thus $a \subseteq R'\beta$ and hence $a \in \mathscr{P}(R'_1\beta) = R'_1(\beta + 1)$. ☐

Proposition 9.13. $\mathrm{Wf}(a)$.

Proof. From the Axiom of Regularity E is a well-founded relation on V. From Proposition 9.12 if

$$A = \{x \mid \mathrm{Wf}(x)\}$$

then $a \subseteq A$ implies $a \in A$. Then by $\in$-induction (Proposition 5.22) $A = V$. ☐

Remark. From Proposition 9.13 we see that in the presence of the Axiom of Regularity the function R_1 determines a class of sets $\{R'_1\alpha \mid \alpha \in \mathrm{On}\}$ whose union is the entire universe. Furthermore, from Proposition 9.10, these sets are nested, i.e., $\alpha < \beta \rightarrow R'_1\alpha \subset R'_1\beta$.

We offer the following pictorial representation of this nesting of sets. The universe is represented as the points in a V-shaped wedge.

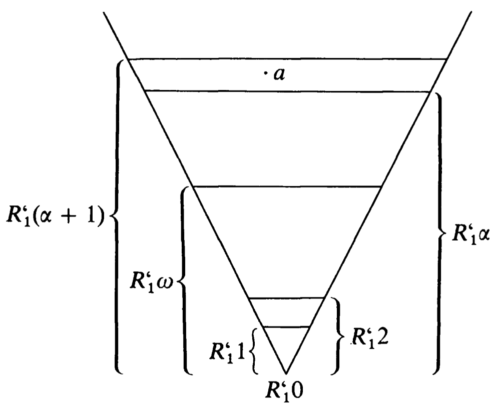

Figure 2

If $\alpha \in K_{\mathrm{II}}$ then $R'_1\alpha = \bigcup_{\beta<\alpha} R'_1\beta$. Thus any set in $R'_1\alpha$ is also in some $R'_1\beta$ with $\beta < \alpha$. Then for each a the smallest ordinal β for which $a \in R'_1\beta$ is a nonlimit ordinal, i.e.,

$$(\exists \alpha)[a \notin R'_1\alpha \wedge a \in R'_1(\alpha + 1)].$$

This particular ordinal α we call the rank of a.

Definition 9.14. $\text{rank}(a) \triangleq \mu_\alpha(a \in R_1`(\alpha + 1))$.

Remark. We read rank(a) as "rank a."

Proposition 9.15.

(1) $\text{rank}(a) \in \text{On}$.

(2) $\alpha = \text{rank}(a) \leftrightarrow a \notin R_1`\alpha \wedge a \in R_1`(\alpha + 1)$.

(3) $\beta \leqq \text{rank}(a) \leftrightarrow a \notin R_1`\beta$.

PROOF. (1) Definition 9.14.

(2) From Definition 9.14 if $\alpha = \text{rank}(a)$ then $a \in R_1`(\alpha + 1)$. If $\alpha = 0$ then since $R_1`0 = 0$ it follows that $a \notin R_1`\alpha$. If $(\exists\, \gamma)[\alpha = \gamma + 1]$ and $a \in R_1`\alpha$, then $\gamma \geqq \alpha$, by Definition 9.14. Consequently $a \notin R_1`\alpha$. If $\alpha \in K_{\text{II}}$ and $a \in R_1`\alpha$ then $(\exists\, \beta < \alpha)[a \in R_1`\beta]$. But $R_1`\beta \subseteq R_1`(\beta + 1)$, hence $a \in R_1`(\beta + 1)$. But then $\beta \geqq \alpha$ and so again we conclude that $a \notin R_1`\alpha$.

Conversely if $a \in R_1`(\alpha + 1)$ then $\alpha \geqq \text{rank}(a)$. If in addition $a \notin R_1`\alpha$ and if $\beta \leqq \alpha$ then $R_1`\beta \subseteq R_1`\alpha$ and hence $a \notin R_1`\beta$. But $a \in R_1`(\text{rank}(a) + 1)$. Therefore $\alpha < \text{rank}(a) + 1$, i.e., $\alpha \leqq \text{rank}(a)$. Thus $\alpha = \text{rank}(a)$.

(3) If $\alpha = \text{rank}(a)$ then by (2) $a \notin R_1`\alpha$. Furthermore if $\beta \leqq \alpha$, then $R_1`\beta \subseteq R_1`\alpha$, and so $a \notin R_1`\beta$. If $\alpha < \beta$, then $R_1`(\alpha + 1) \subseteq R_1`\beta$ and since $a \in R_1`(\alpha + 1)$, it follows that $a \in R`\beta$. □

Proposition 9.16. $a \in b \to \text{rank}(a) < \text{rank}(b)$.

PROOF. By Proposition 9.15, if $\alpha = \text{rank}(a)$, then $a \notin R`\alpha$. If $a \in b$ then since $a \notin R_1`\alpha$ it follows that $b \nsubseteq R_1`\alpha$ and hence $b \notin R_1`(\alpha + 1)$. Thus $\alpha < \text{rank}(b)$. □

Proposition 9.17. $\text{rank}(a) = \mu_\beta((\forall\, x \in a)[\text{rank}(x) < \beta])$.

PROOF. If $x \in a$, then, by Proposition 9.15,

$$\text{rank}(x) < \text{rank}(a).$$

If $x \in a$ and in addition, $\text{rank}(x) < \beta$, then

$$x \in R_1`(\text{rank}(x) + 1) \subseteq R_1`\beta.$$

Consequently $a \subseteq R_1`\beta$ and hence $a \in R_1`(\beta + 1)$. Therefore

$$\text{rank}(a) < \beta. \qquad \square$$

Proposition 9.18. $\text{rank}(\alpha) = \alpha$.

PROOF (By transfinite induction on α). If as our induction hypothesis we have

$$\gamma < \alpha \to \text{rank}(\gamma) = \gamma.$$

Then from Proposition 9.17

$$\operatorname{rank}(\alpha) = \mu_\beta((\forall\, \gamma < \alpha)[\gamma < \beta]) = \mu_\beta(\beta \geqq \alpha) = \alpha. \qquad \square$$

Proposition 9.19. $(\exists\, \alpha)(\forall\, x \in A)[\operatorname{rank}(x) \leqq \alpha] \to \mathscr{M}(A)$.

PROOF. If $\operatorname{rank}(x) \leqq \alpha$, then $x \in R_1`(\alpha + 1)$. Hence $A \subseteq R_1`(\alpha + 1)$. $\square$

Remark. Proposition 9.19 says that any class whose elements have bounded rank is a set.

EXERCISES

(1) $a \subseteq b \to \operatorname{rank}(a) \leqq \operatorname{rank}(b)$.

(2) $\mathscr{P}r(A) \to (\forall\, \alpha)(\exists\, x \in A)[\operatorname{rank}(x) > \alpha]$.

Remark. Earlier we promised to prove the equivalence of the weak and strong forms of the Axiom of Regularity. We redeemed that promise with Proposition 9.4. Indeed Proposition 9.4 is a more general result than the one promised. We now state and prove the specific form of strong regularity that we called Axiom 6'. The purpose of this proof is to illustrate the power and utility of the notion of rank. We point out that this proof is not independent of the first because we will use properties of rank that require Proposition 9.4.

Theorem 9.20 (Axiom 6'). $A \neq 0 \to (\exists\, x \in A)[x \cap A = 0]$.

PROOF. If $B = \{\operatorname{rank}(x) \mid x \in A\}$ and $A \neq 0$, then $B \neq 0$. Thus B is a nonempty class of ordinals, hence, by Proposition 6.26, which was proved using only the weak form of the Axiom of Regularity, B has an E-minimal element α. Since $\alpha \in B$ it follows that

$$(\exists\, x \in A)[\alpha = \operatorname{rank}(x)].$$

Furthermore since α is an E-minimal element of B it follows from Proposition 9.16 that

$$x \cap A = 0. \qquad \square$$

Remark. The simplicity of the proof of Theorem 9.20 illustrates the power of the rank function. Indeed with the aid of the rank function we can prove the following generalization of Proposition 9.4.

Proposition 9.21. $R \operatorname{Fr} A \wedge B \subseteq A \wedge B \neq 0 \to (\exists\, x \in B)[B \cap (R^{-1})``\{x\} = 0]$.

PROOF (By contradiction). Suppose that

$$B_0 = \{x \in B \mid (\forall\, y \in B)[\operatorname{rank}(x) \leqq \operatorname{rank}(y)]\},$$

and

$$B_{n+1} = \{x \in B \mid (\exists\, y \in B_n)[x \mathrel{R} y \wedge (\forall\, z \in B)[z \mathrel{R} y \to \operatorname{rank}(x) \leqq \operatorname{rank}(z)]]\},$$

Then all elements of B_0 have bounded rank, hence B_0 is a set. If B_n is a set then since

$$B_{n+1} = \bigcup_{y \in B_n} \{x \in B \mid x\ R\ y \wedge (\forall\, z \in B)[z\ R\ y \rightarrow \text{rank}(x) \leqq \text{rank}(z)]\}$$

B_{n+1} is a set. Thus $(\forall\, n \geqq 0)[\mathscr{M}(B_n)]$. If

$$b = \bigcup_{n<\omega} B_n$$

then $b \subseteq B \subseteq A$ and $b \neq 0$.

If $(\forall\, x \in B)[B \cap (R^{-1})``\{x\} \neq 0]$ then in particular

$$x \in b \rightarrow B \cap (R^{-1})``\{x\} \neq 0.$$

Furthermore $(\forall\, x \in b)(\exists\, n)[x \in B_n]$. Since $B \cap (R^{-1})``\{x\}$ is not empty it contains an element of minimal rank, i.e.,

$$(\exists\, y \in B)[y\ R\ x \wedge (\forall\, z \in B)[z\ R\ x \rightarrow \text{rank}(y) \leqq \text{rank}(z)]].$$

Since $x \in B_n$, $y \in B_{n+1}$ and hence $y \in b$. But $y\ R\ x$, that is

$$(\forall\, x \in b)[b \cap (R^{-1})``\{x\} \neq 0].$$

This contradicts the fact that R Fr A. □

Proposition 9.22. *If* R Fr $A \wedge B \subseteq A \wedge (\forall\, x \in A)[A \cap (R^{-1})``\{x\} \subseteq B \rightarrow x \in B]$ *then* $A = B$.

The proof is left to the reader.

CHAPTER 10

The Axiom of Choice and Cardinal Numbers

Cantor defined the cardinal number of a set M to be "the general concept which, by means of our active faculty of thought, arises from the set M when we make abstraction of the nature of its various elements m and of the order in which they are given." He denoted this cardinal number by $\bar{\bar{M}}$. The two bars indicate the two levels of abstraction needed to produce the cardinal number from M. With only one level of abstraction, that is, by only abstracting of the nature of its various elements, we obtain the ordinal number $\bar{M}$. Cantor's definition of cardinal number is clearly not an operational one. Indeed Cantor's words suggest that cardinal numbers are psychological entities rather than mathematical objects.

Frege, in 1884, and Russell, independently in 1903, removed cardinal numbers from the psychic realm by defining the cardinal number of a set a to be the class, $\bar{\bar{a}}$, of all sets that can be mapped one-to-one onto a. While this definition has a certain intuitive appeal it has the disadvantage that, at least relative to ZF theory, the objects produced are not sets but proper classes. This is the same problem that we faced with ordinal numbers and, as we did there, we will resolve the problem by defining the cardinal number $\bar{\bar{a}}$ to be a particular set that can be mapped one-to-one onto a and which will then serve as a representative of the class of all such sets. Let us review the situation for ordinal numbers.

The study of ordinal numbers is essentially the study of well-ordered sets. The appropriate mappings for such a study are the one-to-one-onto-order-preserving maps. Under such mappings well-ordered sets divide into equivalence classes, each of which contains exactly one ordinal number. More precisely, each equivalence class contains exactly one set that is transitive and well ordered by the $\in$-relation. From each equivalent class we took this special set as a representative, and we called these representatives ordinal numbers.

We propose to do a similar thing for cardinal numbers. For the study of cardinal numbers the basic mappings are simply the one-to-one-onto maps. Sets a and b are said to be equivalent (or equipollent) if there exists a function f that maps a one-to-one onto b.

Definition 10.1. $a \simeq b \overset{\Delta}{\leftrightarrow} (\exists f)[f\colon a \xrightarrow[\text{onto}]{1-1} b]$.

Remark. We read $a \simeq b$ as "a is equivalent to b."

Set equivalence, as formulated in Definition 10.1, is an equivalence relation:

Proposition 10.2.

(1) $a \simeq a$.

(2) $a \simeq b \rightarrow b \simeq a$.

(3) $a \simeq b \wedge b \simeq c \rightarrow a \simeq c$.

The proof is left to the reader.

Remark. From Proposition 10.2 we know that set equivalence partitions the universe of sets into equivalence classes. Let us call them cardinal equivalence classes. The cardinal equivalence classes are larger than ordinal equivalence classes in the sense that all of the elements of several difference ordinal equivalence classes can belong to the same cardinal equivalence class. This is because there exist well-ordered sets that are not order isomorphic but which are equivalent. For example $\omega + 1$ and ω are not order isomorphic but $\omega + 1$ can be mapped one-to-one onto ω by mapping ω to 0, 0 to 1, 1 to 2, etc. Thus $\omega + 1$ and ω belong to the same cardinal equivalence class. Then why not pick the smallest ordinal in each cardinal equivalence class as a representative of that class? That is exactly what we will do but there is one problem. How do we know that every cardinal equivalence class contains an ordinal? Any cardinal equivalence class that contains an ordinal is a collection of sets that can be well ordered. Indeed any function that maps an ordinal one-to-one onto a set induces a well ordering on that set. Perhaps there exist sets that cannot be well ordered and hence are not equivalent to any ordinal. We are going to assume that this is not the case. More precisely we will assume the Axiom of Choice from which we will prove that every set is equivalent to an ordinal and hence can be well ordered.

Axiom of Choice (weak form).

$$(\exists f)(\forall x \in a)[x \neq 0 \rightarrow f`x \in x].$$

Remark. The Axiom of Choice asserts that for each set a, there exists a function f, that picks an element of each nonempty set in a. This function f is called a choice function for the set a. There is a strong form of the Axiom of Choice that asserts the existence of a universal choice function, that is, a function that picks an element from every nonempty set in the universe of sets. In a language that permits quantification on class symbols this axiom could be stated thus:

Axiom of Choice (strong form).

$$(\exists F)(\forall x)[x \neq 0 \rightarrow F`x \in x].$$

Since in ZF we cannot quantify on class symbols, the strong form can only be expressed in ZF by adding a constant f_0 to the language together with the axiom

$$(\forall x)[x \neq 0 \rightarrow f_0`x \in x].$$

Why have we chosen not to do this? Because it has been proved that ZF plus strong choice is a conservative extension of ZF plus weak choice.[1]

Throughout this text when we refer to the Axiom of Choice (AC) we will mean the weak form. We will use the symbol ZFC to denote the theory obtained by adjoining the Axiom of Choice (weak form) to ZF. When at a later time we prove the relative consistency of ZF and ZFC it will be essential that the Axiom of Choice not be used in the proof. Consequently, in this chapter we will mark, with an asterisk, each theorem whose proof requires AC. The asterisk then warns us of results that cannot be used in the relative consistency proof toward which we are working.

Let us now explore some of the consequences of the Axiom of Choice.

***Theorem 10.3.** $(\exists \alpha)[a \simeq \alpha]$.

PROOF. By AC the power set of a, $\mathscr{P}(a)$, has a choice function f. Using this choice function f we define a class G such that

(1) $G`x = f`(a - \mathscr{W}(x))$.

But clearly $a - \mathscr{W}(x)$ is a subset of a and hence an element of $\mathscr{P}(a)$ and so

(2) $f`(a - \mathscr{W}(x)) \in a - \mathscr{W}(x) \quad \text{if } a - \mathscr{W}(x) \neq 0$.

By the Principle of Transfinite Recursion, Theorem 7.41, there exists a function F defined on On so that

(3) $(\forall \alpha)[F`\alpha = G`(F \restriction \alpha)]$.

From (1), (2), and (3) it follows that

$$F`\alpha \in a - \mathscr{W}(F \restriction \alpha) \quad \text{if } a - \mathscr{W}(F \restriction \alpha) \neq 0.$$

[1] Felgner, Ulrich. Comparison of the Axioms of Local and Universal Choice. *Fundamenta Mathematicae*, **71**, 43–62 (1971).

Since a is not a proper class it follows from Proposition 7.48 that

$$(\exists\, \alpha)[a - \mathscr{W}(F \upharpoonright \alpha) = 0].$$

Let α_0 be the smallest such ordinal and let $g = F \upharpoonright \alpha_0$. Then Proposition 7.49 assures us that $g\colon \alpha_0 \xrightarrow[\text{onto}]{1-1} a$. □

Remark. *Theorem 10.3 assures us that every set can be well ordered and that every cardinal equivalence class contains an ordinal. We can now pick the smallest ordinal in each cardinal equivalence class as the representative of that class.

Definition 10.4. $\bar{\bar{a}} \triangleq \mu_\alpha(a \simeq \alpha)$.

Remark. We read $\bar{\bar{a}}$ as "the cardinal number of a."

***Proposition 10.5.** $a \simeq \bar{\bar{a}}$.

PROOF. *Theorem 10.3 and Definition 10.4. □

Proposition 10.6.

(1) $\bar{\bar{a}} \in \mathrm{On}$.

(2) $(\forall\, \alpha)[\alpha < \bar{\bar{a}} \rightarrow (a \not\simeq \alpha)]$.

(3) $\bar{\bar{\alpha}} \leqq \alpha$.

PROOF. Definition 10.4. □

Definition 10.7. $N \triangleq \{\bar{\bar{x}} \mid x \in \mathrm{V}\}$.

Proposition 10.8. $N \subseteq \mathrm{On}$.

PROOF. Proposition 10.6(1) and Definition 10.7. □

Proposition 10.9. $\alpha \in N \leftrightarrow \alpha = \bar{\bar{\alpha}}$.

PROOF. If $\alpha = \bar{\bar{\alpha}}$, then $\alpha \in N$ by Definition 10.7. Conversely if $\alpha \in N$, then $(\exists\, x)[\alpha = \bar{\bar{x}}]$. Suppose that $\bar{\bar{\alpha}} < \alpha$, then since $\bar{\bar{\alpha}} \simeq \alpha$ and $\alpha \simeq x$ we would have $\bar{\bar{\alpha}} \simeq x$. But this contradicts the fact that α is the smallest ordinal equivalent to x. Consequently $\alpha \leqq \bar{\bar{\alpha}}$. From Proposition 10.6(8) it then follows that $\alpha = \bar{\bar{\alpha}}$. □

***Proposition 10.10.** $a \simeq b \leftrightarrow \bar{\bar{a}} = \bar{\bar{b}}$.

PROOF. If $a \simeq b$, then since $a \simeq \bar{\bar{a}}$ it follows that $\bar{\bar{a}} \simeq b$ and hence $\bar{\bar{b}} \leqq \bar{\bar{a}}$. Similarly, since $b \simeq \bar{\bar{b}}$ it follows that if $a \simeq b$, then $\bar{\bar{b}} \simeq a$ and hence $\bar{\bar{a}} \leqq \bar{\bar{b}}$. Therefore $\bar{\bar{a}} = \bar{\bar{b}}$.

Conversely if $\bar{\bar{a}} = \bar{\bar{b}}$, then since $a \simeq \bar{\bar{a}}$ it follows that $a \simeq \bar{\bar{b}}$. But $\bar{\bar{b}} \simeq b$ and so $a \simeq b$. □

***Proposition 10.11.** $\overline{\overline{(\bar{\bar{a}})}} = \bar{\bar{a}}$.

PROOF. By *Proposition 10.5, $\bar{\bar{a}} \simeq a$. Then by *Proposition 10.10, $\overline{\overline{(\bar{\bar{a}})}} = \bar{\bar{a}}$. □

Proposition 10.12. $a \subseteq \alpha \rightarrow (\exists\, \beta \leqq \alpha)[a \simeq \beta]$.

PROOF. If $a \subseteq \alpha$, then by Corollary 7.54, $(\exists\, \beta)(\exists f)[f \operatorname{Isom}_{E,E}(\beta, a)]$. This function f is then a strictly monotone ordinal function and so it follows that $(\forall\, \gamma \in \beta)[\gamma \leqq f`\gamma \leqq \alpha]$. Therefore $\beta \leqq \alpha$ and $\beta \simeq a$. □

***Proposition 10.13.** $a \subseteq b \rightarrow \bar{\bar{a}} \leqq \bar{\bar{b}}$.

PROOF. If $a \subseteq b$, then since $b \simeq \bar{\bar{b}}$ it follows that $(\exists\, x)[x \subseteq \bar{\bar{b}} \wedge a \simeq x]$. By Proposition 10.12 there exists an ordinal $\beta \leqq \bar{\bar{b}}$ such that $\beta \simeq x$. We then have

$$\bar{\bar{a}} = \bar{\bar{x}} = \bar{\bar{\beta}} \leqq \beta \leqq \bar{\bar{b}}.$$ □

Theorem 10.14 (Cantor–Schröder–Bernstein).

$$a \simeq c \subseteq b \wedge b \simeq d \subseteq a \rightarrow a \simeq b.$$

*PROOF. If $a \simeq c \subseteq b$, then $\bar{\bar{a}} = \bar{\bar{c}} \leqq \bar{\bar{b}}$. If $b \simeq d \subseteq a$, then $\bar{\bar{b}} = \bar{\bar{d}} \leqq \bar{\bar{a}}$. Since $\bar{\bar{a}} = \bar{\bar{b}}$ it follows, from *Proposition 10.10 that $a \simeq b$. □

Remark. The Cantor–Schröder–Bernstein theorem was first proved by Cantor. Like the proof above, Cantor's proof used results that presuppose AC. In 1896 and 1898 respectively, Ernst Schröder and Felix Bernstein, independently, gave proofs that do not require AC. Below we give such a proof, but note how hard we have to work when denied the use of AC.

PROOF (Cantor–Schröder–Bernstein). If $a \simeq c$, then $(\exists f)[f\colon a \xrightarrow[\text{onto}]{1-1} c]$. Similarly if $b \simeq d$, then $(\exists\, g)[g\colon b \xrightarrow[\text{onto}]{1-1} d]$. Let $H`x = (g \circ f)``x$. Then there exists a function h defined on ω such that

$$h`0 = a - d$$

$$h`(n + 1) = H`h`n = (g \circ f)``h`n.$$

Since $h`0 \subseteq a$ and since $g \circ f$ maps $a$ into $a$, it follows, by induction, that $(\forall\, n)[h`n \subseteq a]$. Consequently $(\forall\, n)[f``h`n \subseteq b]$.

We next define a function F on a in the following way

$$\begin{aligned} F`x &= f`x && \text{if } x \in a \wedge (\exists\, n)[x \in h`n] \\ &= (g^{-1})`x && \text{if } x \in a \wedge (\forall\, n)[x \notin h`n]. \end{aligned}$$

Then $F\colon a \rightarrow b$. We wish to prove that $F\colon a \xrightarrow[\text{onto}]{1-1} b$. To prove that F is onto we note that if $y \in b$, then $(\exists\, n)[y \in f``h`n]$ or $(\forall\, n)[y \notin f``h`n]$.

Suppose that $(\exists\, n)[y \in f``h`n]$. Then $(\exists\, x \in h`n)[y = f`x]$. But $x \in h`n$ implies that $x \in a$. Furthermore, from the definition of F, we see that if $x \in a$ and $x \in h`n$, then $F`x = f`x = y$.

On the other hand, suppose that $(\forall\, n)[y \notin f``h`n]$. As we will now show, it then follows that $(\forall\, n)[g`y \notin h`n]$. Assume that this is not the case. Then $(\exists\, n)[g`y \in h`n]$. Since $h`0 = a - d$ and since $g`y \in d$ it follows that $n \neq 0$. Therefore $(\exists\, m)[n = m + 1]$. But $h`(m + 1) = g``f``h`m$ and $g`y \in h`(m + 1)$. Since $g$ is one-to-one it then follows that $y \in f``h`m$; but this is a contradiction. And from this contradiction we conclude that $(\forall\, n)[g`y \notin h`n]$. On the other hand since $y \in b$, it follows that $g`y \in a$. Therefore $F`g`y = (g^{-1})`g`y = y$. From this we conclude that F is onto, that is, $\mathscr{W}(F) = b$.

To prove that F is one-to-one assume that $x \in a, y \in a$, and $F`x = F`y$. From this we will first prove that $(\exists\, m)[x \in h`m]$ iff $(\exists\, m)[y \in h`m]$. The proof is by contradiction: Suppose that $x \in h`m$ and $(\forall\, n)[y \notin h`n]$. Then $F`x = F`y$ implies that $f`x = (g^{-1})`y$ and hence $y = (g \circ f)`x$. Since $x \in h`m$ it then follows that $y \in (g \circ f)``h`m = h`(m + 1)$. This is a contradiction. Similarly we can prove that $y \in h`m$ and $(\forall\, n)[x \notin h`n]$ implies that $x \in h`(m + 1)$. This too is a contradiction. From these contradictions we conclude that $(\exists\, m)[x \in h`m]$ iff $(\exists\, m)[y \in h`m]$. From this and the fact that $F`x = F`y$ it follows that $f`x = f`y$ or $(g^{-1})`x = (g^{-1})`y$. Since both f and g are one-to-one it follows that $x = y$. Thus F is one-to-one. Furthermore since F is a function with domain a, it follows that F is a set. □

Proposition 10.15. $a \simeq b \to \mathscr{P}(a) \simeq \mathscr{P}(b)$.

Proof. If $a \simeq b$, then $(\exists f)[f\colon a \xrightarrow[\text{onto}]{1-1} b]$. Let $F = \{\langle x, f``x\rangle \mid x \in \mathscr{P}(a)\}$. Then $F\colon \mathscr{P}(a) \to \mathscr{P}(b)$. Furthermore, since $f``x = \{f`z \mid z \in x\}$, it follows that if $f``x = f``y$ and $z \in x$, then $f`z \in f``x = f``y$. That is $(\exists\, w \in y)[f`z = f`w]$. But f is one-to-one; therefore $z = w$ and hence $z \in y$. Similarly $z \in y$ implies $z \in x$. Therefore $x = y$ and hence F is one-to-one.

Finally, if $y \in \mathscr{P}(b)$ and if $x = (f^{-1})``y$, then $x \in \mathscr{P}(a)$ and

$$F`x = f``(f^{-1})``y = y.$$

Thus F is onto, that is, $\mathscr{W}(F) = \mathscr{P}(b)$. □

***Theorem 10.16** (Cantor). $\bar{\bar{a}} < \overline{\overline{\mathscr{P}(a)}}$.

Proof. Since $a \simeq b$ implies $\mathscr{P}(a) \simeq \mathscr{P}(b)$, it is sufficient to prove that $\bar{\bar{\alpha}} < \overline{\overline{\mathscr{P}(\alpha)}}$. Since ordinals are transitive, it follows that $\alpha \subseteq \mathscr{P}(\alpha)$ and hence, by *Proposition 10.14, $\bar{\bar{\alpha}} \leq \overline{\overline{\mathscr{P}(\alpha)}}$. If $\bar{\bar{\alpha}} = \overline{\overline{\mathscr{P}(\alpha)}}$, then $\alpha \simeq \mathscr{P}(\alpha)$ and hence $(\exists\, h)[h\colon \alpha \xrightarrow[\text{onto}]{1-1} \mathscr{P}(\alpha)]$. Let $c = \{\beta \in \alpha \mid \beta \notin h`\beta\}$. Then $c \subseteq \alpha$ and hence $c \in \mathscr{P}(\alpha)$. Consequently $(\exists\, \gamma \in \alpha)[c = h`\alpha]$. But then

$$\begin{aligned} \gamma \in c &\leftrightarrow \gamma \notin h`\gamma \\ &\leftrightarrow \gamma \notin c. \end{aligned}$$

This contradiction forces the conclusion that $\bar{\bar{\alpha}} < \overline{\overline{\mathscr{P}(\alpha)}}$. □

Remark. In the foregoing argument the proof that α is not equivalent to $\mathscr{P}(\alpha)$ can be easily modified to prove, of any set a, that a and $\mathscr{P}(a)$ are not equivalent. Furthermore this proof does not require AC.

From Cantor's theorem it is easy to prove that for any set of cardinal numbers there exists a cardinal larger than each cardinal in the given set.

***Proposition 10.17.** $a \subseteq N \rightarrow (\exists\, \beta \in N)(\forall\, \alpha \in a)[\alpha < \beta]$.

PROOF. If $\alpha \in a$, then $\alpha \subseteq \cup(a)$ and hence $\alpha = \overline{\overline{\alpha}} \leqq \overline{\overline{\cup(a)}}$. But by Cantor's theorem

$$\overline{\overline{\cup(a)}} < \overline{\overline{\mathscr{P}(\cup(a))}}. \qquad \square$$

Remark. Cantor's theorem led him to the paradox of the largest cardinal. We formulate that paradox in the following form: Consider N the "set" of all cardinal numbers. By *Proposition 10.17, there exists a cardinal larger than any cardinal in N. But this contradicts the fact that N contains all cardinals.

In ZF we can use this contradiction to conclude that N is not a set.

Theorem 10.18. $\mathscr{P}\imath(N)$.

*PROOF. If N were a set, then *Proposition 10.17 leads us to the contradiction that there exists an element of N that is not in N. $\square$

Remark. It has been claimed that Cantor discovered the paradox of the largest cardinal in 1895 and communicated it to Hilbert in 1896. But the oldest documented evidence we have dates back only to 1899. That year Cantor wrote to his friend Dedekind of his concern for collections that could not be considered as sets because to do so would lead to a contradiction. He called such collections "inconsistent multipicities." It is also interesting that Cantor did not include the troublesome theorem in his two part memoir published in 1895 and 1897.

EXERCISES

Prove the following.

(1) $a \simeq 0 \leftrightarrow a = 0$.

(2) $[a \cup \{b\} = a] \vee [a \cup \{b\} \simeq a \cup \{a\}]$.

(3) $\{b\} \times a \simeq a \times \{b\} \simeq a$.

(4) $a_1 \simeq a_2 \wedge b_1 \simeq b_2 \rightarrow a_1 \times b_1 \simeq a_2 \times b_2$.

(5) $a \simeq b \leftrightarrow a \cup \{a\} \simeq b \cup \{b\}$.

(6) $a \times b \simeq b \times a$.

(7) $\alpha \geqq \omega \rightarrow \alpha \simeq \alpha + 1$.

(8) $\alpha \geqq \omega \rightarrow \alpha \simeq \alpha + n$.

(9) $a \cap b = 0 \wedge a \simeq \alpha \wedge b \simeq \beta \rightarrow a \cup b \simeq \alpha + \beta$.

(10) $\alpha \times \beta \simeq \alpha\beta$.

Remark. Given a set a Cantor's theorem tells us that there exists a set of larger cardinality, namely $\mathscr{P}(a)$. There is another way to produce a set of larger cardinality. Indeed the basic idea was understood by Cantor and used by him to generate cardinal numbers. We consider all possible ways to well order a and its subsets. As we will now prove, the set of all well orderings of a and its subsets, has cardinality larger than a.

Theorem 10.19. *For each set a*

$$(\exists\, \beta \in N)[\beta = \{\alpha \mid (\exists f)[f\colon \alpha \xrightarrow{1-1} a]\} \wedge \neg(\exists f)[f\colon \beta \xrightarrow{1-1} a]].$$

Proof. We consider all ordered pairs $\langle r, x\rangle$ where $x \subseteq a$ and $r \subseteq a \times a$. If r well orders x, then

$$(\exists!\, \beta)(\exists!\, f_{r,x})[f_{r,x} \operatorname{Isom}_{r,E}(x, \beta)].$$

Using $f_{r,x}$ we define a function on $\mathscr{P}(a \times a) \times \mathscr{P}(a)$ in the following way

$$\begin{aligned} F`\langle r, x\rangle &= f_{r,x}``x && \text{If } \; r \text{ We } x \\ &= 0 && \text{otherwise.} \end{aligned}$$

Since F is a function whose domain is a set its range is also a set. Let $b = \mathscr{W}(F)$. Then

$$\begin{aligned} y \in b &\leftrightarrow (\exists\, r \subseteqq a \times a)(\exists\, x \subseteqq a)[y = F`\langle r, x\rangle] \\ &\leftrightarrow y = 0 \vee (\exists\, r \subseteqq a \times a)(\exists\, x \subseteqq a)[r \text{ We } x \wedge y = f_{r,x}``x] \\ &\leftrightarrow y \in \text{On} \wedge (\exists f)[f\colon y \xrightarrow{1-1} a]. \end{aligned}$$

Thus $b = \{\alpha \mid (\exists f)[f\colon \alpha \xrightarrow{1-1} a]\}$.

If $\gamma < \alpha$ and $\alpha \in b$, then $(\exists\, f)[f : \alpha \xrightarrow{1-1} a]$. Furthermore $f \restriction \gamma\colon \gamma \xrightarrow{1-1} a$, and so $\gamma \in b$. Thus b is a transitive set of ordinals. Therefore b is an ordinal, that is

$$(\exists\, \beta)[\beta = \{\alpha \mid (\exists f)[f\colon \alpha \xrightarrow{1-1} a]\}].$$

Furthermore if $\gamma \simeq \beta$ and $\gamma < \beta$, then $(\exists\, g)[g\colon \beta \xrightarrow{1-1} \gamma]$ and $(\exists f)[f\colon \gamma \xrightarrow{1-1} a]$. Consequently $f \circ g\colon \beta \xrightarrow{1-1} a$. But this implies that $\beta \in \beta$. From this contradiction we conclude that if $\gamma \simeq \beta$, then $\beta \leqq \gamma$ and so $\overline{\overline{\beta}} = \beta$, that is, $\beta \in N$.

Finally we conclude that $\neg(\exists f)[f\colon \beta \xrightarrow{1-1} a]$ since otherwise we would have $\beta \in \beta$. □

Remark. From Theorem 10.19 we can provide a second proof that N is a proper class, but we leave that proof as an exercise for the reader.

Proposition 10.20. $m \simeq n \to m = n$.

PROOF (By induction on n). If $m \simeq 0$, then $m = 0$. As our induction hypothesis assume that $(\forall m)[m \simeq n \to m = n]$ and assume that $m \simeq (n + 1)$. It then follows that $m \neq 0$ and hence, for some integer p, we have $m = p + 1$. But $p + 1 \simeq n + 1$ implies that $p \simeq n$. (Exercise 5 above.) Then, from the induction hypothesis it follows that $p = n$; hence $m = p + 1 = n + 1$. □

Corollary 10.21.

(1) $\neg(n \simeq n + 1)$.

(2) $\neg(\exists f)[f: (n + 1) \xrightarrow{1-1} n]$.

The proof is left to the reader.

Proposition 10.22. $\alpha \simeq n \to \alpha = n$.

PROOF. If $\alpha \geq \omega$ then $n < \alpha$ and hence $n + 1 \leq \alpha$. If $\alpha \simeq n$ then, since $n \subset n + 1$, it follows from the Cantor–Schröder–Bernstein theorem that $\alpha \simeq n + 1$. But then $n \simeq n + 1$. This contradicts Corollary 10.21(1) and compels us to conclude that if $\alpha \simeq n$ then $\alpha < \omega$. From Proposition 10.20 it then follows that $\alpha = n$. □

Corollary 10.23. $\bar{\bar{n}} = n$.

PROOF. Proposition 10.24 and Definition 10.4. □

Corollary 10.24. $\omega \subseteq N$.

PROOF. Corollary 10.23 and Definition 10.7. □

Remark. The elements of ω are the finite cardinals. We next introduce special notation for the class of infinite cardinals and its members.

Definition 10.25. $N' \triangleq N - \omega$.

Proposition 10.26. $\Pr(N')$.

PROOF. Since $N = N' \cup \omega$ it follows that if N' is a set, so also is N. □

Remark. Since N' is a proper class of ordinals it is order isomorphic to On. We now give this order isomorphism a special symbol.

Definition 10.27. $\aleph\ \mathrm{Isom}_{E,E}(\mathrm{On}, N')$.

Definition 10.28. $\aleph_\alpha \triangleq \aleph`\alpha$.

EXERCISES

Prove the following.

(1) $\omega \in N'$.

(2) $\alpha \leqq \aleph_\alpha$.

(3) $\aleph_0 = \omega$.

(4) $N' \subseteqq K_{\text{II}}$.

Definition 10.29.

(1) $\text{Fin}(a) \overset{\Delta}{\leftrightarrow} (\exists\, n)[a \simeq n]$.

(2) $\text{Inf}(a) \overset{\Delta}{\leftrightarrow} \neg\ \text{Fin}(a)$.

Remark. We read Fin(a) as "a is finite" and we read Inf(a) as "a is infinite."

Proposition 10.30. $\text{Fin}(a) \wedge b \subseteqq a \to \text{Fin}(b)$.

PROOF. If a is finite then by Definition 10.29, $(\exists\, n)[a \simeq n]$. If $b \subseteqq a$ it then follows from Proposition 10.12 that $(\exists\, \beta \leq n)[b \simeq \beta]$. Since such a β must be in ω it follows that b is finite. □

EXERCISES

Prove the following.

(1) Fin(n).

(2) $\text{Fin}(a) \to \text{Fin}(a \cup \{b\})$.

(3) $\text{Fin}(a) \to \text{Fin}(a - \{b\})$.

(4) $\text{Fin}(a) \to \text{Fin}(\{b\} \times a)$.

(5) $\text{Inf}(a) \to \bar{\bar{a}} = \overline{\overline{a \cup \{b\}}}$.

(6) $\text{Inf}(a) \wedge a \simeq b \to \text{Inf}(b)$.

(7) $\text{Inf}(a) \to (\exists\, x)[x \subset a \wedge x \simeq a]$ (*Hint*: Use AC).

(8) $\text{Inf}(a) \to (\exists\, x)[x \subseteqq a \wedge x \simeq \omega]$ (*Hint*: Use AC).

(9) $\bar{\bar{a}} = n + 1 \wedge b \in a \to \overline{\overline{a - \{b\}}} = n$.

Proposition 10.31. $\text{Fin}(a) \wedge \text{Fin}(b) \to \text{Fin}(a \cup b) \wedge \text{Fin}(a \times b)$.

PROOF (By induction on $\bar{\bar{a}}$). If $\bar{\bar{a}} = 0$, then $a = 0$ and hence $a \cup b = b$ and $a \times b = 0$. But b is finite by hypothesis and 0 is finite by Exercise 1 above. Assume, as our induction hypothesis that $(\forall\, a)[\bar{\bar{a}} = n \wedge \text{Fin}(b) \to \text{Fin}(a \cup b) \wedge \text{Fin}(a \times b)]$. Suppose that $\bar{\bar{a}} = n + 1$. Then $a \neq 0$ and so $(\exists\, x)[x \in a]$. Then $\overline{\overline{a - \{x\}}} = n$, by Exercise 9 above and hence, by the induction hypothesis

$(a - \{x\}) \cup b$ is finite and $(a - \{x\}) \times b$ is finite. But $a \cup b = [(a - \{x\}) \cup b] \cup \{x\}$ which is finite by Exercise 2 above, and $a \times b = [(a - \{x\}) \times b] \cup [\{x\} \times b]$, which is finite because it is the union of two finite sets. □

Proposition 10.32. $\mathrm{Fin}(\alpha) \leftrightarrow \alpha \in \omega$.

PROOF. If α is finite, then $(\exists\, n)[n \simeq \alpha]$. But by Proposition 10.22, this implies that $\alpha = n$ and so $\alpha \in \omega$.

Conversely if $\alpha \in \omega$, then since $\alpha \simeq \alpha$ it follows that α is finite. □

***Proposition 10.33.**

(1) $b \neq 0 \rightarrow \bar{\bar{a}} \leqq \overline{\overline{a \times b}}$.

(2) $\bar{\bar{b}} \leqq \bar{\bar{c}} \rightarrow \overline{\overline{a \times b}} \leqq \overline{\overline{a \times c}}$.

PROOF. (1) If $b \neq 0$, then $(\exists\, y)[y \in b]$. Then the function f defined by

$$f`x = \langle x, y\rangle, \qquad x \in a$$

maps a one-to-one into $a \times b$. Consequently $a \simeq f``a \subseteq a \times b$. Then by *Proposition 10.13,

$$\bar{\bar{a}} \leqq \overline{\overline{a \times b}}.$$

(2) If $\bar{\bar{b}} \leqq \bar{\bar{c}}$, then $(\exists f)[f\colon b \xrightarrow{1-1} c]$. Let g be defined by

$$g`\langle x, y\rangle = \langle x, f`y\rangle, \qquad \langle x, y\rangle \in a \times b.$$

Then $g\colon a \times b \xrightarrow{1-1} a \times c$. Consequently $a \times b \simeq g``(a \times b) \subseteq a \times c$ and hence $\overline{\overline{a \times b}} \leqq \overline{\overline{a \times c}}$. □

***Proposition 10.34.** $\mathscr{U}n(A) \rightarrow \overline{\overline{A``a}} \leqq \bar{\bar{a}}$.

PROOF. Since $a \simeq \bar{\bar{a}}$ it follows that $(\exists\, h)[h\colon \bar{\bar{a}} \xrightarrow[\text{onto}]{1-1} a]$. Let F be defined by

$$F`\alpha = A`h`\alpha, \qquad \alpha \in \bar{\bar{a}}.$$

Then $F\colon \bar{\bar{a}} \xrightarrow[\text{onto}]{} A``a$. Furthermore if

$$B = \{\beta \in \bar{\bar{a}} \mid (\forall\, \alpha < \beta)[A`h`\alpha \neq A`h`\beta]\},$$

then $B \subseteq \bar{\bar{a}}$. Therefore B is a set and $\bar{\bar{B}} \leqq \bar{\bar{a}}$. Since $B \subseteq \bar{\bar{a}}$ it follows that $F \upharpoonright B\colon B \rightarrow A``a$. We wish to prove that this mapping is one-to-one and onto. For this purpose we note that if $\beta \in B$, $\gamma \in B$, and $F`\beta = F`\gamma$, then $A`h`\beta = A`h`\gamma$ and hence, from the definition of B, $\beta = \gamma$. Thus $F \upharpoonright B$ is one-to-one. Furthermore if $x \in A``a$, then $(\exists\, \beta \in \bar{\bar{a}})[x = A`h`\beta]$. There is then a smallest such β and this smallest β is in B. That is, $(\exists\, \beta \in B)[x = F`\beta]$. Therefore $F \upharpoonright B$ is onto.

We then have

$$\overline{\overline{A``a}} = \bar{\bar{B}} \leqq \bar{\bar{a}}.$$ □

***Proposition 10.35.** $a \neq 0 \wedge (\exists f)[f\colon a \xrightarrow[\text{onto}]{} b] \leftrightarrow 0 < \bar{\bar{b}} \leqq \bar{\bar{a}}$.

PROOF. If $f: a \xrightarrow[\text{onto}]{} b$, then from *Proposition 10.34

$$\overline{\overline{b}} = \overline{\overline{f``a}} \leqq \overline{\overline{a}}.$$

Furthermore, if $a \neq 0$, then $b \neq 0$ and hence $\overline{\overline{b}} > 0$.

Conversely if $0 < \overline{\overline{b}} \leqq \overline{\overline{a}}$, then $a \neq 0$ and $b \neq 0$. Moreover we have $b \simeq \overline{\overline{b}} \leqq \overline{\overline{a}} \simeq a$, and so $(\exists\, g)[g: b \xrightarrow{1-1} a]$. Since $b \neq 0$, $(\exists\, y)[y \in b]$. Let f be defined by

$$\begin{aligned} f`x &= (g^{-1})`x, && x \in g``b \\ &= y, && x \in a - g``b. \end{aligned}$$

Then $f: a \xrightarrow[\text{onto}]{} b$. □

***Proposition 10.36.** $\overline{\overline{a}} > 1 \wedge \overline{\overline{b}} > 1 \rightarrow \overline{\overline{a \cup b}} \leqq \overline{\overline{a \times b}}$.

PROOF. If $\overline{\overline{a}} > 1$ and $\overline{\overline{b}} > 1$, then $(\exists\, x_1, x_2 \in a)[x_1 \neq x_2]$ and in addition $(\exists\, y_1, y_2 \in b)[y_1 \neq y_2]$. We then define a function F on $a \times b$:

$$\begin{aligned} F`\langle x_2, y_2\rangle &= x_2, \\ F`\langle x, y_1\rangle &= x, \quad x \neq x_2 \\ F`\langle x, y\rangle &= y, \quad \text{otherwise.} \end{aligned}$$

Then $F: a \times b \xrightarrow[\text{onto}]{} a \cup b$ and by *Proposition 10.34

$$\overline{\overline{a \cup b}} = \overline{\overline{F``(a \times b)}} \leqq \overline{\overline{a \times b}}.$$ □

Proposition 10.37. $\overline{\overline{\alpha}} < \overline{\overline{\beta}} \leftrightarrow \alpha < \overline{\overline{\beta}}$.

*PROOF. Since $\overline{\overline{\alpha}} \leqq \alpha$ it follows that if $\alpha < \overline{\overline{\beta}}$, then $\overline{\overline{\alpha}} < \overline{\overline{\beta}}$. Furthermore if $\overline{\overline{\beta}} \leqq \alpha$, then by *Proposition 10.13 and *Proposition 10.11,

$$\overline{\overline{\beta}} = \overline{\overline{(\overline{\overline{\beta}})}} \leqq \overline{\overline{\alpha}}.$$ □

Remark. To obtain a proof that does not require AC we need only observe that thus far we have only used AC to prove that all sets are well ordered. Not since the proof of *Theorem 10.3 have we made a direct application of AC. This means that all of our starred theorems can be proved without AC if we restrict the statement of the theorem to sets that are well ordered such as the ordinal numbers. This observation applies to the theorems that follow.

Proposition 10.38. $\alpha > 1 \rightarrow \overline{\overline{\alpha + 1}} \leqq \overline{\overline{\alpha \times \alpha}}$.

*PROOF. Let g be a function defined on $\alpha + 1$ by

$$\begin{aligned} g`\beta &= \langle 0, \beta\rangle, \qquad \beta < \alpha \\ g`\alpha &= \langle 1, 0\rangle. \end{aligned}$$

Then $g\colon (\alpha + 1) \xrightarrow{1-1} \alpha \times \alpha$ and hence $(\alpha + 1) \simeq g``(\alpha + 1) \subseteq \alpha \times \alpha$. Then by *Proposition 10.13

$$\overline{\overline{\alpha + 1}} \leqq \overline{\overline{\alpha \times \alpha}}. \qquad \square$$

Proposition 10.39. $\alpha \geqq \omega \rightarrow \overline{\overline{\alpha \times \alpha}} = \overline{\overline{\alpha}}$.

*Proof (By transfinite induction on α). As our induction hypothesis we have

$$\mu < \alpha \rightarrow [\mu < \omega \vee \overline{\overline{\mu \times \mu}} = \overline{\overline{\mu}}].$$

By *Proposition 10.13 if $\mu < \alpha$, then $\overline{\overline{\mu}} \leqq \overline{\overline{\alpha}}$. If $(\exists\, \mu < \alpha)[\overline{\overline{\mu}} = \overline{\overline{\alpha}}]$ then $\mu \simeq \alpha$ and

$$\overline{\overline{\alpha \times \alpha}} = \overline{\overline{\mu \times \mu}} = \overline{\overline{\mu}} = \overline{\overline{\alpha}}.$$

However if $(\forall\, \mu < \alpha)[\overline{\overline{\mu}} < \overline{\overline{\alpha}}]$, then since $\mu < \omega \vee \overline{\overline{\mu + 1}} = \overline{\overline{\mu}}$ it follows from Proposition 10.37 that $\mu + 1 < \overline{\overline{\alpha}} \leqq \alpha$. Therefore by our induction hypothesis

$$\alpha > \mu \geqq \omega \rightarrow \overline{\overline{(\mu + 1) \times (\mu + 1)}} = \overline{\overline{\mu + 1}}.$$

Recall the relation R_0 of Definition 7.57. By Theorem 7.58, R_0 well orders On^2. Consequently there is an order isomorphism J_0 such that

$$J_0 \operatorname{Isom}_{R_0, E}(\mathrm{On}^2, \mathrm{On}).$$

We wish to show that $J_0``(\alpha \times \alpha) \subseteq \overline{\overline{\alpha}}$. First we recall that an order isomorphism maps initial segments onto initial segments:

$$J_0``(R_0^{-1})``\{\langle \beta, \gamma \rangle\} = (E^{-1})``\{J_0`\langle \beta, \gamma \rangle\} = J_0`\langle \beta, \gamma \rangle.$$

then since J_0 is one-to-one

$$\overline{\overline{J_0`\langle \beta, \gamma \rangle}} = \overline{\overline{J_0``(R_0^{-1})``\{\langle \beta, \gamma \rangle\}}} = \overline{\overline{(R_0^{-1})``\{\langle \beta, \gamma \rangle\}}}.$$

But if $\langle \beta, \gamma \rangle \in \alpha \times \alpha$ and if $\mu = \max(\beta, \gamma)$, then $\mu < \alpha$ and

$$\begin{aligned} \langle \eta, \theta \rangle \in (R_0^{-1})``\{\langle \beta, \gamma \rangle\} &\rightarrow \langle \eta, \theta \rangle\, R_0\, \langle \beta, \gamma \rangle \\ &\rightarrow \max(\eta, \theta) \leqq \mu \\ &\rightarrow \eta \leqq \mu \wedge \theta \leqq \mu \\ &\rightarrow \langle \eta, \theta \rangle \in (\mu + 1) \times (\mu + 1). \end{aligned}$$

Thus $(R_0^{-1})``\{\langle \beta, \gamma \rangle\} \subseteq (\mu + 1) \times (\mu + 1)$.

Since $\mu < \alpha$ it follows from the induction hypothesis that if $\mu \geqq \omega$ then

$$\overline{\overline{(R_0^{-1})``\{\langle \beta, \gamma \rangle\}}} \leqq \overline{\overline{(\mu + 1) \times (\mu + 1)}} = \overline{\overline{\mu + 1}} < \overline{\overline{\alpha}}.$$

Therefore $\overline{\overline{J_0`\langle \beta, \gamma \rangle}} < \overline{\overline{\alpha}}$ and hence $J_0`\langle \beta, \gamma \rangle < \overline{\overline{\alpha}}$.

If $\mu < \omega$, then $(\mu + 1) \times (\mu + 1)$ is finite. Hence $(R_0^{-1})``\{\langle \beta, \gamma \rangle\}$ is finite and $J_0`\langle \beta, \gamma \rangle$ is finite. Since $\alpha \geqq \omega$, $J_0`\langle \beta, \gamma \rangle < \overline{\overline{\alpha}}$.

Therefore

$$J_0``(\alpha \times \alpha) \subseteq \overline{\overline{\alpha}}.$$

Since J_0 is one-to-one

$$\overline{\overline{\alpha \times \alpha}} = \overline{\overline{J_0``(\alpha \times \alpha)}} \leqq \overline{\overline{\alpha}}.$$

But from Proposition 10.38

$$\overline{\overline{\alpha}} = \overline{\overline{\alpha + 1}} \leqq \overline{\overline{\alpha \times \alpha}}.$$

Therefore $\overline{\overline{\alpha \times \alpha}} = \overline{\overline{\alpha}}$. ☐

***Proposition 10.40.** $\overline{\overline{a}} \geqq \omega \rightarrow \overline{\overline{a \times a}} = \overline{\overline{a}}$.

Proof. Since $a \simeq \overline{\overline{a}}$ we have $a \times a \simeq \overline{\overline{a}} \times \overline{\overline{a}}$. Then from Proposition 10.39

$$\overline{\overline{a \times a}} = \overline{\overline{\overline{\overline{a}} \times \overline{\overline{a}}}} = \overline{\overline{a}}.$$ ☐

***Proposition 10.41.** $\overline{\overline{a}} \geqq \omega \wedge \overline{\overline{b}} > 0 \rightarrow \overline{\overline{a \times b}} = \overline{\overline{a \cup b}} = \max(\overline{\overline{a}}, \overline{\overline{b}})$.

Proof. If $\alpha = \max(\overline{\overline{a}}, \overline{\overline{b}})$, then since $a \simeq \overline{\overline{a}}$ and $b \simeq \overline{\overline{b}}$

$$a \times b \simeq \overline{\overline{a}} \times \overline{\overline{b}} \subseteq \alpha \times \alpha.$$

Then

$$\overline{\overline{a \times b}} \leqq \overline{\overline{\alpha \times \alpha}} = \alpha.$$

If $\overline{\overline{b}} = 1$, then since $\overline{\overline{a}} \geqq \omega$ we have $a \times b \simeq a \cup b$ and so $\overline{\overline{a \times b}} = \overline{\overline{a \cup b}} = \overline{\overline{a}} = \max(\overline{\overline{a}}, \overline{\overline{b}})$.

If $\overline{\overline{b}} > 1$, then by *Proposition 10.36, $\overline{\overline{a \cup b}} \leqq \overline{\overline{a \times b}} \leqq \max(\overline{\overline{a}}, \overline{\overline{b}})$. Furthermore since $a \subseteq a \cup b$ and $b \subseteq a \cup b$ we have $\overline{\overline{a}} \leqq \overline{\overline{a \cup b}}$ and $\overline{\overline{b}} < \overline{\overline{a \cup b}}$ and so

$$\max(\overline{\overline{a}}, \overline{\overline{b}}) \leqq \overline{\overline{a \cup b}} = \overline{\overline{a \times b}} \leqq \max(\overline{\overline{a}}, \overline{\overline{b}}).$$ ☐

Remark. The natural notion of cardinal addition is that the sum $\overline{\overline{a}} + \overline{\overline{b}}$ should be $\overline{\overline{a \cup b}}$, provided a and b are disjoint. Similarly, the product $\overline{\overline{a}} \cdot \overline{\overline{b}}$ should be $\overline{\overline{a \times b}}$. *Proposition 10.41 tells us that for infinite cardinals, addition and multiplication are quite uninteresting. In the infinite case the problem of finding the sum or the product of $\overline{\overline{a}}$ and $\overline{\overline{b}}$ is simply the problem of finding the larger of $\overline{\overline{a}}$ and $\overline{\overline{b}}$.

Something interesting does come up with cardinal exponentation. For its definition we turn to an idea introduced by Cantor.

Definition 10.42. $a^b \triangleq \{f \mid f: b \rightarrow a\}$.

Remark. We will read a^b as "the set of mapping from b into a." That reading however presupposes the following result.

Proposition 10.43. $\mathscr{M}(a^b)$.

Proof. $a^b \subseteq \mathscr{P}(b \times a)$. ☐

Remark. Earlier we used a^n to mean the n-fold cross product of a and in our discussion of ordinal arithmetic we used α^β to mean the βth power of α. Now we propose a third use, namely that a^b shall denote the set of mapping from b into a. We will rely upon the context to make clear what we intend when we use this notation. That will be much simplier than creating new notation.

It is a temptation to add a fourth usage and define cardinal powers by

$$\bar{\bar{a}}^{\bar{\bar{b}}} = \overline{\overline{a^b}}.$$

But we will suppress that urge and not add an addition ambiguity to an already overworked notation. We will however talk about cardinal powers even though we will not introduce a notation for them. Thus when we write $\aleph_\alpha^{\aleph_\beta}$ we will always mean the set of functions from $\aleph_\beta$ into $\aleph_\alpha$. For the associated cardinal number we will write

$$\overline{\overline{\aleph_\alpha^{\aleph_\beta}}}.$$

Proposition 10.44. $2^a \simeq \mathscr{P}(a)$.

Proof. We define a function h on 2^a in the following way

$$h\text{`}f = \{x \in a \mid f\text{`}x = 1\}, \qquad f \in 2^a.$$

Then $h: 2^a \to \mathscr{P}(a)$. We wish to prove that h is one-to-one and onto.

If $b \in \mathscr{P}(a)$ and if f is defined on a by

$$\begin{aligned} f\text{`}x &= 1, \qquad x \in b \\ &= 0, \qquad x \in a - b. \end{aligned}$$

Then $f \in 2^a$ and $h\text{`}f = b$. Thus h is onto.

Suppose that $f \in 2^a$, $g \in 2^a$, and $h\text{`}f = h\text{`}g$, then

$$\{x \in a \mid f\text{`}x = 1\} = \{x \in a \mid g\text{`}x = 1\}.$$

Since f and g each take the value 1 at the same points in a they must also take the value 0 at the same points in a and so $f = g$. Therefore h is one-to-one and hence $2^a \simeq \mathscr{P}(a)$. □

Proposition 10.45. $(a^b)^c \simeq a^{b \times c}$.

Proof. We define a function F on $(a^b)^c$ in the following way. If $f \in (a^b)^c$ and $y \in c$, then $f\text{`}y \in a^b$, that if, $f\text{`}y: b \to a$. Thus if $x \in b$, then $(f\text{`}y)\text{`}x \in a$. We then specify that $F\text{`}f$ is a function defined on $b \times c$ by

$$(F\text{`}f)\text{`}\langle x, y\rangle = (f\text{`}y)\text{`}x.$$

Then $F\text{`}f: b \times c \to a$ and hence $F\text{`}f \in a^{b \times c}$. Thus $F: (a^b)^c \to a^{b \times c}$. We will prove that F is one-to-one and onto.

If $g \in a^{b \times c}$ and if $\forall\, y \in c$

$$f\text{`}_y x = g\text{`}\langle x, y\rangle, \qquad x \in b$$

then $\tilde{f}_y \in a^b$. Therefore if

$$f`y = \tilde{f}_y, \qquad y \in c,$$

then $f \in (a^b)^c$ and $g`\langle x, y\rangle = (f`y)`x$. Consequently $F`f = g$ and F is onto.

If $f_1 \in (a^b)^c, f_2 \in (a^b)^c$, and $F`f_1 = F`f_2$, then

$$(\forall\, y \in c)(\forall\, x \in b)[(f_1`y)`x = (f_2`y)`x]$$
$$(\forall\, y \in c)[f_1`y = f_2`y]$$
$$f_1 = f_2.$$

Therefore F is one-to-one.

Since F maps $(a^b)^c$ one-to-one onto $a^{b \times c}$ it follows that $(a^b)^c \simeq a^{b \times c}$. $\square$

Remark. Proposition 10.45 tells us that powers of cardinals obey a law of finite cardinals that we have known since we first studied powers of integers:

$$\overline{\overline{(\aleph_\alpha^{\aleph_\beta})^{\aleph_\gamma}}} = \overline{\overline{\aleph_\alpha^{\aleph_\beta \times \aleph_\gamma}}}.$$

From our studies of integers we know that, at least in principle, we can compute any finite power of any finite cardinal. But alas we cannot compute even such a simple infinite power as

$$(1) \qquad \overline{\overline{2^{\aleph_0}}}.$$

Indeed Cohen has shown that the question of what (1) is, is undecidable in ZF.

In the next section we will introduce the Generalized Continuum Hypothesis from which we will show how to compute powers of cardinals. But before we do that we wish to prove two more results on the cardinality of unions. For their proof we need the following result.

***Theorem 10.46.** $(\forall\, x \in a)(\exists\, y)\varphi(x, y) \rightarrow (\exists f)(\forall\, x \in a)\varphi(x, f`x)).$

Proof. Let G be defined on a by

$$G`x = \mu_\alpha((\exists\, y)[\alpha = \operatorname{rank}(y) \wedge \varphi(x, y)]).$$

Then $G``a$ is a set of ordinals. Let $\alpha = \cup(G``a) + 1$. It follows from this that if

$$(\forall\, x \in a)(\exists\, y)\varphi(x, y)$$

then

$$(\exists\, y)[\varphi(x, y) \wedge \operatorname{rank}(y) = G`x].$$

Furthermore if $\operatorname{rank}(y) = G`x$ then $\operatorname{rank}(y) < \alpha$ and so $y \in R`_1\alpha$.

To complete the proof we use AC to well order $R`_1\alpha$ and we define $f`x$ to be the first element in $R`_1\alpha$ for which $\varphi(x, f`x)$. Here are the details.

By *Theorem 10.3

$$(\exists \beta)(\exists h)[h: \beta \xrightarrow[\text{onto}]{1-1} R_1`\alpha].$$

Since $(\forall x \in a)(\exists y \in R_1`\alpha)\varphi(x, y)$ and since h is onto,

$$(\forall x \in a)(\exists \gamma \in \beta)\varphi(x, h`\gamma).$$

There is then a smallest such γ. Let f be a function defined on a by

$$f`x = h`\mu_\gamma(\varphi(x, h`\gamma)), x \in a$$

then

$$(\forall x \in a)\varphi(x, f`x). \qquad \square$$

Remark. Theorem 10.46 is a generalization of AC to classes. It asserts that given any collection of nonempty classes

$$A_x, \quad x \in a,$$

there exists a choice function f such that $f`x \in A_x$ for each x in a.

***Theorem 10.47.** $(\forall x \in a)[\bar{\bar{x}} \leqq \bar{\bar{b}}] \rightarrow \overline{\overline{\cup(a)}} \leqq \overline{\overline{a \times b}}$.

Proof. If $(\forall x \in a)[\bar{\bar{x}} \leqq \bar{\bar{b}}]$, then

$$(\forall x \in a)(\exists \tilde{f}_x)[\tilde{f}_x: x \xrightarrow{1-1} b].$$

By *Theorem 10.46

(1) $(\exists f)(\forall x \in a)[f`x: x \xrightarrow{1-1} b]$.

Furthermore, if $x \in \cup(a)$ then

$$(\exists y)[x \in y \wedge y \in a].$$

Again by *Theorem 12.46

(2) $(\exists h)(\forall x \in \cup(a))[x \in h`x \wedge h`x \in a]$.

We then define a function F on $\cup(a)$ in the following way

$$F`x = \langle h`x, (f`h`x)`x\rangle, x \in \cup(a).$$

From (1) and (2) it then follows that if $x \in \cup(a)$ then

$$x \in h`x \wedge h`x \in a$$

and so

$$(f`h`x)`x \in b.$$

Thus $F: \cup(a) \rightarrow a \times b$. We wish to prove that F is one-to-one.
If $x \in \cup(a)$, $y \in \cup(a)$, and $F`x = F`y$ then

$$h`x = h`y \wedge (f`h`x)`x = (f`h`y)`y.$$

But if $h`x = h`y$, then $f`h`x = f`h`y$. Furthermore, since $f`h`x$ is one-to-one

$$(f`h`x)`x = (f`h`y)`y \rightarrow x = y.$$

Thus $F\colon \cup(a) \xrightarrow{1-1} a \times b$ and so

$$\overline{\overline{\cup(a)}} \leqq \overline{\overline{a \times b}}. \qquad \square$$

***Proposition 10.48.** $\mathscr{U}n(F) \wedge (\forall\, y \in b)[\overline{\overline{F``y}} \leqq \bar{\bar{a}}] \rightarrow \overline{\overline{\cup(F``b)}} \leqq \overline{\overline{a \times b}}$.

Proof. By *Proposition 10.34

$$\mathscr{U}n(F) \rightarrow \overline{\overline{F``b}} < \bar{\bar{b}}.$$

Furthermore if $x \in F``b$ then $(\exists\, y \in b)[x = F`y]$. Consequently if $\overline{\overline{F`y}} \leqq \bar{\bar{a}}$ for each y in b, then by *Proposition 10.47

$$\overline{\overline{\cup F``b}} \leqq \overline{\overline{a \times F``b}} \leqq \overline{\overline{a \times b}}. \qquad \square$$

CHAPTER 11

Cofinality, the Generalized Continuum Hypothesis, and Cardinal Arithmetic

As we promised in the last section we now take up the problem of computing cardinal powers. For this purpose we introduce the idea of *cofinality*: An ordinal α is cofinal with an ordinal β provided that $\beta \leqq \alpha$ and there exists a strictly monotone ordinal function f that maps β into α in such a way that every element of α is less than or equal to some element in the range of f:

Definition 11.1.

$$\mathrm{cof}(\alpha, \beta) \overset{\Delta}{\leftrightarrow} \beta \leqq \alpha \wedge (\exists f)[\mathrm{Smo}(f) \wedge f: \beta \to \alpha \wedge (\forall \gamma < \alpha)(\exists \delta < \beta)[f`\delta \geqq \gamma]].$$

Remark. We read cof(α, β) as "α is cofinal with β."

EXERCISES

Prove the following.

(1) $\mathrm{cof}(\alpha, 0) \leftrightarrow \alpha = 0$.

(2) $\alpha \in K_{\mathrm{I}} \wedge \beta \in K_{\mathrm{I}} \wedge 0 < \beta \leqq \alpha \to \mathrm{cof}(\alpha, \beta)$.

(3) $1 \leqq \alpha \wedge \alpha \in K_{\mathrm{I}} \to \mathrm{cof}(\alpha, 1)$.

Remark. The fact that every nonzero K_{I} ordinal is cofinal with every smaller nonzero K_{I} ordinal and 0 is only cofinal with itself, tells us that the cofinality properties of K_{I} ordinals are not very interesting. Indeed some authors formulate the definition in such a way as to exclude the K_{I} ordinals from consideration.

Proposition 11.2. $\operatorname{cof}(\alpha, \beta) \to [\alpha \in K_{\mathrm{II}} \leftrightarrow \beta \in K_{\mathrm{II}}]$.

The proof is left to the reader.

Proposition 11.3. $\alpha \in K_{\mathrm{II}} \wedge \operatorname{cof}(\alpha, \beta) \to (\exists f)[f \mathscr{F}\!n\, \beta \wedge \alpha = \cup(f``\beta).]$

The proof is left to the reader.

Remark. If α is cofinal with β and $\beta < \alpha$, then α can be "reached" by a mapping from "below" in a sense made clear by the definition. Proposition 11.3 gives us another perspective on cofinality for limit ordinals. It tells us that if a limit ordinal α is cofinal with β, then α is the union of β sets of ordinals each of which is bounded by an element of α. This tells us, for example, that ω is not cofinal with any integer n for if it were then ω would be the union of n bounded sets of integers.

Proposition 11.4. $\operatorname{cof}(\alpha, \alpha)$.

The proof is left to the reader.

Proposition 11.5. $\operatorname{cof}(\alpha, \beta) \wedge \operatorname{cof}(\beta, \gamma) \to \operatorname{cof}(\alpha, \gamma)$.

The proof is left to the reader.

Proposition 11.6. $\alpha \in K_{\mathrm{II}} \to \operatorname{cof}(\aleph_\alpha, \alpha)$.

The proof is left to the reader.

Exercises

Prove the following.

(1) $\operatorname{cof}(\aleph_\omega, \omega)$.

(2) $\operatorname{cof}(\aleph_{\aleph_\omega}, \omega)$.

Proposition 11.7.

$$\beta \leqq \alpha \wedge (\exists f)[f\colon \beta \to \alpha \wedge (\forall \gamma < \alpha)(\exists \delta < \beta)[f`\delta \geqq \gamma] \to (\exists \eta \leqq \beta)[\operatorname{cof}(\alpha, \eta)]].$$

Proof. If $a = \{\delta < \beta \mid (\forall \gamma < \delta)[f`\gamma < f`\delta]\}$, then $a \subseteq \beta$. Therefore

$$(\exists \eta \leqq \beta)(\exists h)[h \operatorname{Isom}_{E,E}(\eta, a)].$$

If $g = f \circ h$, then $g\colon \eta \to \alpha$. To prove that g is strictly monotone we note that if $\delta < \gamma < \eta$, then $h`\delta < h`\gamma$, $h`\delta \in a$, and $h`\gamma \in a$. From the definition of a it follows that $f`h`\delta < f`h`\gamma$. That is, $g`\delta < g`\gamma$, and hence g is strictly monotone.

Since by hypothesis $(\forall \gamma < \alpha)(\exists \delta < \beta)[f`\delta \geqq \gamma]$ it follows that there is a smallest such $\delta$. This smallest $\delta$ is an element of $a$ and so $(\exists \nu < \eta)[h`\nu = \delta]$. Consequently

$$g`\nu = f`h`\nu = f`\delta \geqq \gamma.$$

Thus α is cofinal with η. □

Corollary 11.8. $\beta \leqq \alpha \wedge \beta \simeq \alpha \rightarrow (\exists \eta \leqq \beta)[\mathrm{cof}(\alpha, \eta)]$.

PROOF. If $\beta \simeq \alpha$, then $(\exists f)[f: \beta \xrightarrow[\text{onto}]{1-1} \alpha]$ and so $(\forall \gamma < \alpha)(\exists \delta < \beta)[f`\delta = \gamma]$. Therefore, by Proposition 11.7, $(\exists \eta \leqq \beta)[\mathrm{cof}(\alpha, \eta)]$. □

Proposition 11.9. $\mathrm{cof}(\alpha, \beta) \wedge \mathrm{cof}(\alpha, \gamma) \wedge \gamma \leqq \beta \rightarrow (\exists \eta \leqq \gamma)[\mathrm{cof}(\beta, \eta)]$.

PROOF. If α is confinal with β and with γ, then

$$(\exists f)[f: \beta \rightarrow \alpha \wedge (\forall \tau < \alpha)(\exists \delta < \beta)[f`\delta \geqq \tau] \wedge \mathrm{Smo}(f)]$$

$$(\exists g)[g: \gamma \rightarrow \alpha \wedge (\forall \tau < \alpha)(\exists \delta < \gamma)[g`\delta \geqq \tau] \wedge \mathrm{Smo}(g)].$$

In particular if $\delta < \gamma$, then $g`\delta < \alpha$ and so $(\exists \tau < \beta)[f`\tau \geqq g`\delta]$. There is then a smallest such τ.

If

$$F`\delta = \mu_\tau(f`\tau \geqq g`\delta), \qquad \delta < \gamma.$$

Then $F: \gamma \rightarrow \beta$. We wish to prove that $(\forall \nu < \beta)(\exists \delta < \gamma)[F`\delta \geqq \nu]$. For this purpose we note that if $\nu < \beta$, then $\nu \leqq f`\nu < \alpha$. Therefore $(\exists \delta < \gamma)[g`\delta \geqq f`\nu]$. Since f is strictly monotone, if $\tau < \nu$, then $f`\tau < f`\nu \leqq g`\delta$. Thus the smallest ordinal $\tau$ for which $f`\tau \geqq g`\delta$ is greater than or equal to ν. That is

$$F`\delta = \mu_\tau(f`\tau \geqq g`\delta) \geqq \nu.$$

It then follows from Proposition 11.7 that $(\exists \eta \leqq \gamma)[\mathrm{cof}(\beta, \eta)]$. □

Definition 11.10. $\mathrm{cf}(\alpha) \triangleq \mu_\beta(\mathrm{cof}(\alpha, \beta))$.

Remark. We read $\mathrm{cf}(\alpha)$ as "the character of cofinality of α."

EXERCISES

Prove the following.

(1) $\mathrm{cf}(\alpha) \leqq \alpha$.

(2) $\mathrm{cf}(0) = 0$.

(3) $\mathrm{cf}(\alpha + 1) = 1$.

(4) $\mathrm{cf}(\omega) = \omega$.

(5) $\mathrm{cof}(\alpha, \mathrm{cf}(\alpha))$.

Remark. From the results we have now proven, it is easy to show that the character of cofinality of any ordinal α, is a cardinal. That is, the smallest ordinal with which α is cofinal is a cardinal.

Proposition 11.11. $\mathrm{cf}(\alpha) \in N$.

PROOF. Suppose that $\beta = \mathrm{cf}(\alpha)$. To prove that β is a cardinal we need only prove that if $\gamma \simeq \beta$ then $\gamma \geqq \beta$. We argue by contradiction. Suppose that $\gamma < \beta$. Then by Corollary 11.8 $(\exists\, \eta \leqq \gamma)[\mathrm{cof}(\beta, \eta)]$. But from this it follows that $\beta = \mathrm{cf}(\alpha) \leqq \eta \leqq \gamma$. This is a contradiction. □

Proposition 11.12. $\alpha \in N' \to \mathrm{cf}(\alpha) \in N'$.

PROOF. If $\alpha \in N'$, then $\alpha \in K_{\mathrm{II}}$. Since α is cofinal with $\mathrm{cf}(\alpha)$ it follows, from Proposition 11.2 that $\mathrm{cf}(\alpha) \in K_{\mathrm{II}}$. But then $\mathrm{cf}(\alpha) \geqq \omega$ and so from Proposition 11.11, $\mathrm{cf}(\alpha) \in N'$. □

Proposition 11.13. $\alpha \in K_{\mathrm{II}} \to \mathrm{cf}(\alpha) = \mathrm{cf}(\aleph_\alpha)$.

PROOF. If $\alpha \in K_{\mathrm{II}}$, then $\aleph_\alpha$ is cofinal with α by Proposition 11.6. Since α is cofinal with $\mathrm{cf}(\alpha)$ it follows from Proposition 11.5 that $\aleph_\alpha$ is cofinal with $\mathrm{cf}(\alpha)$. Thus

(1) $\mathrm{cf}(\aleph_\alpha) \leqq \mathrm{cf}(\alpha)$.

But $\aleph_\alpha$ is also cofinal with $\mathrm{cf}(\aleph_\alpha)$ and so by Proposition 11.9

$$(\exists\, \eta \leqq \mathrm{cf}(\aleph_\alpha))[\mathrm{cof}(\mathrm{cf}(\alpha), \eta].$$

But since α is cofinal with $\mathrm{cf}(\alpha)$ it then follows that α is cofinal with η and so

(2) $\mathrm{cf}(\alpha) \leqq \eta \leqq \mathrm{cf}(\aleph_\alpha)$.

Then from (1) and (2), $\mathrm{cf}(\alpha) = \mathrm{cf}(\aleph_\alpha)$. □

Definition 11.14.

(1) $\mathrm{Reg}(\alpha) \overset{\Delta}{\leftrightarrow} \alpha \in N' \wedge \mathrm{cf}(\alpha) = \alpha$.

(2) $\mathrm{Sing}(\alpha) \overset{\Delta}{\leftrightarrow} \alpha \in N' \wedge \mathrm{cf}(\alpha) < \alpha$.

Remark. We read $\mathrm{Reg}(\alpha)$ as "α is a regular cardinal" and we read $\mathrm{Sing}(\alpha)$ as "α is a singular cardinal."

From Proposition 11.6 we see that $\aleph_\omega$ is cofinal with ω and so $\mathrm{cf}(\aleph_\omega) = \omega < \aleph_\omega$. Thus $\aleph_\omega$ is a singular cardinal. So also are $\aleph_{\omega 2}$, $\aleph_{\omega 3}$, etc. Furthermore, since $\mathrm{cf}(\aleph_0) = \aleph_0$ it follows that $\aleph_0$ is a regular cardinal. Are there other regular cardinals? Yes, in fact it is easy to prove that $\aleph_\alpha$ is regular if $\alpha \in K_{\mathrm{I}}$:

***Theorem 11.15.** *$\aleph_{\alpha+1}$ is regular.*

PROOF (By contradiction). Suppose that $\aleph_{\alpha+1}$ is singular. That is, suppose that $\aleph_\beta = \mathrm{cf}(\aleph_{\alpha+1}) < \aleph_{\alpha+1}$. Then $(\exists\, h)[h\colon \aleph_\beta \to \aleph_{\alpha+1} = \cup(h``\aleph_\beta)]$. Furthermore $(\forall\, \delta < \aleph_\beta)[h`\delta < \aleph_{\alpha+1}]$ and so $\overline{\overline{h`\delta}} \leqq \aleph_\alpha$. Then from Proposition 10.48

$$\aleph_{\alpha+1} = \overline{\overline{\cup(h``\aleph_\beta)}} \leqq \overline{\overline{\aleph_\alpha \times \aleph_\beta}} = \aleph_\alpha.$$

From this contradiction we conclude that $\mathrm{cf}(\aleph_{\alpha+1}) = \aleph_{\alpha+1}$. □

Remark. Let us now summarize what we know about regular and singular cardinals. We know that $\aleph_\alpha$ is regular if $\alpha \in K_{\mathrm{I}}$. If $\alpha \in K_{\mathrm{II}}$ we know that $\aleph_\alpha$ is cofinal with α and so $\mathrm{cf}(\aleph_\alpha) \leqq \alpha$. We also know that $\alpha \leqq \aleph_\alpha$. If $\alpha < \aleph_\alpha$, then $\aleph_\alpha$ is singular. But if $\alpha = \aleph_\alpha$ we do not know whether $\aleph_\alpha$ is regular or singular. Do there exist ordinals α for which $\alpha = \aleph_\alpha$? Yes. To prove this we first prove the following result.

Proposition 11.16. $[f\colon a \to N] \to \cup(f``a) \in N$.

PROOF. Since $\mathscr{D}(f)$ is a set, $f``a$ is a set, indeed it is a set of ordinals. If $\beta = \cup(f``a)$, then $\overline{\overline{\beta}} \leqq \beta$. We wish to prove that $\overline{\overline{\beta}} = \beta$. Suppose not. Suppose that $\overline{\overline{\beta}} < \beta$. Then $(\exists\, x \in a)[\overline{\overline{\beta}} < f`x \in f``a]$. Therefore $f`x \leqq \beta$ and since $f`x \in N$

$$f`x = \overline{\overline{f`x}} \leqq \overline{\overline{\beta}}.$$

This is a contradiction. Therefore $\overline{\overline{\beta}} = \beta$ and $\cup(f``a) \in N$. □

Corollary 11.17.

$$[f\colon a \to N] \wedge (\exists\, x \in a)[f`x \in N'] \to \cup(f``a) \in N'.$$

PROOF. By Proposition 11.16, $\cup(f``a) \in N$. If $(\exists\, x \in a)[f`x \in N']$, then since $f`x \leqq \cup(f``a)$ it follows that $\cup(f``a) \in N'$. □

Proposition 11.18. $(\exists\, \alpha)[\alpha = \aleph_\alpha]$.

PROOF. If we define h recursively by

$$h`0 = \aleph_0$$

$$h`(n + 1) = \aleph_{h`n},$$

then $h\colon \omega < N'$. By Corollary 11.17, $\cup(h``\omega) \in N'$. Thus $(\exists\, \alpha)[\aleph_\alpha = \cup(h``\omega) \wedge \alpha \leqq \aleph_\alpha]$. If $\alpha < \aleph_\alpha$, then $(\exists\, n)[\alpha < h`n]$. Therefore $\aleph_\alpha < \aleph_{h`n} = h`(n + 1) \leqq \aleph_\alpha$. From this contradiction we conclude that $\alpha = \aleph_\alpha$. □

Definition 11.19.

(1) $\mathrm{Inacc}_w(\aleph_\alpha) \overset{\Delta}{\leftrightarrow} \alpha \in K_{\mathrm{II}} \wedge \mathrm{Reg}(\aleph_\alpha)$.

(2) $\mathrm{Inacc}(\aleph_\alpha) \overset{\Delta}{\leftrightarrow} \mathrm{Inacc}_w(\aleph_\alpha) \wedge (\forall\, x)[\overline{\overline{x}} < \aleph_\alpha \to \overline{\overline{\mathscr{P}(x)}} < \aleph_\alpha]$.

Remark. We read $\mathrm{Inacc}_w(\aleph_\alpha)$ as "$\aleph_\alpha$ is weakly inaccessible" and we read $\mathrm{Inacc}(\aleph_\alpha)$ as "$\aleph_\alpha$ is inaccessible."

Does there exist a weakly inaccessible cardinal? We do not know. We do know that we cannot prove the existence of a weakly inaccessible cardinal in ZF. How this is proved we will discuss later. For the moment let us be content to discuss what it means. For one thing it means that if we chose to do so we could add to ZF an axiom asserting that there does not exist a weakly inaccessible cardinal and be assured that the resulting system is consistent if ZF is consistent. But we do not like axioms that say that things do not exist. We prefer axioms that enrich rather than impoverish. Perhaps we would like to add an axiom that says that weakly inaccessible cardinals do exist. May we do so without fear that we will produce an inconsistent theory? Probably so but we do not know.

Let us now turn to the problem of computing cardinal powers. It may come as a surprise that in ZF we cannot compute such a simple power as

$$\overline{\overline{2^{\aleph_0}}}. \tag{1}$$

Let us review the problem. By definition, (1) is the cardinality of the set of all functions that map ω into 2. We can think of any such function as a sequence of zeros and ones. But any such sequence can also be thought of as the binary representation of a real number that lies between 0 and 1. Thus (1) is the cardinality of the set of real numbers that lie between 0 and 1. But that is also the cardinality of the set of all real numbers.

About the set $2^{\aleph_0}$ we know that $2^{\aleph_0} \simeq \mathscr{P}(\omega)$ and from Cantor's Theorem we know that $\overline{\overline{\mathscr{P}(\omega)}} > \aleph_0$. Thus

$$\overline{\overline{2^{\aleph_0}}} > \aleph_0.$$

The question then is whether there exist cardinalities intermediate between

$$\overline{\overline{2^{\aleph_0}}}$$

and $\aleph_0$. Is every infinite set of reals either equivalent to ω or to the set of all reals? If the answer to that questions is yes, then

$$\overline{\overline{2^{\aleph_0}}} = \aleph_1.$$

If the answer is no, then

$$\overline{\overline{2^{\aleph_0}}} > \aleph_1.$$

For the purposes at hand we assume the answer is yes. Thus we assume the Continuum Hypothesis.

CH. $\overline{\overline{2^{\aleph_0}}} = \aleph_1$.

But this assumption alone is not enough to settle all questions about cardinal powers. So we also assume the Generalized Continuum Hypothesis.

GCH. $\overline{\overline{2^{\aleph_\alpha}}} = \aleph_{\alpha+1}$.

Theorem 11.20. GCH $\to$ [Inacc$_w(\aleph_\alpha) \leftrightarrow$ Inacc$(\aleph_\alpha)$].

PROOF. By definition $\aleph_\alpha$ inaccessible implies $\aleph_\alpha$ weakly inaccessible. Conversely

$$\text{Inf}(x) \wedge \bar{\bar{x}} < \aleph_\alpha \to (\exists\, \beta)[\bar{\bar{x}} = \aleph_\beta \wedge \beta < \alpha].$$

Since $\alpha \in K_{\text{II}}$, $\beta + 1 < \alpha$ hence $\aleph_{\beta+1} < \aleph_\alpha$. But by GCH

$$\overline{\overline{\mathscr{P}(x)}} = \aleph_{\beta+1}. \qquad \square$$

***Theorem 11.21.** $\text{Inf}(b) \wedge 2 \leqq \bar{\bar{a}} \leqq \overline{\overline{\mathscr{P}(b)}} \to \overline{\overline{a^b}} = \overline{\overline{\mathscr{P}(b)}}$.

PROOF. By Proposition 10.44 $\overline{\overline{\mathscr{P}(b)}} = \overline{\overline{2^b}}$. Therefore if $\bar{\bar{a}} \leqq \overline{\overline{\mathscr{P}(b)}}$, then $\bar{\bar{a}} \leqq \overline{\overline{2^b}}$ and so

$$\overline{\overline{a^b}} \leqq \overline{\overline{(2^b)^b}} = \overline{\overline{2^{b \times b}}} = \overline{\overline{2^b}} = \overline{\overline{\mathscr{P}(b)}}$$

i.e., $\overline{\overline{a^b}} \leqq \overline{\overline{\mathscr{P}(b)}}$.

On the other hand if $2 \leqq \bar{\bar{a}}$, then $\overline{\overline{2^b}} \leqq \overline{\overline{a^b}}$. Therefore

$$\overline{\overline{a^b}} = \overline{\overline{\mathscr{P}(b)}}. \qquad \square$$

***Corollary 11.22.**

(1) $\aleph_\alpha \leqq \aleph_\beta \to \overline{\overline{\aleph_\alpha^{\aleph_\beta}}} = \overline{\overline{2^{\aleph_\beta}}}$.

(2) $\overline{\overline{\aleph_\alpha^{\aleph_\alpha}}} = \overline{\overline{2^{\aleph_\alpha}}}$.

(3) $\aleph_\alpha \leqq \aleph_\beta \to \overline{\overline{\aleph_\beta^{\aleph_\alpha}}} \leqq \overline{\overline{2^{\aleph_\beta}}}$.

PROOF. (1) Since $\aleph_\beta$ is infinite and $2 \leqq \aleph_\alpha \leqq \aleph_\beta < \overline{\overline{\mathscr{P}(\aleph_\beta)}}$ we have from Theorem 11.21

$$\overline{\overline{\aleph_\alpha^{\aleph_\beta}}} = \overline{\overline{\mathscr{P}(\aleph_\beta)}} = \overline{\overline{2^{\aleph_\beta}}}.$$

(2) Obvious from (1) with $\alpha = \beta$.

(3) $\overline{\overline{\aleph_\beta^{\aleph_\alpha}}} \leqq \overline{\overline{\aleph_\beta^{\aleph_\beta}}} = \overline{\overline{2^{\aleph_\beta}}}$. $\square$

***Theorem 11.23.** $\alpha \in K_{\text{II}} \wedge (\forall\, \gamma < \alpha)\, [\overline{\overline{2^{\aleph_\gamma}}} < \aleph_\alpha] \wedge [\aleph_\beta < \text{cf}(\aleph_\alpha)] \to \overline{\overline{\aleph_\alpha^{\aleph_\beta}}} = \aleph_\alpha$.

PROOF. If $a = \{\aleph_\gamma^{\aleph_\beta} \mid \gamma < \alpha\}$ and if $f \in \cup\,(a)$ then

$$(\exists\, \gamma < \alpha)\, [f: \aleph_\beta \to \aleph_\gamma].$$

Since $\aleph_\gamma \subset \aleph_\alpha$ it follows that $f \in \aleph_\alpha^{\aleph_\beta}$ and so

$$\cup\,(a) \subseteq \aleph_\alpha^{\aleph_\beta}.$$

If $f \in \aleph_\alpha^{\aleph_\beta}$ then $f: \aleph_\beta \to \aleph_\alpha$. Since $\aleph_\beta < \text{cf}(\aleph_\alpha)$ it follows that

$$(\exists\, \delta < \aleph_\alpha)\, [\mathscr{W}(f) \subseteq \delta].$$

Furthermore since $\alpha \in K_{\mathrm{II}}$

$$(\exists\, \gamma < \alpha)\,[\delta \leqq \aleph_\gamma < \aleph_\alpha].$$

Therefore $f \in \aleph_\gamma^{\aleph_\beta}$, i.e., $f \in \cup\,(a)$.

Thus $\aleph_\alpha^{\aleph_\beta} = \cup(a)$ and hence

$$\overline{\overline{\aleph_\alpha^{\aleph_\beta}}} = \overline{\overline{\cup(a)}}.$$

Further if $x \in a$, then $(\exists\, \gamma < \alpha)\,[x = \aleph_\gamma^{\aleph_\beta}]$. If in addition $\gamma < \beta$

$$\overline{\overline{\aleph_\gamma^{\aleph_\beta}}} \leqq \overline{\overline{\aleph_\beta^{\aleph_\beta}}} = \overline{\overline{2^{\aleph_\beta}}} < \aleph_\alpha.$$

On the other hand if $\beta \leqq \gamma$ then by *Corollary 11.22

$$\overline{\overline{\aleph_\gamma^{\aleph_\beta}}} \leqq \overline{\overline{2^{\aleph_\gamma}}} < \aleph_\alpha.$$

Finally since $\bar{\bar{a}} \leqq \bar{\bar{\alpha}} \leqq \aleph_\alpha$ we have from *Theorem 10.47

$$\overline{\overline{\cup(a)}} \leqq \overline{\overline{\aleph_\alpha \times \aleph_\alpha}} = \aleph_\alpha.$$

Therefore

$$\overline{\overline{\aleph_\alpha^{\aleph_\beta}}} = \aleph_\alpha. \qquad \square$$

Definition 11.24. $\prod_{x \in a} c`x \triangleq \{g \mid g\, \mathscr{F}\!n\, a \wedge (\forall\, x \in a)\,[g`x \in c`x]\}$.

Remark. We read $\prod_{x\in a} c`x$ as "the cross product of $c`x$ for $x \in a$." To see that this is a reasonable generalization of the cross product of two sets note that if $a = 2$ and if $g \in \prod_{x\in 2} c`x$ then $g\, \mathscr{F}\!n\, 2 \wedge g`0 \in c`0 \wedge g`1 \in c`1$, i.e., $\langle g`0, g`1\rangle \in c`0 \times c`1$. Conversely if $\langle x, y\rangle \in c`0 \times c`1$ and if we define $g$ on 2 by $g`0 = x \wedge g`1 = y$ then $g \in \prod_{x\in 2} c`x$. Clearly there is a natural one-to-one correspondence between $\prod_{x\in 2} c`x$ and $c`0 \times c`1$.

Proposition 11.25. $\mathscr{M}(\prod_{x\in a} c`x)$.

PROOF. $\prod_{x\in a} c`x \subseteq [\cup(c``a)]^a$. $\square$

***Theorem 11.26** (Zermelo). $(\forall\, x \in a)\,[\overline{\overline{b`x}} < \overline{\overline{c`x}}] \rightarrow \overline{\overline{\cup(b``a)}} < \overline{\overline{\prod_{x\in a} c`x}}$.

PROOF. (By contradiction). Otherwise

$$(\exists\, f)\left[f : \cup(b``a) \xrightarrow[\text{onto}]{} \prod_{x\in a} c`x\right].$$

If we define d on a by

$$d`x = \{(f`z)`x \mid z \in b`x\},\ x \in a$$

then $\overline{\overline{d`x}} \leqq \overline{\overline{b`x}}$ and $d`x \subseteq c`x$. Since by hypothesis $\overline{\overline{b`x}} < \overline{\overline{c`x}}$ it follows that $\overline{\overline{d`x}} < \overline{\overline{c`x}}$ and hence $c`x - d`x \neq 0$. Therefore by AC

$$(\exists\, e)\,(\forall\, x \in a)\,[e`x \in c`x - d`x].$$

Since $e`x \in c`x$ and since $e \mathscr{F}n\ a$, $e \in \prod_{x \in a} c`x$. Then

$$(\exists z \in \cup(b``a))[f`z = e].$$

But $z \in \cup(b``a)$ implies $(\exists x \in a)[z \in b`x]$. Consequently

$$(f`z)`x \in d`x \wedge (f`z)`x \in c`x - d`x. \qquad \square$$

Definition 11.27. $a^+ \triangleq \mu_\alpha(\bar{\bar{\alpha}} > \bar{\bar{a}})$.

Remark. Note that a^+ is a cardinal number.

***Theorem 11.28.** $\aleph_\alpha < \overline{\overline{\aleph_\alpha^{\mathrm{cf}(\aleph_\alpha)}}}$.

PROOF. If $\alpha \in K_{\mathrm{I}}$ then $\mathrm{cf}(\aleph_\alpha) = \aleph_\alpha$. Therefore

$$\aleph_\alpha < \overline{\overline{2^{\aleph_\alpha}}} = \overline{\overline{\aleph_\alpha^{\aleph_\alpha}}} = \overline{\overline{\aleph_\alpha^{\mathrm{cf}(\aleph_\alpha)}}}.$$

If $\alpha \in K_{\mathrm{II}}$ and if $\beta = \mathrm{cf}(\aleph_\alpha)$ then

$$(\exists f)\,[\mathrm{Smo}(f) \wedge [f : \beta \to \aleph_\alpha] \wedge \aleph_\alpha = \cup(f``\beta)].$$

If

$$c`\gamma = (f`\gamma)^+, \qquad \gamma < \beta$$

then $(\forall \gamma < \beta)\,[\overline{\overline{f`\gamma}} < \overline{\overline{c`\gamma}}]$. Therefore by *Theorem 11.26

$$\aleph_\alpha = \overline{\overline{\cup(f``\beta)}} < \overline{\overline{\prod_{\gamma<\beta} c`\gamma}}.$$

Since

$$\prod_{\gamma<\beta} c`\gamma \subseteq (\cup(c``\beta))^\beta,$$

it follows that

$$\overline{\overline{\prod_{\gamma<\beta} c`\gamma}} \leqq \overline{\overline{(\cup(c``\beta))^\beta}}.$$

Furthermore $\gamma < \beta$ implies $c`\gamma < \aleph_\alpha$ and $\beta \leqq \aleph_\alpha$. Therefore

$$\overline{\overline{(\cup(c``\beta))}} \leqq \overline{\overline{\aleph_\alpha \times \aleph_\alpha}} = \aleph_\alpha$$

and hence

$$\overline{\overline{(\cup(c``\beta))^\beta}} \leqq \overline{\overline{\aleph_\alpha^\beta}} = \overline{\overline{\aleph_\alpha^{\mathrm{cf}(\aleph_\alpha)}}}.$$

Thus

$$\aleph_\alpha < \overline{\overline{\aleph_\alpha^{\mathrm{cf}(\aleph_\alpha)}}}. \qquad \square$$

***Theorem 11.29.** $\aleph_\beta < \mathrm{cf}(\overline{\overline{\aleph_\alpha^{\aleph_\beta}}})$.

PROOF. (By contradiction). If $\mathrm{cf}(\overline{\overline{\aleph_\alpha^{\aleph_\beta}}}) = \aleph_\gamma \leqq \aleph_\beta$ then by Theorem 11.28, Proposition 10.45, and *Proposition 10.41

$$\overline{\overline{\aleph_\alpha^{\aleph_\beta}}} < \overline{\overline{(\overline{\overline{\aleph_\alpha^{\aleph_\beta}}})^{\aleph_\gamma}}} = \overline{\overline{\aleph_\alpha^{\aleph_\beta \times \aleph_\gamma}}} = \overline{\overline{\aleph_\alpha^{\aleph_\beta}}}. \qquad \square$$

***Corollary 11.30** (König). $\aleph_\alpha < \mathrm{cf}(\overline{\overline{2^{\aleph_\alpha}}})$.

PROOF. $\aleph_\alpha < \mathrm{cf}(\overline{\overline{\aleph_0^{\aleph_\alpha}}}) = \mathrm{cf}(\overline{\overline{2^{\aleph_\alpha}}})$. $\square$

Theorem 11.31. GCH $\rightarrow$

$$\begin{aligned} \overline{\overline{\aleph_\alpha^{\aleph_\beta}}} &= \aleph_\alpha && \textit{if } \aleph_\beta < \mathrm{cf}(\aleph_\alpha) \\ &= \aleph_{\alpha+1} && \textit{if } \mathrm{cf}(\aleph_\alpha) \leqq \aleph_\beta \leqq \aleph_\alpha \\ &= \aleph_{\beta+1} && \textit{if } \aleph_\alpha \leqq \aleph_\beta. \end{aligned}$$

PROOF. If $\aleph_\beta < \mathrm{cf}(\aleph_\alpha)$ then $\alpha \neq 0$. If $(\exists \gamma)[\alpha = \gamma + 1]$ then $\aleph_\alpha$ is regular and hence

$$\aleph_\beta < \mathrm{cf}(\aleph_\alpha) = \aleph_\alpha = \aleph_{\gamma+1}.$$

Therefore $\aleph_\beta \leqq \aleph_\gamma$. Since by GCH, $\aleph_{\gamma+1} = \overline{\overline{2^{\aleph_\gamma}}}$

$$\overline{\overline{\aleph_\alpha^{\aleph_\beta}}} = \overline{\overline{\aleph_{\gamma+1}^{\aleph_\beta}}} = \overline{\overline{(2^{\aleph_\gamma})^{\aleph_\beta}}} = \overline{\overline{2^{\aleph_\gamma \times \aleph_\beta}}} = \overline{\overline{2^{\aleph_\gamma}}} = \aleph_{\gamma+1} = \aleph_\alpha.$$

If $\alpha \in K_{\mathrm{II}}$ then since $\gamma < \alpha$ implies $\overline{\overline{2^{\aleph_\gamma}}} = \aleph_{\gamma+1} < \aleph_\alpha$ we have from *Theorem 11.23

$$\overline{\overline{\aleph_\alpha^{\aleph_\beta}}} = \aleph_\alpha.$$

If $\mathrm{cf}(\aleph_\alpha) \leqq \aleph_\beta \leqq \aleph_\alpha$ then from *Theorem 11.28 and *Corollary 11.22

$$\aleph_\alpha < \overline{\overline{\aleph_\alpha^{\mathrm{cf}(\aleph_\alpha)}}} \leqq \overline{\overline{\aleph_\alpha^{\aleph_\beta}}} \leqq \overline{\overline{2^{\aleph_\alpha}}} = \aleph_{\alpha+1}.$$

That is $\aleph_\alpha < \overline{\overline{\aleph_\alpha^{\aleph_\beta}}} \leqq \aleph_{\alpha+1}$. Therefore

$$\overline{\overline{\aleph_\alpha^{\aleph_\beta}}} = \aleph_{\alpha+1}.$$

If $\aleph_\alpha \leqq \aleph_\beta$ then since $\aleph_\beta < \aleph_{\beta+1} = \overline{\overline{2^{\aleph_\beta}}}$

$$\overline{\overline{\aleph_\alpha^{\aleph_\beta}}} \leqq \overline{\overline{(\overline{\overline{2^{\aleph_\beta}}})^{\aleph_\beta}}} = \overline{\overline{2^{\aleph_\beta \times \aleph_\beta}}} = \overline{\overline{2^{\aleph_\beta}}} = \aleph_{\beta+1}.$$

By *Theorem 11.29

$$\aleph_\beta < \mathrm{cf}(\overline{\overline{\aleph_\alpha^{\aleph_\beta}}}) \leqq \overline{\overline{\aleph_\alpha^{\aleph_\beta}}}.$$

Therefore

$$\overline{\overline{\aleph_\alpha^{\aleph_\beta}}} = \aleph_{\beta+1}. \qquad \square$$

Remark. With the aid of AC we can also improve on Proposition 9.6.

***Proposition 11.32.** $\mathrm{Inf}(a) \wedge \mathrm{Cl}(R_1, A) \wedge \cdots \wedge \mathrm{Cl}(R_m, A) \wedge \mathrm{Cl}_2(S_1, A) \wedge \cdots \wedge \mathrm{Cl}_2(S_n, A) \wedge \mathscr{U}\!n(R_1) \wedge \cdots \wedge \mathscr{U}\!n(R_m) \wedge \mathscr{U}\!n(S_1) \wedge \cdots \wedge \mathscr{U}\!n(S_n) \wedge a \subset A$ then there exists a set b such that $a \subseteqq b \subseteqq A$, $\overline{\overline{a}} = \overline{\overline{b}}$, and

$$\mathrm{Cl}(R_1, b) \wedge \cdots \wedge \mathrm{Cl}(R_m, b) \wedge \mathrm{Cl}_2(S_1, b) \wedge \cdots \wedge \mathrm{Cl}_2(S_n, b).$$

Proof. The proof proceeds as for Proposition 9.6. We then note that since R_i and S_i are single valued

$$\overline{\overline{R_i``f`k}} \leqq \overline{\overline{f`k}} \qquad i = 1, \dots, m$$

$$\overline{\overline{S_i``(f`k)^2}} \leqq \overline{\overline{(f`k)^2}} = \overline{\overline{f`k}}, \qquad i = 1, \dots, n.$$

Therefore $f`(k + 1)$ is the union of a finite number of sets each of cardinality not greater than $\overline{\overline{f`k}}$. Then by *Proposition 10.41

$$\overline{\overline{f`(k+1)}} = \overline{\overline{f`k}} = \cdots = \overline{\overline{f`0}} = \overline{\overline{a}}.$$

Then from *Proposition 10.48, and the fact that a is infinite

$$\overline{\overline{b}} \leqq \overline{\overline{a \times \omega}} = \overline{\overline{a}}.$$

Furthermore since $a \subseteqq b$, $\overline{\overline{a}} \leqq \overline{\overline{b}}$. Therefore $\overline{\overline{a}} = \overline{\overline{b}}$. $\square$

CHAPTER 12
Models

We turn now to the very interesting subject of models of set theory. Intuitively by a model of set theory we mean a system in which the axioms and theorems of ZF are true. Such a system must consist of a domain of objects that we interpret as the universe V of our theory and a binary relation that we interpret as the $\in$-relation of our theory.

Assuming consistency there is a model of ZF consisting of a universe of "sets" V on which there is defined an "$\in$-relation." Given such a universe V it is possible that some subclass A of V together with some relation R on A is also a model of ZF. With $A \subseteq V$ and $R \subseteq A \times A$ the language of ZF is adequate for the development of a theory of such internal models. Our next objective is to make the foregoing ideas precise and thereby compel ZF to tell us about some of its models.

In order to define "model" we first introduce the idea of a structure or relational system. For each nonempty class A and each relation $R \subseteq A \times A$ we introduce the term

$$[A, R]$$

which we call a structure (or relational system); A is the universe of this structure and the elements of A we call individuals.

We next define "the structure $[A, R]$ satisfies the wff φ." Our definition is by induction on the number of logical symbols in φ. For this purpose we assume $\neg$, $\wedge$, and $\forall$ as primitive.

Definition 12.1.

(1) $[A, R] \models a \in b \overset{\Delta}{\leftrightarrow} a \in A \wedge b \in A \wedge a R b.$

(2) $[A, R] \models \neg \psi \overset{\Delta}{\leftrightarrow} \neg [[A, R] \models \psi].$

(3) $[A, R] \models \psi \wedge \eta \overset{\triangle}{\leftrightarrow} [[A, R] \models \psi] \wedge [[A, R] \models \eta]$.
(4) $[A, R] \models (\forall x)\psi(x) \overset{\triangle}{\leftrightarrow} (\forall x \in A)[[A, R] \models \psi(x)]$.

Remark. We read $[A, R] \models \varphi$ as "the structure $[A, R]$ satisfies φ." With the understanding that each term $[A, R]$ and the satisfaction symbol $\models$ occur only in contexts covered by Definition 12.1 it is clear that these symbols can be eliminated from our language, that is,

$$[A, R] \models \varphi$$

is an abbreviation for a wff of our language.

If $[A, R] \models \varphi$ we say that $[A, R]$ is a model of φ. Moreover $[A, R]$ is a model of a collection of wffs provided $[A, R] \models \varphi$ for each φ in the collection. In order to prove that a certain structure $[A, R]$ is a model of a given wff we must prove a certain wff in ZF namely the well-formed formula $[A, R] \models \varphi$. We will be particularly interested in structures for which R is the usual $\in$-relation. Such a structure we call a standard structure.

Definition 12.2. $[A, R]$ is a *standard structure* iff $R = E \cap A^2$.

Definition 12.3. $A \models \varphi \overset{\triangle}{\leftrightarrow} [A, E \cap A^2] \models \varphi$.

Definition 12.4.
(1) $[a \in b]^A \overset{\triangle}{\leftrightarrow} a \in b$.
(2) $[\neg\psi]^A \overset{\triangle}{\leftrightarrow} \neg\psi^A$.
(3) $[\psi \wedge \eta]^A \overset{\triangle}{\leftrightarrow} \psi^A \wedge \eta^A$.
(4) $[(\forall x)\psi(x)]^A \overset{\triangle}{\leftrightarrow} (\forall x \in A)[\psi^A(x)]$.

Remark. From Definition 12.4 we see that φ^A is simply the wff obtained from φ by replacing each occurrence of a quantified variable $(\forall x)$ by $(\forall x \in A)$.

Proposition 12.5. (1) *If φ is closed then*

$$[A \models \varphi] \leftrightarrow \varphi^A.$$

(2) *If all free variables occurring in φ are among $a_1, \ldots, a_n$ then*

$$a_1 \in A \wedge \cdots \wedge a_n \in A \rightarrow [[A \models \varphi] \leftrightarrow \varphi^A].$$

Proof. We consider (1) to be the special case of (2) with $n = 0$, and so we need only prove (2). This we do by induction on the number of logical symbols in φ. We assume that $a_1, \ldots, a_n \in A$.

If φ is of the form $a \in b$ and if a and b are among $a_1, \ldots, a_n$ then

$$a \in b \wedge a \in A \wedge b \in A \leftrightarrow a \in b$$

and so

$$[A \models \varphi] \leftrightarrow \varphi^A.$$

If φ is of the form $\neg\psi$, then all of the free variables of ψ are among $a_1, \ldots, a_n$. From the induction hypothesis

$$[A \models \psi] \leftrightarrow \psi^A.$$

Therefore

$$\neg[A \models \psi] \leftrightarrow \neg\psi^A$$

and hence

$$[A \models \varphi] \leftrightarrow \varphi^A.$$

If φ is of the form $\psi \wedge \eta$, then all of the free variables of ψ and of η are among $a_1, \ldots, a_n$. As our induction hypothesis we have

$$[A \models \psi] \leftrightarrow \psi^A,$$

and

$$[A \models \eta] \leftrightarrow \eta^A.$$

Therefore

$$[A \models \psi] \wedge [A \models \eta] \leftrightarrow \psi^A \wedge \eta^A.$$

Hence

$$[A \models \varphi] \leftrightarrow \varphi^A.$$

If φ is of the form $(\forall x)\psi(x)$, then there is an x that is not among $a_1, \ldots, a_n$ and all of the free variables of $\psi(x)$ are among $a_1, \ldots, a_n, x$. From the induction hypothesis

$$x \in A \rightarrow [[A \models \psi(x)] \leftrightarrow \psi^A(x)]$$

and hence

$$[x \in A \rightarrow A \models \psi(x)] \leftrightarrow [x \in A \rightarrow \psi^A(x)].$$

Since x is not among $a_1, \ldots, a_n$ we may generalize on x and from properties of logic we conclude that

$$(\forall x \in A)[A \models \psi(x)] \leftrightarrow (\forall x \in A)\psi^A(x).$$

Hence

$$[A \models \varphi] \leftrightarrow \varphi^A. \qquad \square$$

Remark. Proposition 12.5 is a basic result. It assures us that if φ is closed A is a model of φ if and only if φ^A is a theorem in ZF.

Suppose that $A \models \varphi$ and φ is equivalent to ψ, i.e., $\varphi \leftrightarrow \psi$. Does it follow that $A \models \psi$? To answer this question we need a result from logic for which we review the axioms for our logic and the rules of inference.

Logical Axioms

(1) $\varphi \to [\psi \to \varphi]$.

(2) $[\varphi \to [\psi \to \eta]] \to [[\varphi \to \psi] \to [\varphi \to \eta]]$.

(3) $[\neg\varphi \to \neg\psi] \to [\psi \to \varphi]$.

(4) $(\forall x)[\varphi \to \psi] \to [\varphi \to (\forall x)\psi]$ where the free variable on which we quantify does not occur in φ.

(5) $(\forall x)\varphi(x) \to \varphi(a)$.

Rules of Inference

(1) From $\varphi \to \psi$ and φ to infer ψ.

(2) From φ to infer $(\forall x)\varphi$.

Theorem 12.6. *If* $\vdash \varphi$ *and if* A *is a nonempty class that satisfies each nonlogical axiom that occurs in some proof of* φ *then*

(1) $\vdash \varphi^A$

if φ *is closed.*

(2) $\vdash a_1, \ldots, a_n \in A \to \varphi^A$

if all of the free variable of φ *are among* $a_1, \ldots, a_n$.

Proof. We regard (1) as the special case of (2) with $n = 0$. Since φ is a theorem it has a proof and indeed by hypothesis a proof in which each nonlogical axiom is satisfied by A. Suppose that the sequence of wff

$$\eta_1, \ldots, \eta_m$$

is such a proof. Then η_m is φ and each η_k is either an axiom or is inferred from previous formulas in the sequence by one of the rules of inference. Our procedure is to show that the sequence

$$\eta_1, \ldots, \eta_m$$

can be modified to produce a proof of (2). More precisely we will prove by induction that for each η_k, $k = 1, \ldots, m$, if all of the free variables in η_k are among $b_1, \ldots, b_p$ then

$$b_1, \ldots, b_p \in A \to \eta_k^A.$$

Case 1. Suppose that η_k is an axiom. If η_k is of the form $\psi \to [\eta \to \psi]$ then η_k^A is

$$\psi^A \to [\eta^A \to \psi^A]$$

i.e., η_k^A is an axiom. Hence

$$b_1, \ldots, b_p \in A \to \eta_k^A.$$

If η_k is of the form $[\psi \to [\theta \to \zeta]] \to [[\psi \to \theta] \to [\psi \to \zeta]]$ or $[\neg\psi \to \neg\theta] \to [\theta \to \psi]$ an argument similar to the foregoing leads to

$$b_1, \ldots, b_p \in A \to \eta_k^A.$$

If η_k is of the form $(\forall x)[\psi \to \theta] \to [\psi \to (\forall x)\theta]$ where x is not free in ψ then from the tautology $[p \to [q \to r]] \to [q \to [p \to r]]$ we have

$$[x \in A \to [\psi^A \to \theta^A]] \to [\psi^A \to [x \in A \to \theta^A]].$$

By generalization and Axiom 4

$$(\forall x)[x \in A \to [\psi^A \to \theta^A]] \to [\psi^A \to (\forall x)[x \in A \to \theta^A]],$$

and hence

$$b_1, \ldots, b_p \in A \to \eta_k^A.$$

If η_k is of the form $(\forall x)\psi(x) \to \psi(a)$ then as an instance of this same axiom we have

$$(\forall x)[x \in A \to \psi^A(x)] \to [a \in A \to \psi^A(a)].$$

Therefore

$$a \in A \to [(\forall x \in A)\psi^A(x) \to \psi^A(a)]$$

and hence

$$b_1, \ldots, b_p \in A \to \eta_k^A.$$

If η_k is an axiom of ZF then by hypothesis $A \models \eta_k$, and from Proposition 12.5

$$b_1, \ldots, b_p \in A \to \eta_k^A.$$

Case 2. If η_k is inferred by modus ponens from η_i and $\eta_i \to \eta_k$ and if all of the free variables of η_i are among $b_1, \ldots, b_p, c_1, \ldots, c_q$ with $c_1, \ldots, c_q$ all distinct and none of them occur among $b_1, \ldots, b_p$ then from our induction hypothesis

$$b_1, \ldots, b_p \in A \wedge c_1, \ldots, c_q \in A \to \eta_i^A,$$

$$b_1, \ldots, b_p \in A \wedge c_1, \ldots, c_q \in A \to [\eta_i^A \to \eta_k^A].$$

From the self-distributive law of implication and modus ponens

$$b_1, \ldots, b_p \in A \wedge c_1, \ldots, c_{q-1} \in A \to [c_q \in A \to \eta_k^A].$$

Since c_q does not occur among $b_1, \ldots, b_p, c_1, \ldots, c_{q-1}$ we have by generalization and Axiom 4

$$b_1, \ldots, b_p \in A \wedge c_1, \ldots, c_{q-1} \in A \to (\forall x)[x \in A \to \eta_k^A]$$
$$\to [(\exists x)[x \in A] \to \eta_k^A]$$
$$(\exists x)[x \in A] \to [b_1, \ldots, b_p \in A \wedge c_1, \ldots, c_{q-1} \in A \to \eta_k^A].$$

Since $A \neq 0$

$$b_1, \ldots, b_p \in A \wedge c_1, \ldots, c_{q-1} \in A \to \eta_k^A.$$

With $q - 1$ repetitions we obtain

$$b_1, \ldots, b_p \in A \to \eta_k^A.$$

Case 3. If η_k is inferred from η_i by generalization then there is an a not among $b_1, \ldots, b_p$. From the induction hypothesis

$$b_1, \ldots, b_p \in A \wedge a \in A \to \eta_i^A(a).$$

Since a is not among $b_1, \ldots, b_p$ we have by generalization and Axiom 4

$$b_1, \ldots, b_p \in A \to (\forall x \in A)\eta_i^A(x). \qquad \square$$

Remark. From Theorem 12.6 we see that if a proof of a wff φ requires only the logical axioms then every nonempty class A will be a model of φ. In particular every nonempty class A is a model of the logical axioms and if two wffs are logically equivalent, i.e.,

$$\vdash_{\mathrm{LA}} \varphi \leftrightarrow \psi$$

then a nonempty class A is a model of φ iff it is a model of ψ.

We are interested in classes A that are models of ZF. Since there are infinitely many axioms for ZF the assertion that A is a model of ZF is the assertion that each wff in a certain infinite collection of wffs is a theorem in ZF. This assertion we abbreviate as the metastatement, $A \models \mathrm{ZF}$.

From Theorem 12.6 we see that if $A \models \mathrm{ZF}$ then every theorem of ZF holds in A, that is, A satisfies each theorem of ZF. In the next section we will give conditions on A that assure that $A \models \mathrm{ZF}$. One requirement for most results of that section is that A be transitive.

Definition 12.7. $\mathrm{STM}(A, \varphi) \stackrel{\Delta}{\leftrightarrow} \mathrm{Tr}(A) \wedge A \models \varphi$.

Remark. By a standard transitive model of ZF we mean a nonempty transitive class A that satisfies each axiom of ZF, i.e., for each axiom φ

$$\mathrm{STM}(A, \varphi).$$

Although we restrict our discussion to standard transitive models of ZF this theory nevertheless encompasses a large class of models of ZF as we see from the following theorem.

Theorem 12.8 (Mostowski). *If* $R \subseteq A^2 \wedge R \text{ Wfr } A \wedge (\forall x \in A)(\forall y \in A)[(\forall z)[zRx \leftrightarrow zRy] \to x = y]$, *then there exists a B and F such that*

(1) $\mathrm{Tr}(B)$.

(2) $F \, \mathrm{Isom}_{R,E}(A, B)$.

(3) $[A, R] \models \varphi \leftrightarrow B \models \varphi$ *if* φ *is closed.*

(4) *If all of the free variables of* φ *are among* $a_1, \ldots, a_n$ *and ij*

$$a_1, \ldots, a_n \in A$$

then

$$[A, R] \models \varphi(a_1, \ldots, a_n) \leftrightarrow B \models \varphi(F`a_1, \ldots, F`a_n).$$

PROOF. (1) If R Wfr $A \wedge x \in A$ it then follows that $(R^{-1})``\{x\}$ is a set. Therefore if $f$ $\mathscr{F}n(R^{-1})``\{x\}$ then $\mathscr{W}(f)$ is a set. Let

$$K = \{f \mid (\exists z \subseteqq A)[f \ \mathscr{F}n \ z \wedge (\forall x \in z)[f`x = \{f`y \mid y \, R \, x\} \wedge (R^{-1})``\{x\} \subseteqq z]\}.$$

Then any two functions in K have the same values at any point common to their domains: Otherwise there would exist an f and g in K and an x in $\mathscr{D}(f) \cap \mathscr{D}(g)$ such that $f(x) \neq g(x)$. If $c = \{x \in \mathscr{D}(f) \cap \mathscr{D}(g) \mid f`x \neq g`x\}$ then $c \neq 0 \wedge c \subseteqq A$. Therefore $(\exists x \in c)[c \cap (R^{-1})``\{x\} = 0]$. Since $x \in c$

$$(R^{-1})``\{x\} \subseteqq \mathscr{D}(f) \wedge (R^{-1})``\{x\} \subseteqq \mathscr{D}(g).$$

Then $y \, R \, x$ implies $f`y = g`y$ and hence

$$f`x = \{f`y \mid y \, R \, x\} = \{g`y \mid y \, R \, x\} = g`x.$$

This is a contradiction.

Furthermore each f in K is one-to-one, for otherwise

$$(\exists f \in K)(\exists x \in \mathscr{D}(f))\,(\exists y \in \mathscr{D}(f))\,[x \neq y \wedge f`x = f`y].$$

If $c = \{x \in \mathscr{D}(f) \mid (\exists y \in \mathscr{D}(f))[x \neq y \wedge f`x = f`y]\}$ then $c \neq 0 \wedge c \subseteqq A$. Therefore c has an R-minimal element, i.e., $(\exists x \in c)[c \cap (R^{-1})``\{x\} = 0]$. Since $x \in c$,

$$(\exists y \in \mathscr{D}(f))\,[x \neq y \wedge f`x = f`y].$$

But

$$f`x = \{f`z \mid z \, R \, x\} \quad \text{and} \quad f`y = \{f`w \mid w \, R \, y\}.$$

Therefore if $z \, R \, x$, then

$$f`z \in f`x = f`y$$

and hence

$$(\exists w)[w \, R \, y \wedge f`w = f`z].$$

Since x is an R-minimal element of c and $z \, R \, x$, $f`w = f`z$ implies that $w = z$, that is,

$$z \, R \, x \rightarrow z \, R \, y.$$

By a similar argument we obtain $z \, R \, y \rightarrow z \, R \, x$ and hence from the hypotheses of our theorem we conclude that $x = y$. This is a contradiction.

If

$$F = \cup(K)$$

and if $\langle a, x\rangle \in F \wedge \langle a, y\rangle \in F$ then $(\exists f \in K)(\exists g \in K)[x = f`a \wedge y = g`a]$. But since $a \in \mathscr{D}(f) \cap \mathscr{D}(g), f`a = g`a$, i.e., $x = y$. Therefore F is a function.

Furthermore $(\forall f \in K)(\forall x \in \mathscr{D}(f))\,[F`x = f`x]$; consequently

$$F`x = f`x = \{f`y \mid y\, R\, x\} = \{F`y \mid y\, R\, x\}.$$

From this it follows that F is one-to-one, for if not there is an x and a y in $\mathscr{D}(F)$ for which $F`x = F`y$ but $x \neq y$. From Proposition 9.4 it then follows that there is an R-minimal x in $\mathscr{D}(F)$ for which $(\exists y \in \mathscr{D}(F))\,[x \neq y \wedge F`x = F`y]$. Then

$$F`x = \{F`z \mid z\, R\, x\} = \{F`w \mid w\, R\, y\} = F`y.$$

From this and the defining property of x it then follows that $z\, R\, y$ if and only if $z\, R\, x$ and hence $x = y$. This is a contradiction.

Since

$$\mathscr{D}(F) = \bigcup_{f \in K} \mathscr{D}(f)$$

and $f \in K$ implies $\mathscr{D}(f) \subseteqq A$ it follows that $\mathscr{D}(F) \subseteqq A$. If $A - \mathscr{D}(F) \neq 0$ then by Proposition 9.4, $A - \mathscr{D}(F)$ has an R-minimal element, that is

$$(\exists x \in A - \mathscr{D}(F))\,[A - \mathscr{D}(F) \cap (R^{-1})``\{x\} = 0].$$

By Proposition 9.3 there is a subset a of A that is the R^{-1} closure of $\{x\}$, i.e.,

$$[\{x\} \subseteqq a \subseteqq A \wedge (\forall y)(\forall z)[y\, R\, z \wedge z \in a \rightarrow y \in a]].$$

Furthermore each element of a is "connected" to x by a finite R-chain, i.e.,

$$(\forall y \in a)(\exists n)(\exists f)[f : n + 1 \rightarrow a \wedge f`0 = x \wedge f`n = y \wedge \\ (\forall i < n)[f`(i + 1) R f`i]].$$

Since x is an R-minimal element of $A - \mathscr{D}(F)$

$$A \cap (R^{-1})``\{x\} \subseteqq \mathscr{D}(F).$$

By definition of K if $z \in \mathscr{D}(F)$ then $(R^{-1})``\{z\} \subseteqq \mathscr{D}(F)$. Since each element of a is connected to x by a finite R-chain it follows by induction on the length of such chains that $a - \{x\} \subseteqq \mathscr{D}(F)$. We then define g by

$$g = (F \upharpoonright (a - \{x\})) \cup \{\langle x, F``(R^{-1})``\{x\}\rangle\}.$$

Thus $\mathscr{D}(g) \subseteqq A \wedge (\forall z \in \mathscr{D}(g))\,[z = x \vee z \neq x]$.

If $z = x$ then

$$g`x = F``(R^{-1})``\{x\} = \{F`y \mid y\, R\, x\}.$$

Furthermore $y \, R \, x$ implies $y \in a - \{x\}$ and hence $g`y = F`y$. Therefore

$$g`x = \{g`y \mid y \, R \, x\}.$$

If $z \neq x$ then since $z \in \mathscr{D}(g)$, $g`z = F`z$. But

$$F`z = \{F`y \mid y \, R \, z\}.$$

If $y \, R \, z$ and $z \in a$, then $y \in a$. Furthermore since z is connected to x by a finite R-chain and since R is well founded, $y \neq x$. Then $g`y = F`y$ and

$$g`z = \{g`y \mid y \, R \, z\}.$$

Since a is the domain of g and a is closed under R^{-1} it follows that $g \in K$. Hence $x \in a \subseteq \mathscr{D}(F)$. This is a contradiction from which we conclude that $\mathscr{D}(F) = A$.

Thus if $B = F``A$ then

$$F : A \xrightarrow[\text{onto}]{1-1} B.$$

Furthermore $a \in b \wedge b \in B$ implies that $(\exists \, x \in A)[a \in b \wedge b = F`x]$. But

$$F`x = \{F`y \mid y \, R \, x\}.$$

Thus $(\exists \, y)[y \, R \, x \wedge a = F`y]$, i.e., $a \in B$ and hence B is transitive.

Also $a \in A \wedge b \in A \wedge a \, R \, b \rightarrow F`a \in \{F`y \mid y \, R \, b\} = F`b$. Therefore

$$F \ \mathrm{Isom}_{R, E}(A, B).$$

We have now proved (1) and (2). Since (3) is the special case of (4) with $n = 0$ it is sufficient to prove (4). This we do by induction on the number, n, of logical symbols in φ. If $n = 0$, then φ is of the form $a \in b$ and

$$[A, R] \models a \in b \leftrightarrow a \in A \wedge b \in A \wedge a \, R \, b.$$

Since $F \ \mathrm{Isom}_{R, E}(A, B)$

$$a, b \in A \rightarrow [a \, R \, b \leftrightarrow F`a \in F`b].$$

But since $F`a \in B$ and $F`b \in B$ we have

$$a, b \in A \rightarrow [F`a \in F`b \leftrightarrow B \models F`a \in F`b].$$

Therefore

$$a, b \in A \rightarrow [[A, R] \models a \in b \leftrightarrow B \models F`a \in F`b].$$

If φ is of the form $\neg \psi$ and all of the free variables of φ are among $a_1, \ldots, a_n$ then so are the free variables of ψ. From the induction hypothesis if $a_1 \in A \wedge \cdots \wedge a_n \in A$ then

$$[A, R] \models \psi(a_1, \ldots, a_n) \leftrightarrow B \models \psi(F`a_1, \ldots, F`a_n),$$

$$\neg [A, R] \models \psi(a_1, \ldots, a_n) \leftrightarrow \neg B \models \psi(F`a_1, \ldots, F`a_n),$$

$$[A, R] \models \neg \psi(a_1, \ldots, a_n) \leftrightarrow B \models \neg \psi(F`a_1, \ldots, F`a_n).$$

If φ is of the form $\psi \wedge \eta$ and all of the free variables of φ are among $a_1, \ldots, a_n$ then so are the free variables of ψ and of η. From the induction hypothesis if $a_1 \in A \wedge \cdots \wedge a_n \in A$ then

$$[A, R] \models \psi(a_1, \ldots, a_n) \leftrightarrow B \models \psi(F`a_1, \ldots, F`a_n)$$

$$[A, R] \models \eta(a_1, \ldots, a_n) \leftrightarrow B \models \eta(F`a_1, \ldots, F`a_n).$$

Therefore

$$[A, R] \models \psi(a_1, \ldots, a_n) \wedge [A, R] \models \eta(a_1, \ldots, a_n) \leftrightarrow B \models \psi(F`a_1, \ldots, F`a_n) \wedge B \models \eta(F`a_1, \ldots, F`a_n).$$

Hence

$$[A, R] \models [\psi(a_1, \ldots, a_n) \wedge \eta(a_1, \ldots, a_n)] \leftrightarrow B \models [\psi(F`a_1, \ldots, F`a_n) \wedge \eta(F`a_1, \ldots, F`a_n)].$$

If φ is of the form $(\forall x)\psi(x)$ and if all of the free variables of φ are among $a_1, \ldots, a_n$ then there is an x not among $a_1, \ldots, a_n$ and all of the free variables of $\psi(x)$ are among $a_1, \ldots, a_n$, x. From the induction hypothesis if $a_1 \in A \wedge \cdots \wedge a_n \in A$ then

$$x \in A \rightarrow [[A, R] \models \psi(x, a_1, \ldots, a_n) \leftrightarrow B \models \psi(F`x, F`a_1, \ldots, F`a_n)].$$

From the self-distributive law for implication

$$[x \in A \rightarrow [A, R] \models \psi(x, a_1, \ldots, a_n)] \leftrightarrow [x \in A \rightarrow B \models \psi(F`x, F`a_1, \ldots, F`a_n)]$$

Since x is not among $a_1, \ldots, a_n$ we have on generalizing and distributing

$$[(\forall x)[x \in A \rightarrow [A, R] \models \psi(x, a_1, \ldots, a_n)] \leftrightarrow (\forall x)[x \in A \rightarrow B \models \psi(F`x, F`a_1, \ldots, F`a_n)].$$

Since F maps A one-to-one onto B

$$(\forall x)[x \in A \rightarrow B \models \psi(F`x, F`a_1, \ldots, F`a_n)] \leftrightarrow (\forall x)[x \in B \rightarrow B \models \psi(x, F`a_1, \ldots, F`a_n)].$$

Therefore

$$[A, R] \models (\forall x)\psi(x, a_1, \ldots, a_n) \leftrightarrow B \models (\forall x)\psi(x, F`a_1, \ldots, F`a_n). \qquad \square$$

CHAPTER 13
Absoluteness

A basic part of the interpretation of our theory is that each wff $\varphi(x)$ expresses a property that a given individual a has or does not have according as $\varphi(a)$ holds or does not hold. Then $\varphi^A(x)$ expresses the "same" or "corresponding" property for the universe A.

Consider, for example, the existence of an empty set. Earlier we proved that there exists an individual a, called the empty set, having the property

$$(\forall x)[x \notin a].$$

From the Axiom of Regularity it follows that every nonempty class A, as a universe, has this property. In particular, the class of infinite cardinal numbers N' contains an individual a with the property

$$(\forall x \in N')[x \notin a].$$

The set in N' that plays the role of the empty set is $\aleph_0$, a set that is far from empty. Thus when viewed from within the universe N', $\aleph_0$ is empty but when viewed from "without," i.e., in V, $\aleph_0$ is not empty.

There are however properties $\varphi(x)$ and universes A such that an individual of A has the property when viewed from within A iff it has the property when viewed from without. Such a property is said to be absolute with respect to A.

Definition 13.1.

(1) $\varphi \text{ Abs } A \overset{\triangle}{\leftrightarrow} [\varphi^A \leftrightarrow \varphi]$

if φ is closed.

(2) $\varphi \text{ Abs } A \overset{\triangle}{\leftrightarrow} a_1 \in A \wedge \cdots \wedge a_n \in A \rightarrow [\varphi^A \leftrightarrow \varphi]$

where $a_1, \ldots, a_n$ is a complete list of all of the free variables in φ.

Remark. We read φ Abs A as "φ is absolute with respect to A."

Proposition 13.2. *If* φ Abs $A \wedge \psi$ Abs A *then*

(1) $\varphi \wedge \psi$ Abs A,

(2) $\neg\varphi$ Abs A.

The proofs are left to the reader.

Remark. From Proposition 13.2 we see that if φ and ψ are each absolute with respect to A then $\varphi \vee \psi$, $\varphi \to \psi$, and $\varphi \leftrightarrow \psi$ are also absolute with respect to A. The interesting questions about absoluteness center around quantifiers.

Proposition 13.3. *If* φ Abs A, *if* ψ Abs A, *if* $a_1, \ldots, a_m, b_1, \ldots, b_n$ *is a list of distinct variables containing all of the free variables of* φ *and of* ψ, *and if*

$$b_1, \ldots, b_n \in A \wedge \varphi(a_1, \ldots, a_m, b_1, \ldots, b_n) \to a_1, \ldots, a_m \in A$$

then generalizing on $a_1, \ldots, a_m$,

$$(\forall x_1, \ldots, x_m)[\varphi \to \psi] \text{ Abs } A.$$

PROOF. Clearly if $b_1, \ldots, b_n \in A$ then

$$[(\forall x_1, \ldots, x_m)[\varphi \to \psi] \to (\forall x_1, \ldots, x_m \in A)[\varphi \to \psi]].$$

The formal details consist of observing that on the hypothesis

$$(\forall x_1, \ldots, x_m)[\varphi \to \psi]$$

we can deduce $\varphi \to \psi$ and hence

$$a_m \in A \to [\varphi \to \psi].$$

Then by generalization

$$(\forall x_m)[x_m \in A \to [\varphi \to \psi]].$$

By iteration

$$(\forall x_1, \ldots, x_m \in A)[\varphi \to \psi].$$

On the other hand we can deduce from the hypotheses

$$b_1, \ldots, b_n \in A, (\forall x_1, \ldots, x_m \in A)[\varphi \to \psi], \varphi$$

the following wffs

$$(\forall x_1, \ldots, x_m \in A)[\varphi \to \psi],$$

$$a_1 \in A \to (\forall x_2, \ldots, x_m \in A)[\varphi \to \psi].$$

But a basic hypothesis of our theorem is that under the hypotheses listed above

$$a_1 \in A.$$

Hence by modus ponens we can deduce

$$(\forall x_2, \ldots, x_m \in A)[\varphi \to \psi].$$

Repeating this we arrive finally at

$$\psi$$

from which one application of the deduction theorem gives that on the hypotheses

$$b_1, \ldots, b_n \in A, (\forall x_1, \ldots, x_m \in A)[\varphi \to \psi]$$

we can deduce

$$\varphi \to \psi.$$

Then by generalization we deduce

$$(\forall x_1) \ldots (\forall x_m)[\varphi \to \psi]$$

and finally, by the deduction theorem,

$$b_1, \ldots, b_n \in A \to [(\forall x_1, \ldots, x_m \in A)[\varphi \to \psi] \to (\forall x_1, \ldots, x_m)[\varphi \to \psi]].$$

We then have

$$b_1, \ldots, b_n \in A \to [(\forall x_1, \ldots, x_m)[\varphi \to \psi] \leftrightarrow (\forall x_1, \ldots, x_m \in A)[\varphi \to \psi]].$$

Since φ and ψ are each absolute w.r.t. A

$$b_1, \ldots, b_n \in A \wedge a_1, \ldots, a_m \in A \to [\varphi \leftrightarrow \varphi^A],$$

$$b_1, \ldots, b_n \in A \wedge a_1, \ldots, a_m \in A \to [\psi \leftrightarrow \psi^A].$$

Hence by generalization, properties of equivalence, and the self-distributive law, if $b_1, \ldots, b_n \in A$ then

$$[(\forall x_1, \ldots, x_m \in A)[\varphi \to \psi] \leftrightarrow (\forall x_1, \ldots, x_m \in A)[\varphi^A \to \psi^A]].$$

We then conclude that if $b_1, \ldots, b_n \in A$,

$$[(\forall x_1, \ldots, x_m)[\varphi \to \psi] \leftrightarrow (\forall x_1, \ldots, x_m \in A)[\varphi^A \to \psi^A]]. \qquad \square$$

Corollary 13.4. *If φ Abs A, if ψ Abs A, if $a_1, \ldots, a_m, b_1, \ldots, b_n$ is a list of distinct variables containing all of the free variables in φ and in ψ, and if*

$$b_1, \ldots, b_n \in A \wedge \varphi(a_1, \ldots, a_m, b_1, \ldots, b_n) \to a_1, \ldots, a_m \in A,$$

$$b_1, \ldots, b_n \in A \wedge \psi(a_1, \ldots, a_m, b_1, \ldots, b_n) \to a_1, \ldots, a_m \in A$$

then generalizing on $a_1, \ldots, a_m$, $(\forall x_1, \ldots, x_m)[\varphi \leftrightarrow \psi]$ Abs A.

The proof is left to the reader.

Proposition 13.5. *If* φ Abs A, *if* $a_1, \ldots, a_m, b_1, \ldots, b_n$ *is a list of distinct variables containing all of the free variables in* φ *and if*

$$b_1, \ldots, b_n \in A \wedge \varphi(a_1, \ldots, a_m, b_1, \ldots, b_n) \to a_1, \ldots, a_m \in A$$

then quantifying on $a_1 \ldots a_m$,

$$(\exists x_1, \ldots, x_m)\varphi \text{ Abs } A.$$

Proof. Since

$$b_1, \ldots, b_n \in A \wedge \varphi(a_1, \ldots, a_m, b_1, \ldots, b_n) \to a_1, \ldots, a_m \in A$$

we have that

$$b_1, \ldots, b_n \in A \to [(\exists x_1, \ldots, x_m)\varphi \leftrightarrow (\exists x_1, \ldots, x_m \in A)\varphi].$$

Since φ is absolute w.r.t. A

$$b_1, \ldots, b_n \in A \wedge a_1, \ldots, a_m \in A \to [\varphi \leftrightarrow \varphi^A].$$

Therefore if $b_1, \ldots, b_n \in A$ then

$$[(\exists x_1, \ldots, x_m \in A)\varphi \leftrightarrow (\exists x_1, \ldots, x_m \in A)\varphi^A]$$

and hence

$$[(\exists x_1, \ldots, x_m)\varphi \leftrightarrow (\exists x_1, \ldots, x_m \in A)\varphi^A]. \qquad \square$$

Proposition 13.6. *If* $\vdash [\varphi \leftrightarrow \psi]$ *and if* A *is a nonempty class that satisfies each nonlogical axiom in some proof of* $\varphi \leftrightarrow \psi$ *then*

$$\varphi \text{ Abs } A \leftrightarrow \psi \text{ Abs } A.$$

Proof. If all of the free variables of φ and of ψ are among $b_1, \ldots, b_n$ then by Theorem 12.6

$$b_1, \ldots, b_n \in A \to [\varphi^A \leftrightarrow \psi^A].$$

Therefore if $b_1, \ldots, b_n \in A$

$$[\varphi \leftrightarrow \varphi^A] \leftrightarrow [\psi \leftrightarrow \psi^A]. \qquad \square$$

Theorem 13.7. *If* $\vdash (\exists x)\varphi(x)$, *if* A *is a nonempty class that satisfies each nonlogical axiom in some proof of* $(\exists x)\varphi(x)$, *and if* $\varphi(x)$ Abs A then

$$(\exists x)\varphi(x) \text{ Abs } A.$$

Proof. If all of the free variables of $(\exists x)\varphi(x)$ are among $a_1, \ldots, a_n$ then by Theorem 12.6

$$a_1, \ldots, a_n \in A \to (\exists x \in A)\varphi^A(x).$$

Then

$$a_1, \ldots, a_n \in A \to [(\exists x)\varphi(x) \to (\exists x \in A)\varphi^A(x)].$$

Furthermore there exists an x distinct from $a_1, \ldots, a_n$. Then since $\varphi(x)$ is absolute with respect to A.

$$a_1, \ldots, a_n \in A \ \wedge \ x \in A \to [\varphi(x) \leftrightarrow \varphi^A(x)].$$

Therefore

$$\begin{aligned} a_1, \ldots, a_n \in A &\to [(\exists\, x \in A)\varphi(x) \leftrightarrow (\exists\, x \in A)\varphi^A(x)] \\ &\to [(\exists\, x \in A)\varphi^A(x) \to (\exists\, x)\varphi(x)]. \end{aligned}$$

□

Theorem 13.8. *If* $\vdash [\varphi \leftrightarrow (\forall\, x)\psi(x)]$ *and* $\vdash [\varphi \leftrightarrow (\exists\, x)\eta(x)]$, *if* A *is a nonempty class that satisfies each nonlogical axiom in some proof of* $\varphi \leftrightarrow (\forall\, x)\psi(x)$ *and some proof of* $\varphi \leftrightarrow (\exists\, x)\eta(x)$, *if* $\psi(x)$ Abs A *and if* $\eta(x)$ Abs A *then*

$$\varphi \text{ Abs } A.$$

Proof. If all of the free variables of φ, $(\forall\, x)\psi(x)$, and $(\exists\, x)\eta(x)$ are among $a_1, \ldots, a_n$ then by Theorem 12.6

$$a_1, \ldots, a_n \in A \to [\varphi^A \leftrightarrow (\forall\, x \in A)\psi^A(x)].$$
$$a_1, \ldots, a_n \in A \to [\varphi^A \leftrightarrow (\exists\, x \in A)\eta^A(x)].$$

Also since $\psi(x)$ and $\eta(x)$ are absolute

$$a_1, \ldots, a_n \in A \ \wedge \ x \in A \to [\psi(x) \leftrightarrow \psi^A(x)],$$
$$a_1, \ldots, a_n \in A \ \wedge \ x \in A \to [\eta(x) \leftrightarrow \eta^A(x)].$$

From this, choosing x distinct from $a_1, \ldots, a_n$

$$a_1, \ldots, a_n \in A \to [(\forall\, x \in A)\psi(x) \leftrightarrow (\forall\, x \in A)\psi^A(x)],$$
$$a_1, \ldots, a_n \in A \to [(\exists\, x \in A)\eta(x) \leftrightarrow (\exists\, x \in A)\eta^A(x)].$$

Then since $\varphi \leftrightarrow (\forall\, x)\psi(x)$ is a theorem

$$\begin{aligned} a_1, \ldots, a_n \in A \ \wedge \ \varphi &\to (\forall\, x)\psi(x) \\ &\to (\forall\, x \in A)\psi(x) \\ &\to (\forall\, x \in A)\psi^A(x) \\ &\to \varphi^A. \end{aligned}$$

Also

$$\begin{aligned} a_1, \ldots, a_n \in A \ \wedge \ \varphi^A &\to (\exists\, x \in A)\eta^A(x) \\ &\to (\exists\, x \in A)\eta(x) \\ &\to (\exists\, x)\eta(x) \\ &\to \varphi. \end{aligned}$$

□

Remark. Satisfaction and absoluteness have been defined for wffs. Most of our theorems in ZF are however wffs in the wider sense (Definition 4.1). It is therefore convenient to extend our definitions to wffs in the wider sense.

Definition 13.9. If φ is a wff in the wider sense then

(1) $[A, R] \models \varphi \overset{\Delta}{\leftrightarrow} [A, R] \models \varphi^*$,

(2) $\varphi^A \triangleq (\varphi^*)^A$,

(3) φ Abs $A \overset{\Delta}{\leftrightarrow} \varphi^*$ Abs A.

It would however be helpful to be able to determine the absoluteness of wffs in the wider sense without first reducing them to primitive terms. For this purpose the following substitution theorem is useful.

Proposition 13.10. *If* $A \neq 0$, *if* $\varphi(b_1, \ldots, b_n)$ Abs A *and if* $\mathscr{M}(B_1) \wedge \cdots \wedge \mathscr{M}(B_n) \wedge b_1 = B_1$ Abs $A \wedge \cdots \wedge b_n = B_n$ Abs A *then* $\varphi(B_1, \ldots, B_n)$ Abs A.

PROOF (By induction on n). If $n = 1$ then

$$\begin{aligned}\varphi(B_1) &\leftrightarrow (\forall x_1)[x_1 = B_1 \to \varphi(b_1)]\\ &\leftrightarrow (\exists x_1)[x_1 = B_1 \wedge \varphi(b_1)].\end{aligned}$$

If $\varphi(b_1)$ Abs A and $b_1 = B_1$ Abs A then $[b_1 = B_1 \to \varphi(b_1)]$ Abs A and $[b_1 = B_1 \wedge \varphi(b_1)]$ Abs A. Then by Theorem 13.8, $\varphi(B_1)$ Abs A.

The induction step is obvious and hence omitted. □

Definition 13.11.

(1) $x^A \triangleq x \cap A$.

(2) $\{x \mid \varphi(x)\}^A \triangleq \{x \in A \mid \varphi^A(x)\}$.

Proposition 13.12. *If A is transitive and $x \in A$ then*

(1) $x^A = x$,

(2) $[x \in y]^A \leftrightarrow x^A \in y^A$,

(3) $[[x \in \{y \mid \varphi(y)\}]^A \leftrightarrow [x^A \in \{y \mid \varphi(y)\}^A]]$,

(4) $[[\{y \mid \varphi(y)\} \in x]^A \leftrightarrow [\{y \mid \varphi(y)\}^A \in x^A]]$,

(5) $[\{x \mid \varphi(x)\} \in \{y \mid \psi(y)\}]^A \leftrightarrow [\{x \mid \varphi(x)\}^A \in \{y \mid \psi(y)\}^A]$.

PROOF. (1) If A is transitive and $x \in A$, then $x \subseteq A$. Therefore $x^A = x \cap A = x$.

(2) Obvious from (1).

(3) $[x \in \{y \mid \varphi(y)\}]^* \leftrightarrow \varphi(x)$.

Hence

$$[x \in \{y \mid \varphi(y)\}]^A \leftrightarrow \varphi^A(x).$$

Then $x \in A$ implies

$$\begin{aligned}[x \in \{y \mid \varphi(y)\}]^A &\leftrightarrow x \in A \wedge \varphi^A(x) \\ &\leftrightarrow x \in \{y \in A \mid \varphi^A(y)\} \\ &\leftrightarrow x^A \in \{y \mid \varphi(y)\}^A.\end{aligned}$$

(4) $[\{y \mid \varphi(y)\} \in x]^* \leftrightarrow (\exists z)[z \in x \wedge (\forall y)[y \in z \leftrightarrow \varphi(y)]]$.

Hence

$$[\{y \mid \varphi(y)\} \in x]^A \leftrightarrow (\exists z \in A)[z \in x \wedge (\forall y \in A)[y \in z \leftrightarrow \varphi^A(y)]].$$

Then A transitive and $x \in A$ implies

$$\begin{aligned}[\{y \mid \varphi(y)\} \in x]^A &\leftrightarrow (\exists z \in A)[z \in x \wedge (\forall y \in A)[y \in z \leftrightarrow \varphi^A(y)]] \\ &\leftrightarrow (\exists z \in A)[z \in x \wedge z = \{y \in A \mid \varphi^A(y)\}] \\ &\leftrightarrow \{y \in A \mid \varphi^A(y)\} \in x \\ &\leftrightarrow \{y \mid \varphi(y)\}^A \in x^A.\end{aligned}$$

(5) $[\{x \mid \varphi(x)\} \in \{y \mid \psi(y)\}]^*$

$$\leftrightarrow (\exists z)[(\forall x)[x \in z \leftrightarrow \varphi(x)] \wedge z \in \{y \mid \psi(y)\}].$$

Therefore from (1) and (3)

$$\begin{aligned}&[\{x \mid \varphi(x)\} \in \{y \mid \psi(y)\}]^A \\ &\quad\leftrightarrow (\exists z \in A)(\forall x \in A)[x \in z \leftrightarrow \varphi^A(x) \wedge z \in \{y \mid \psi(y)\}^A] \\ &\quad\leftrightarrow (\exists z \in A)[z = \{x \in A \mid \varphi^A(x)\} \wedge z \in \{y \mid \psi(y)\}^A] \\ &\quad\leftrightarrow \{x \in A \mid \varphi^A(x)\} \in \{y \mid \psi(y)\}^A \\ &\quad\leftrightarrow \{x \mid \varphi(x)\}^A \in \{y \mid \psi(y)\}^A.\end{aligned}$$

□

Definition 13.13.

(1) $B \text{ Abs } A \overset{\triangle}{\leftrightarrow} [B^A = B]$

if B is a term containing no free variables.

(2) $B \text{ Abs } A \overset{\triangle}{\leftrightarrow} a_1, \ldots, a_n \in A \to [B^A = B]$

where $a_1, \ldots, a_n$ is a complete list of all the free variables in B.

Proposition 13.14. *If $\varphi(x)$ Abs A and if all of the free variables of $\varphi(x)$ are among $a_1, \ldots, a_n, x$ then*

$$a_1, \ldots, a_n \in A \to [\{x \mid \varphi(x)\}^A = \{x \mid \varphi(x)\} \cap A].$$

PROOF. If $a_1, \ldots, a_n \in A$ then since $\varphi(x)$ is absolute with respect to A

$$\begin{aligned} x \in \{x \mid \varphi(x)\}^A &\leftrightarrow x \in A \wedge \varphi^A(x) \\ &\leftrightarrow x \in A \wedge \varphi(x) \\ &\leftrightarrow x \in \{x \mid \varphi(x)\} \cap A. \end{aligned}$$ □

Remark. The class $\{x \mid \varphi(x)\}^A$ is the class $\{x \mid \varphi(x)\}$ relativized to A. By definition $\{x \mid \varphi(x)\}^A$ is the class of individuals in A for which $\varphi^A(x)$ holds. From Proposition 13.14 we see that if $\varphi(x)$ is absolute with respect to A then $\{x \mid \varphi(x)\}^A$ is simply the class of individuals in A for which $\varphi(x)$ holds.

Proposition 13.15. *If A is nonempty and transitive and if $\{y \mid \varphi(y)\}$ is a set, then*

$$[a = \{y \mid \varphi(y)\}] \text{ Abs } A \qquad \textit{iff} \quad \{y \mid \varphi(y)\} \text{ Abs } A.$$

PROOF. If all of the free variables of $\varphi(y)$ are among $a_1, \ldots, a_n$, y then

$$\begin{aligned} &[a = \{y \mid \varphi(y)\}] \text{ Abs } A \\ &\leftrightarrow a_1, \ldots, a_n, a \in A \rightarrow [(\forall y)[y \in a \leftrightarrow \varphi(y)] \\ &\qquad \leftrightarrow (\forall y \in A)[y \in a \leftrightarrow \varphi^A(y)]] \\ &\leftrightarrow a_1, \ldots, a_n, a \in A \rightarrow [(\forall y)[y \in a \leftrightarrow \varphi(y)] \\ &\qquad \leftrightarrow (\forall y)[y \in a \leftrightarrow y \in A \wedge \varphi^A(y)]] \\ &\leftrightarrow a_1, \ldots, a_n, a \in A \rightarrow [a = \{y \mid \varphi(y)\} \leftrightarrow a = \{y \in A \mid \varphi^A(y)\}] \\ &\leftrightarrow a_1, \ldots, a_n \in A \rightarrow [\{y \mid \varphi(y)\} = \{y \mid \varphi(y)\}^A] \\ &\leftrightarrow \{y \mid \varphi(y)\} \text{ Abs } A. \end{aligned}$$ □

Remark. We turn now to the problem of establishing the absoluteness properties of certain wffs and terms. Our ultimate goal is to find conditions on A that will assure us that A is a standard transitive model of ZF.

Proposition 13.16. *If φ is quantifier free then φ* Abs A.

PROOF. If φ is quantifier free then $\varphi^A \leftrightarrow \varphi$. □

Proposition 13.17. $a \in b$ Abs A.

PROOF. The formula $a \in b$ is quantifier free. □

Proposition 13.18. *If A is nonempty and transitive then*

(1) $a \subseteq b$ Abs A,

(2) $a = b$ Abs A.

PROOF.

(1) $a \subseteqq b \leftrightarrow (\forall x)[x \in a \rightarrow x \in b]$.

Since A is transitive $a \in A$ implies $a \subseteqq A$, i.e., $x \in a \wedge a \in A$ implies $x \in A$. From Propositions 13.17, 13.3, and 13.6 it then follows that

$$a \subseteqq b \text{ Abs } A.$$

(2) $a = b \leftrightarrow a \subseteqq b \wedge b \subseteqq a$. □

Remark. The requirement in Proposition 13.18 that A be transitive cannot be dropped. For example if $A = \{0, 1, \{0, 1, 2\}, \{0, 1, 3\}\}$ then from an internal vantage point the sets $\{0, 1, 2\}$ and $\{0, 1, 3\}$ are indistinguishable, i.e.,

$$(\forall x \in A)[x \in \{0, 1, 2\} \leftrightarrow x \in \{0, 1, 3\}].$$

Since the membership property is absolute with respect to any class (Proposition 13.17) if $b \in A$ then those individuals in A that play the role of elements of b are individuals in V that *are* elements in b. But not conversely. In the foregoing example we have, relative to A

$$0 \underset{A}{\in} \{0, 1, 2\}, \qquad 1 \underset{A}{\in} \{0, 1, 2\}, \qquad 2 \underset{A}{\notin} \{0, 1, 2\}.$$

Similarly with subsets, if A is a nonempty transitive class then containment is absolute with respect to A. This means that if $b \in A$ then every element of A that is a subset of b relative to A is a subset of b in the "real" universe V. But not conversely. There may be a subset of b that is not an element of A. Indeed if A is transitive but not supertransitive there must be at least one element of A having a subset that is not in A.

Proposition 13.19. *If A is nonempty and transitive then*

(1) 0 Abs A,

(2) $[a \cup b]$ Abs A,

(3) $\{a, b\}$ Abs A,

(4) $\cup(a)$ Abs A,

(5) $[a - b]$ Abs A.

PROOF. (1) Since $a \neq a$ Abs A we have

$$0^A = \{x \in A \mid [x \neq x]^A\} = \{x \in A \mid x \neq x\} = 0.$$

(2) If $a, b \in A$ then

$$\begin{aligned}[a \cup b]^A &= \{x \in A \mid [x \in a \vee x \in b]^A\}\\ &= \{x \in A \mid x \in a \vee x \in b\}\\ &= \{x \mid x \in a \vee x \in b\}\\ &= a \cup b.\end{aligned}$$

(3) If $a, b \in A$ then

$$\begin{aligned}\{a, b\}^A &= \{x \in A \mid [x = a \vee x = b]^A\}\\ &= \{x \mid x = a \vee x = b\}\\ &= \{a, b\}.\end{aligned}$$

(4) If $a \in A$ then

$$\begin{aligned}[\cup(a)]^A &= \{x \in A \mid (\exists y \in A)[x \in y \wedge y \in a]^A\}\\ &= \{x \in A \mid (\exists y \in A)[x \in y \wedge y \in a]\}\\ &= \{x \mid (\exists y)[x \in y \wedge y \in a]\}\\ &= \cup(a).\end{aligned}$$

(5) If $a, b \in A$ then

$$\begin{aligned}[a - b]^A &= \{x \in A \mid [x \in a \wedge x \notin b]^A\}\\ &= \{x \in A \mid x \in a \wedge x \notin b\}\\ &= a - b.\end{aligned}$$

□

Remark. The proofs of several of the theorems to follow are similar to the proof of Proposition 13.18 involving repeated applications of foregoing theorems on absoluteness. To avoid rather dull repetitions we omit most of the details.

Proposition 13.20. *If A is nonempty and transitive then*

(1) $\mathrm{Tr}(a)$ Abs A,

(2) $\mathrm{Ord}(a)$ Abs A.

PROOF.

(1) $\mathrm{Tr}(a) \leftrightarrow (\forall x)[x \in a \to x \subset a]$.

(2) $\mathrm{Ord}(a) \leftrightarrow \mathrm{Tr}(a) \wedge (\forall x, y)[x \in a \wedge y \in a \to x \in y \vee x = y \vee y \in x]$. □

Remark. Proposition 13.20 assures us that restricting the definition of ordinal number to a nonempty transitive class does not enable any new objects to qualify as ordinals. Consequently if A is a standard transitive model of ZF then the class of ordinals "in" A is a subclass of On, i.e.,

$$\mathrm{On}^A = \{x \in A \mid [\mathrm{Ord}(x)]^A\} = \{x \in A \mid \mathrm{Ord}(x)\} = \mathrm{On} \cap A.$$

Proposition 13.21. *If A is nonempty and transitive then*

(1) $[a \in \omega]$ Abs A,

(2) $[a = \omega]$ Abs A if $\omega \subseteq A$.

PROOF.

(1) $a \in \omega \leftrightarrow a \cup \{a\} \subseteq K_1$

$$\leftrightarrow (\forall x)[x \in a \vee x = a \rightarrow x \in K_I]$$

$$\leftrightarrow (\forall x)[x \in a \vee x = a \rightarrow x = 0 \vee (\exists y)[\mathrm{Ord}(y) \wedge x = y \cup \{y\}]].$$

(2) $a = \omega \leftrightarrow (\forall x)[x \in a \leftrightarrow x \in \omega]$. □

Proposition 13.22. *If A is nonempty and transitive then*

(1) $[\alpha < \beta]$ Abs A,

(2) $[\alpha = \beta]$ Abs A,

(3) $[\gamma = \max(\alpha, \beta)]$ Abs A.

PROOF.

(1) $[\alpha < \beta] \leftrightarrow \mathrm{Ord}(\alpha) \wedge \mathrm{Ord}(\beta) \wedge \alpha \in \beta$.

(2) $[\alpha = \beta] \leftrightarrow \mathrm{Ord}(\alpha) \wedge \mathrm{Ord}(\beta) \wedge \alpha = \beta$.

(3) $[\gamma = \max(\alpha, \beta)] \leftrightarrow \mathrm{Ord}(\gamma) \wedge \mathrm{Ord}(\alpha) \wedge \mathrm{Ord}(\beta) \wedge \gamma = \alpha \cup \beta$. □

Proposition 13.23. *If A is nonempty and transitive then*

(1) $[\langle\alpha, \beta\rangle \,\mathrm{Le}\, \langle\gamma, \delta\rangle]$ Abs A,

(2) $[\langle\alpha, \beta\rangle \, R_0 \langle\gamma, \delta\rangle]$ Abs A.

PROOF.

(1) $[\langle\alpha, \beta\rangle \,\mathrm{Le}\, \langle\gamma, \delta\rangle] \leftrightarrow [\alpha < \gamma \vee [\alpha = \gamma \wedge \beta < \delta]]$.

(2) $[\langle\alpha, \beta\rangle \, R_0 \langle\gamma, \delta\rangle] \leftrightarrow [\max(\alpha, \beta) < \max(\gamma, \delta) \vee$

$$[\max(\alpha, \beta) = \max(\gamma, \delta) \wedge \langle\alpha, \beta\rangle \,\mathrm{Le}\, \langle\gamma, \delta\rangle]].$$ □

EXERCISES

In Exercises 1–29 determine whether or not the given predicate is absolute with respect to A, A being nonempty and transitive.

(1) $\mathscr{M}(a)$.

(2) $\mathscr{P}r(a)$.

(3) $\mathscr{R}el(a)$.

(4) $\mathscr{U}n(a)$.

(5) $\mathscr{U}n_2(a)$.

(6) $\mathscr{F}nc(a)$.

(7) $\mathscr{F}nc_2(a)$.

(8) $a \mathscr{F}n\, b$.

(9) $a \mathscr{F}n_2\, b$.

(10) r Fr a.

(11) r Wfr a.

(12) r We a.

(13) $f : a \rightarrow b$.

(14) $f : a \xrightarrow{1-1} b$.

(15) $f : a \xrightarrow[\text{onto}]{1-1} b$.

(16) $f : a \xrightarrow[\text{onto}]{} b$.

(17) $f \ \mathrm{Isom}_{r_1, r_2}(a_1, a_2)$.

(18) $\mathrm{Orf}(f)$.

(19) $\mathrm{Smo}(f)$.

(20) $a \simeq b$.

(21) $\mathrm{Fin}(a)$.

(22) $\mathrm{Inf}(a)$.

(23) $\mathrm{cof}(\alpha, \beta)$.

(24) $\mathrm{Reg}(\alpha)$.

(25) $\mathrm{Inacc}_w(\aleph_\alpha)$.

(26) $\mathrm{Inacc}(\aleph_\alpha)$.

(27) $\mathrm{Cl}(r, a)$.

(28) $\mathrm{Cl}_2(r, a)$.

(29) $\mathrm{St}(a)$.

In Exercise 30–55 determine whether or not the given term is absolute with respect to A, A nonempty and transitive.

(30) $x = a \cap b$.

(31) $x = \mathscr{P}(a)$.

(32) $x = \cap(a)$.

(33) $x = a \times b$.

(34) $x = a^{-1}$.

(35) $x = \mathscr{D}(a)$.

(36) $x = \mathscr{W}(a)$.

(37) $x = a \upharpoonright b$.

(38) $x = a``b$.

(39) $x = a \circ b$.

(40) $x = a`b$.

(41) $\gamma = \alpha + 0$.

(42) $\gamma = \alpha + 1$.

(43) $\gamma = \alpha + \beta$.

(44) $\gamma = \alpha \cdot 0$.

(45) $\gamma = \alpha \cdot 1$.

(46) $\gamma = \alpha \cdot \beta$.

(47) $\gamma = \alpha^0$.

(48) $\gamma = \alpha^1$.

(49) $\gamma = \alpha^\beta$.

(50) $\alpha = \bar{\bar{a}}$.

(51) $f = \aleph$.

(52) $c = a^b$.

(53) $\beta = \mathrm{cf}(\alpha)$.

(54) $x = R`_1\alpha$.

(55) $\alpha = \mathrm{rank}(x)$.

Remark. We turn now to an investigation of conditions on a class A that are necessary for A to be a model of ZF, that is, for A to be a model of the following wffs.

Axiom 1 (Extensionality). $(\forall x, y, z)[x = y \wedge x \in z \rightarrow y \in z]$.

Axiom 2 (Pairing). $(\forall x, y)\mathscr{M}(\{x, y\})$.

Axiom 3 (Unions). $(\forall x)\mathscr{M}(\cup(x))$.

Axiom 4 (Powers). $(\forall x)\mathscr{M}(\mathscr{P}(x))$.

Axiom 5 (Schema of Replacement).

$$(\forall x)[(\forall y, z, w)[\varphi(y, z) \wedge \varphi(y, w) \rightarrow z = w] \rightarrow \mathscr{M}(\{z \mid (\exists y \in x)\varphi(y, z)\})].$$

Axiom 6 (Regularity). $(\forall x)[x \neq 0 \rightarrow (\exists y)[y \in x \wedge y \cap x = 0]]$.

Axiom 7 (Infinity). $\mathscr{M}(\omega)$.

Proposition 13.24. $A \neq 0 \wedge \mathrm{Tr}(A) \rightarrow \mathrm{STM}(A, \mathrm{Ax.1})$.

PROOF. Since Axiom 1 assures us that

(1) $x = y \wedge x \in z \rightarrow y \in z$

holds for all x, y, and z, it also assures us that (1) holds for all $x, y, z \in A$, that is

$$(\forall x, y, z \in A)[x = y \wedge x \in z \rightarrow y \in z].$$

But since (1) is absolute with respect to A, it follows that

$$(\forall x, y, z \in A)[x = y \wedge x \in z \rightarrow y \in z]^A.$$

But this is Axiom 1 relativized to A, i.e., $[\mathrm{Ax.}\ 1]^A$.

Since Axiom 1 is closed we have from Proposition 12.5

$$A \models \mathrm{Ax.}\ 1 \leftrightarrow [\mathrm{Ax.}\ 1]^A.$$

Hence A satisfies Axiom 1, i.e.,

$$A \models \mathrm{Ax.}\ 1.$$

Since A is nonempty and transitive we conclude that A is a standard transitive model of Axiom 1. □

Proposition 13.25. *If A is nonempty and transitive then A is a standard transitive model of the Axiom of Pairing iff*

$$(\forall x, y \in A)[\{x, y\} \in A].$$

PROOF. Ax. $2 \leftrightarrow (\forall x, y)(\exists z)[z = \{x, y\}]$. Since A is nonempty and transitive A is a standard transitive model of Axiom 2 if and only if

(1) $(\forall x, y \in A)(\exists z \in A)[z = \{x, y\}]^A$.

However since $c = \{a, b\}$ is absolute w.r.t. A, (1) holds if and only if

$$(\forall x, y \in A)(\exists z \in A)[z = \{x, y\}]$$

that is, (1) holds if and only if

$$(\forall x, y \in A)[\{x, y\} \in A].$$ □

Remark. Earlier we proved that the empty set, the union of two sets, unordered pairs, and the union of a set are absolute terms with respect to a nonempty transitive class A. Curiously ordered pairs are not absolute with respect to such classes.

Suppose that A is the nonempty transitive class $\{0, 1\}$ then

$$\begin{aligned}[\langle 0, 1\rangle]^A &= \{x \in A \mid [x = \{0\} \vee x = \{0, 1\}]^A\} \\ &= \{x \in A \mid x = \{0\} \vee x = \{0, 1\}\} = \{\{0\}\} = \langle 0, 0\rangle.\end{aligned}$$

This pathology disappears if A is a nonempty transitive model of the Axiom of Pairing.

Proposition 13.26. STM(A, Ax. 2) $\rightarrow \langle a, b\rangle$ Abs A.

Proof. If A is a standard transitive model of Axiom 2 and if $a, b \in A$, then $\{a\} \in A$ and $\{a, b\} \in A$. Then

$$\begin{aligned}[\langle a, b\rangle]^A &= \{x \in A \mid [x = \{a\} \vee x = \{a, b\}]^A\} \\ &= \{x \in A \mid x = \{a\} \vee x = \{a, b\}\} \\ &= \{x \mid x = \{a\} \vee x = \{a, b\}\} \\ &= \langle a, b\rangle.\end{aligned}$$

□

Remark. From Proposition 13.26 and the substitution property, Proposition 13.10 it follows that ordered triples, ordered quadruples, etc. are absolute with respect to standard transitive models of the Axiom of Pairing.

Proposition 13.27. *If A is a nonempty, transitive model of the Axiom of Pairing then*

(1) $\mathscr{R}el(a)$ Abs A,
(2) $\mathscr{U}n(a)$ Abs A,
(3) $\mathscr{U}n_2(a)$ Abs A,
(4) $\mathscr{F}nc(a)$ Abs A,
(5) $\mathscr{F}nc_2(a)$ Abs A.

Proof.

(1) $\mathscr{R}el(a) \leftrightarrow (\forall x)[x \in a \rightarrow (\exists y, z)[x = \langle y, z\rangle]]$.
(2) $\mathscr{U}n(a) \leftrightarrow (\forall x, y, z)[\langle x, y\rangle, \langle x, z\rangle \in a \rightarrow y = z]$.
(3) $\mathscr{U}n_2(a) \leftrightarrow \mathscr{U}n(a) \wedge$
$(\forall x, y, z)[\langle x, z\rangle, \langle y, z\rangle \in a \rightarrow x = y]$.
(4) $\mathscr{F}nc(a) \leftrightarrow \mathscr{R}el(a) \wedge \mathscr{U}n(a)$.
(5) $\mathscr{F}nc_2(a) \leftrightarrow \mathscr{R}el(a) \wedge \mathscr{U}n_2(a)$.

□

Theorem 13.28. *If A is a nonempty transitive model of the Axiom of Pairing then*

(1) $a \times b$ Abs A,

(2) a^{-1} Abs A,

(3) $\mathscr{D}(a)$ Abs A,

(4) $\mathscr{W}(a)$ Abs A,

(5) $a`b$ Abs A,

(6) $a``b$ Abs A,

(7) $a \upharpoonright b$ Abs A.

PROOF. If $a, b \in A$ then

(1)
$$\begin{aligned}[a \times b]^A &= \{x \in A \mid (\exists\, y, z \in A)[y \in a \wedge z \in b \wedge x = \langle y, z\rangle]^A\}\\ &= \{x \in A \mid (\exists\, y, z \in A)[y \in a \wedge z \in b \wedge x = \langle y, z\rangle]\}\\ &= \{x \mid (\exists\, y, z)[y \in a \wedge z \in b \wedge x = \langle y, z\rangle]\}\\ &= a \times b.\end{aligned}$$

(2)
$$\begin{aligned}[a^{-1}]^A &= \{x \in A \mid (\exists\, y, z \in A)[x = \langle y, z\rangle \wedge \langle z, y\rangle \in a]^A\}\\ &= \{x \in A \mid (\exists\, y, z \in A)[x = \langle y, z\rangle \wedge \langle z, y\rangle \in a]\}\\ &= \{x \in A \mid (\exists\, y, z)[x = \langle y, z\rangle \wedge \langle z, y\rangle \in a]\}\\ &= \{x \mid (\exists\, y, z)[x = \langle y, z\rangle \wedge \langle z, y\rangle \in a]\}\\ &= a^{-1}.\end{aligned}$$

(3)
$$\begin{aligned}[\mathscr{D}(a)]^A &= \{x \in A \mid (\exists\, y \in A)[\langle x, y\rangle \in a]^A\}\\ &= \{x \in A \mid (\exists\, y \in A)[\langle x, y\rangle \in a]\}\\ &= \{x \mid (\exists\, y)[\langle x, y\rangle \in a]\}\\ &= \mathscr{D}(a).\end{aligned}$$

(4)
$$[\mathscr{W}(a)]^A = \{x \in A \mid (\exists\, y \in A)[\langle y, x\rangle \in a]^A\} = \mathscr{W}(a).$$

(5)
$$\begin{aligned}[a`b]^A &= \{x \in A \mid (\exists\, y \in A)[x \in y \wedge \langle b, y\rangle \in a]^A\\ &\qquad \wedge (\exists!\, y \in A)[\langle b, y\rangle \in a]^A\}\\ &= \{x \mid (\exists\, y)[x \in y \wedge \langle b, y\rangle \in a] \wedge (\exists!\, y)[\langle b, y\rangle \in a]\}.\\ &= a`b.\end{aligned}$$

(6)
$$\begin{aligned}[a``b]^A &= \{x \in A \mid (\exists\, y \in A)[y \in b \wedge \langle y, x\rangle \in a]^A\}\\ &= \{x \mid (\exists\, y)[y \in b \wedge \langle y, x\rangle \in a]\} = a``b.\end{aligned}$$

(7) The proof is left to the reader. ☐

Proposition 13.29. *If A is a nonempty transitive model of the Axiom of Pairing then*

(1) $f\ \mathscr{F}n\ a$ Abs A,

(2) $f\ \mathscr{F}n_2\ a$ Abs A.

PROOF

(1) $f\ \mathscr{F}n\ a \leftrightarrow \mathscr{F}nc(f) \wedge \mathscr{D}(f) = a$.

(2) $f\ \mathscr{F}n_2\ a \leftrightarrow \mathscr{F}nc_2(f) \wedge \mathscr{D}(f) = a$. □

Proposition 13.30. *If A is a nonempty transitive model of the Axiom of Pairing and if $a\ R_1\ b$ and $a\ R_2\ b$ are each absolute with respect to A then* f $\text{Isom}_{R_1, R_2}(a, b)$ Abs A.

PROOF

$$f\,\text{Isom}_{R_1, R_2}(a, b) \leftrightarrow f\ \mathscr{F}n_2\, a \wedge \mathscr{W}(f) = b$$
$$\wedge\ (\forall x, y)[x \in a \wedge y \in a \wedge \langle x, y\rangle \in R_1 \rightarrow \langle f`x, f`y\rangle \in R_2].$$ □

Theorem 13.31. *If A is nonempty and transitive then A is a standard transitive model of the Axiom of Unions iff*

$$(\forall x \in A)[\cup(x) \in A].$$

PROOF. Ax. 3 $\leftrightarrow (\forall x)(\exists y)[y = \cup(x)]$.

Since A is nonempty and transitive we have

$$\text{STM}(A, \text{Ax. } 3)$$

if and only if

$$(\forall x \in A)(\exists y \in A)[y = \cup(x)]^A.$$

Since $b = \cup(a)$ is absolute w.r.t. A, this holds if and only if

$$(\forall x \in A)(\exists y \in A)[y = \cup(x)].$$

But this is true if and only if

$$(\forall x \in A)[\cup(x) \in A].$$ □

Remark. Axiom 1 is absolute w.r.t. any transitive class. As a consequence every transitive class is a model of this axiom. Furthermore we note that if $b \in A$ then the extent of b as an individual in the universe A is $b \cap A$, that is, the collection of objects in A that play the role of elements of b is precisely the collection of objects in A that *are* elements of b. If A is transitive then $b \in A$ implies $b \subseteq A$ and so $b \cap A = b$, that is, b as an individual in the universe A has the same extent it has as an individual in V. Consequently if A is transitive and a and b are in A then $\{a, b\} \cap A = \{a, b\}$ and $\cup(a) \cap A = \cup(a)$. Therefore in order for A to be a model of the Axiom of Pairing and the

Axiom of Unions we must have

$$\{a, b\} \in A \quad \text{for } a, b \in A,$$

$$\cup(a) \in A \quad \text{for } a \in A.$$

The Power Set Axiom however presents a different situation. If A is transitive but not super transitive then $a \in A$ does not imply that all subsets of a are in A thus

$$\mathscr{P}(a) \cap A \subseteqq \mathscr{P}(a)$$

and equality need not hold. That is, even if A is to be a transitive model of the Power Set Axiom, the object in A that plays the role of the power set of a need not be $\mathscr{P}(a)$.

Proposition 13.32. *If A is nonempty and transitive then A is a standard transitive model of the Axiom of Powers iff*

$$(\forall x \in A)[\mathscr{P}(x) \cap A \in A].$$

Proof. Ax. $4 \leftrightarrow (\forall x)(\exists y)(\forall z)[z \in y \leftrightarrow z \subseteqq x]$. Since A is nonempty and transitive we have

$$\text{STM}(A, \text{Ax. } 4)$$

if and only if

$$(\forall x \in A)(\exists y \in A)(\forall z \in A)[z \in y \leftrightarrow z \subseteqq x]^A.$$

But since $b \in c \leftrightarrow b \subseteqq a$ is absolute w.r.t. A, this holds if and only if

$$(\forall x \in A)(\exists y \in A)(\forall z \in A)[z \in y \leftrightarrow z \subseteqq x].$$

Since A is transitive it follows that if $y \in A$ then

$$(\forall z \in A)[z \in y \leftrightarrow z \subseteqq x]$$

holds if and only if

$$(\forall z)[z \in y \leftrightarrow z \subseteqq x \wedge z \in A]$$

i.e., if and only if

$$(\forall z)[z \in y \leftrightarrow z \in \mathscr{P}(x) \cap A].$$

Thus A is a transitive model of Axiom 4 if and only if

$$(\forall x \in A)(\exists y \in A)(\forall z)[z \in y \leftrightarrow z \in \mathscr{P}(x) \cap A]$$

that is, if and only if

$$(\forall x \in A)[\mathscr{P}(x) \cap A \in A]. \qquad \square$$

Remark. Axiom 5 is of course not *an* axiom but an axiom schema. For each wff $\varphi(u, v)$ we have an instance of Axiom 5 that we will denote by Axiom 5_φ.

Proposition 13.33. *If A is nonempty and transitive then A is a standard transitive model of Axiom 5_φ, iff for $a \in A$*

$$(\forall x, y, z \in A)[\varphi^A(x, y) \wedge \varphi^A(x, z) \to y = z]$$

implies that

$$\{y \in A \mid (\exists x \in a)\varphi^A(x, y)\} \in A.$$

PROOF. Since A is nonempty and transitive A is a standard transitive model of Axiom 5_φ iff for $a \in A$

$$(\forall x, y, z \in A)[\varphi^A(x, y) \wedge \varphi^A(x, z) \to y = z]$$

implies that

$$(\exists z \in A)(\forall y \in A)[y \in z \leftrightarrow (\exists x \in A)[\varphi^A(x, y) \wedge x \in a]].$$

But since A is transitive this is the case iff

$$(\exists z \in A)(\forall y)[y \in z \leftrightarrow (\exists x)[\varphi^A(x, y) \wedge x \in a] \wedge y \in A]$$

i.e.

$$(\exists z \in A)[z = \{y \in A \mid (\exists x \in a)[\varphi^A(x, y)]\}].$$

Hence

$$\{y \in A \mid (\exists x \in a)[\varphi^A(x, y)]\} \in A. \qquad \square$$

Proposition 13.34. $\mathrm{STM}(A, \text{Ax. } 5_\psi) \to (\forall x \in A)[\{y \in x \mid \varphi^A(y)\} \in A]$ *where* $\psi(a, b)$ *is* $\varphi(a) \wedge a = b$.

PROOF. If A is a transitive model of Axiom 5_ψ then A is a nonempty transitive class that satisfies all of the nonlogical axioms required to prove the instance of Zermelo's Schema of Separation:

$$(\forall x)(\exists y)[y = \{z \in x \mid \varphi(z)\}].$$

(See Proposition 5.11.) $\square$

Proposition 13.35. $A \neq 0 \wedge \mathrm{Tr}(A) \to \mathrm{STM}(A, \text{Ax. } 6)$.

PROOF. Ax. $6 \leftrightarrow (\forall x)[x \neq 0 \to (\exists y)[y \in x \wedge y \cap x = 0]]$. In particular, from Axiom 6.

$$(\forall x \in A)[x \neq 0 \to (\exists y)[y \in x \wedge y \cap x = 0]].$$

But since A is transitive $y \in x \wedge x \in A \to y \in A$. Therefore

$$(\forall x \in A)[x \neq 0 \to (\exists y \in A)[y \in x \wedge y \cap x = 0]].$$

Furthermore $y \in x \wedge y \cap x = 0$ is absolute with respect to A. Therefore if $a \in A$ then

$$(\exists y \in A)[y \in a \wedge y \cap a = 0] \leftrightarrow (\exists y \in A)[y \in a \wedge y \cap a = 0]^A.$$

Since $a \neq 0$ is absolute with respect to A it follows that

$$(\forall x \in A)[x \neq 0 \rightarrow (\exists y)[y \in x \wedge y \cap x = 0]]^A$$

i.e., A is a model of Axiom 6. □

Proposition 13.36. $A \neq 0 \wedge \text{Tr}(A) \rightarrow 0 \in A$.

PROOF. By the Axiom of Regularity

$$\begin{aligned} A \neq 0 \wedge \text{Tr}(A) &\rightarrow (\exists x)[x \in A \wedge x \cap A = 0] \\ &\rightarrow (\exists x)[x \in A \wedge x = 0] \\ &\rightarrow 0 \in A. \end{aligned}$$ □

Proposition 13.37. STM(A, Ax. 2) $\wedge$ STM(A, Ax. 3) $\rightarrow \omega \subseteq A$.

PROOF (By induction). From Proposition 13.36, $0 \in A$. If $k \in A$, then since A is a model of Ax. 2, $\{k\} \in A$ and hence $\{k, \{k\}\} \in A$. Since A is also a model of Ax. 3 $\cup \{k, \{k\}\} \in A$, that is $k + 1 \in A$. □

Proposition 13.38. *If A is a standard transitive model of the Axiom of Pairing and the Axiom of Unions then A is a standard transitive model of the Axiom of Infinity* iff $\omega \in A$.

PROOF. Since A is nonempty and transitive A is a standard transitive model of the Axiom of Infinity iff

$$(\exists x \in A)[x = \omega]^A.$$

But Propositions 13.37 and 13.21 establish that $x = \omega$ is absolute w.r.t. A. Therefore A is a standard transitive model of the Axiom of Infinity iff

$$(\exists x \in A)[x = \omega]$$

i.e., iff

$$\omega \in A.$$ □

Remark. We next prove the relative consistency of the Axiom of Regularity and the other axioms of ZF. In addition to being of interest in itself the proof gives an excellent illustration of Gödel's method of proving the relative consistency of ZF with AC and GCH. Our procedure is to prove without using the Axiom of Regularity that the class of well-founded sets is a model of ZF (see Definition 9.11). The proof requires a modification of our theory of ordinals in which we reverse the role of Definition 7.3 and Proposition 7.4.

We redefine ordinal classes thus:

$$\text{Ord}(A) \overset{\Delta}{\leftrightarrow} \text{Tr}(A) \wedge E \text{ We } A.$$

From this definition it is obvious that

$$\text{Ord}(A) \rightarrow \text{Tr}(A) \wedge (\forall x, y \in A)[x \in y \vee x = y \vee y \in x].$$

The properties of the ordinals as redefined, can now be deduced without the Axiom of Regularity. We leave it to the reader to verify that the background results, used in the proof to follow, do not depend upon the Axiom of Regularity.

Proposition 13.39. *If $M = \{x \mid \mathrm{Wf}(x)\}$ then M is a standard transitive model of* ZF.

Proof. Since $0 \in R_1`1$ we have $0 \in M$, i.e., $M \neq 0$. If $x \in M$ then $(\exists\, \alpha)[x \in R_1`\alpha]$. Since $R_1`\alpha$ is transitive $x \in R_1`\alpha \to x \subseteq R_1`\alpha$ and hence $x \subseteq M$. Thus M is transitive. Since M is nonempty and transitive M is a model of Axiom 1.

Recall that a set is well founded if each of its elements is well founded. Consequently if $a \in M \wedge b \in M$ then $\{a, b\} \in M$. Thus M is a model of Axiom 2.

If $a \in M$ then $b \in \cup(a) \to (\exists\, x)[b \in x \wedge x \in a]$. Since M is transitive it follows that $\cup(a) \subseteq M$, hence $\cup(a) \in M$. Thus M is a model of Axiom 3.

If $a \in M \wedge b \subseteq a$ then since M is transitive $b \subseteq M$ and hence $b \in M$. Thus $\mathscr{P}(a) \subseteq M$ and hence $\mathscr{P}(a) \in M$. Thus M is a model of Axiom 4.

If $(\forall\, x \in M)(\forall\, y \in M)(\forall\, z \in M)[\varphi^M(x, y) \wedge \varphi^M(x, z) \to y = z]$ and if $a \in M$ then

$$\{y \in M \mid (\exists\, x \in a)\varphi^M(x, y)\} \subseteq M$$

and hence $\{y \in M \mid (\exists\, x \in a)\varphi^M(x, y)\} \in M$. Thus M is a model of Axiom 5.

If $a \in M$, then since M is transitive $a \subseteq M$. Thus each element of a has a rank. If $a \neq 0$ then a contains an element of smallest rank, i.e.,

$$\begin{aligned} a \neq 0 &\to (\exists\, x \in a)(\forall\, y \in a)[\operatorname{rank} x \leqq \operatorname{rank} y] \\ &\to (\exists\, x \in a)(x \cap a = 0). \end{aligned}$$

Again since M is transitive $b \in a \to b \in M$, i.e.,

$$(\forall\, x \in M)[x \neq 0 \to (\exists\, y \in M)[y \in x \wedge y \cap x = 0]].$$

Thus, since $[b \in a \wedge b \cap a = 0]$ Abs M, M is a standard transitive model of Axiom 6.

Since M is a standard transitive model of Axioms 2 and 3 it follows that $\omega \subseteq M$ (Proposition 13.37). From Proposition 9.12 it then follows that $\omega \in M$. Thus M is a model of Axiom 7.

***Theorem 13.40.** *If $\aleph_\alpha$ is inaccessible then $R_1`\aleph_\alpha$ is a standard transitive model of* ZF.

Proof. Since $R_1`\aleph_\alpha$ is nonempty and transitive it is a model of Axioms 1 and 6. If $a \in R_1`\aleph_\alpha \wedge b \in R_1`\aleph_\alpha$ then

$$\operatorname{rank}(\{a, b\}) = \max(\operatorname{rank}(a), \operatorname{rank}(b)) + 1 < \aleph_\alpha.$$

Hence $\{a, b\} \in R_1`\aleph_\alpha$ and $R_1`\aleph_\alpha$ is a model of Axiom 2.

If $a \in R_1`\aleph_\alpha$ then

$$\text{rank}(\cup(a)) = \mu_\gamma((\forall x \in \cup(a))[\text{rank}(x) < \gamma]) \leqq \text{rank}(a).$$

Therefore $\cup(a) \in R_1`\aleph_\alpha$ and $R_1`\aleph_\alpha$ is a model of Axiom 3.

If $a \in R_1`\aleph_\alpha$ then $\mathscr{P}(a) \subseteq R_1`\aleph_\alpha$ and hence $\mathscr{P}(a) \cap R_1`\aleph_\alpha = \mathscr{P}(a)$. Furthermore since $b \subseteq a$ implies $\text{rank}(b) \leqq \text{rank}(a)$ and since $a \in \mathscr{P}(a)$

$$\text{rank}(\mathscr{P}(a)) = \text{rank}(a) + 1 < \aleph_\alpha.$$

Thus $\mathscr{P}(a) \in R_1`\aleph_\alpha$ and hence $R_1`\aleph_\alpha$ is a model of Axiom 4.

From the AC it follows by induction that

$$(\forall \beta < \aleph_\alpha)[\overline{\overline{R_1`\beta}} < \aleph_\alpha].$$

Indeed $\overline{\overline{R_1`0}} = 0 < \alpha$. If $\overline{\overline{R_1`\beta}} < \aleph_\alpha$ for $\beta < \aleph_\alpha$ then since $\aleph_\alpha$ is inaccessible

$$\overline{\overline{R_1`(\beta + 1)}} = \overline{\overline{\mathscr{P}(R_1`\beta)}} < \aleph_\alpha.$$

If $\beta \in K_{\text{II}}$ and if $\gamma < \beta \to R_1`\gamma < \aleph_\alpha$, then by Theorem 10.47

$$\overline{\overline{R_1`\beta}} = \overline{\overline{\bigcup_{\gamma<\beta} R_1`\gamma}} \leqq \overline{\overline{\aleph_\alpha \times \overline{\overline{\beta}}}} = \aleph_\alpha.$$

If $\overline{\overline{R_1`\beta}} = \aleph_\alpha$ and if $f`\gamma \triangleq \overline{\overline{R_1`\gamma}}$ then $f : \beta \to \aleph_\alpha$, f is monotone increasing and $\cup(f``\beta) = \aleph_\alpha$. Hence $\aleph_\alpha$ is cofinal with β. This is a contradiction from which we conclude that

$$\overline{\overline{R_1`\beta}} < \aleph_\alpha.$$

If for some $a \in R_1`\aleph_\alpha$ and some wff φ we have

$$(\forall x \in R_1`\aleph_\alpha)(\forall y \in R_1`\aleph_\alpha)(\forall z \in R_1`\aleph_\alpha)[\varphi^{R_1`\aleph_\alpha}(x, y) \wedge \varphi^{R_1`\aleph_\alpha}(x, z) \to y = z]$$

and

$$\{y \in R_1`\aleph_\alpha \mid (\exists x \in a)\varphi^{R_1`\aleph_\alpha}(x, y)\} \notin R_1`\aleph_\alpha$$

then since $(\exists \beta < \aleph_\alpha)[a \subseteq R_1`\beta], \overline{\overline{a}} \leqq \overline{\overline{R_1`\beta}} < \aleph_\alpha$, i.e., $(\exists \gamma < \aleph_\alpha)[\gamma \simeq a]$. Since

$$\text{rank}(\{y \in R_1`\aleph_\alpha \mid (\exists x \in a)\varphi^{R_1`\aleph_\alpha}(x, y)\}) = \aleph_\alpha$$

it then follows that $\aleph_\alpha$ is cofinal with some ordinal smaller than or equal to γ. This is a contradiction and hence $R_1`\aleph_\alpha$ is a model of Axiom 5.

Since $\omega \in R_1`\aleph_\alpha$, $R_1`\aleph_\alpha$ is a model of Axiom 7. □

Exercises

Prove the following.

(1) $\text{STM}(R_1`\alpha, \text{Ax. } 1) \leftrightarrow \alpha > 0$.

(2) $\text{STM}(R_1`\alpha, \text{Ax. } 2) \leftrightarrow \alpha \in K_{\text{II}}$.

(3) $\text{STM}(R_1`\alpha, \text{Ax. } 3) \leftrightarrow \alpha > 0$.

(4) $\mathrm{STM}(R_1`\alpha, \text{Ax. } 4) \leftrightarrow \alpha \in K_{II}$.

(5) $\mathrm{STM}(R_1`\omega, \text{Ax. } 5) \wedge \neg\, \mathrm{STM}\,(R_1`(\omega 2), \text{Ax. } 5)$.

(6) $\mathrm{STM}(R_1`\alpha, \text{Ax. } 6) \leftrightarrow \alpha > 0$.

(7) $\mathrm{STM}(R_1`\alpha, \text{Ax. } 7) \leftrightarrow \alpha > \omega$.

In Exercises 8–15 assume $\aleph_\alpha$ an inaccessible cardinal.

(8) $\mathscr{P}(a)$ Abs $R_1`\aleph_\alpha$.

(9) $a \simeq b$ Abs $R_1`\aleph_\alpha$.

(10) $\mathrm{cf}(\beta)$ Abs $R_1`\aleph_\alpha$.

(11) $\beta \in N$ Abs $R_1`\aleph_\alpha$.

(12) $\beta \in N'$ Abs $R_1`\aleph_\alpha$.

(13) $\mathrm{Reg}(\beta)$ Abs $R_1`\aleph_\alpha$.

(14) $\mathrm{Inacc}_w(\beta)$ Abs $R_1`\aleph_\alpha$.

(15) $\mathrm{Inacc}(\beta)$ Abs $R_1`\aleph_\alpha$.

Remark. There are several interesting conclusions to be drawn from the foregoing exercises and Proposition 13.40. First we have that if $\aleph_\alpha$ is inaccessible then $R_1`\aleph_\alpha$ is a standard transitive model of ZF. The proof requires AC. From this and Exercise 15 it follows that it is consistent with ZF to assume that there are no inaccessible cardinals. We argue in the following way. Suppose the statement "there exists an inaccessible cardinal" were provable in ZF. Let $\aleph_\alpha$ be the smallest such cardinal. Then $R_1`\aleph_\alpha$ is a standard transitive model of ZF. But from Exercise 15 above, if there were an inaccessible cardinal in $R_1`\aleph_\alpha$ that cardinal would be inaccessible in V and smaller than the smallest inaccessible cardinal. This is a contradiction. Therefore it is not possible to prove in ZF that inaccessible cardinals exist. It may, however, be possible to prove in ZF that there does not exist an inaccessible cardinal. No proof was known at the time of this writing.

From Exercises 1–7, $R_1`\omega$ is a standard transitive model of Axioms 1–6 but not of Axiom 7. Thus Axiom 7 is independent of Axioms 1–6. Also $R_1`(\omega 2)$ is a standard transitive model of all axioms except Axiom 5.

CHAPTER 14

The Fundamental Operations

Gödel proved the relative consistency of AC and GCH by showing that a certain class L is a model of ZF + AC + GCH. This class L he defined initially as the union of a sequence of sets A_α, $\alpha \in \mathrm{On}$ which were so defined that $a \in A_{\alpha+1}$ iff there exists a wff $\varphi(a_0, a_1, \ldots, a_n)$ having no free variables other than $a_0, a_1, \ldots, a_n$ and there exist $a_1, \ldots, a_n \in A_\alpha$ such that $a = \{y \mid A_\alpha \models \varphi(y, a_1, \ldots, a_n)\}$.

The foregoing condition describes a sense in which $A_{\alpha+1}$ is the collection of sets that are definable from A_α. However, to properly define "definable" we must avoid quantification on wffs.[1]

Later Gödel discovered that his class, L, of constructible sets, could be defined as the range of a certain function F defined on On by transfinite recursion from eight basic operations. In Chapter 15, we will follow Gödel's second development. In anticipation of that development, we now establish certain conditions involving Gödel's eight fundamental operations to be defined below that are sufficient for a class M to be a standard transitive model of ZF.

Definition 14.1.

(1) $\mathrm{Cnv}_2(A) \triangleq \{\langle x, y, z\rangle \mid \langle z, x, y\rangle \in A\}$.

(2) $\mathrm{Cnv}_3(A) \triangleq \{\langle x, y, z\rangle \mid \langle x, z, y\rangle \in A\}$.

Remark. $\mathrm{Cnv}_2(A)$ and $\mathrm{Cnv}_3(A)$ are read "the second converse of A" and "the third converse of A" respectively.

[1] For a more detailed discussion of definability see Takeuti and Zaring: *Axiomatic Set Theory*. New York: Springer-Verlag 1973.

Definition 14.2 (The Fundamental Operations).

$$\mathscr{F}_1(a, b) \triangleq \{a, b\}.$$

$$\mathscr{F}_2(a, b) \triangleq a \cap E.$$

$$\mathscr{F}_3(a, b) \triangleq a - b.$$

$$\mathscr{F}_4(a, b) \triangleq a \restriction b.$$

$$\mathscr{F}_5(a, b) \triangleq a \cap \mathscr{D}(b).$$

$$\mathscr{F}_6(a, b) \triangleq a \cap b^{-1}.$$

$$\mathscr{F}_7(a, b) \triangleq a \cap \mathrm{Cnv}_2(b).$$

$$\mathscr{F}_8(a, b) \triangleq a \cap \mathrm{Cnv}_3(b).$$

Proposition 14.3. *If M is a standard transitive model of* ZF *then M is closed under the eight fundamental operations.*

Proof. From Proposition 13.25

$$a \in M \wedge b \in M \wedge \mathrm{STM}(M, \mathrm{Ax.}\ 2) \to \mathscr{F}_1(a, b) \in M.$$

Since M is a model of Axiom 2, $c = \langle a, b\rangle$ and $d = \langle a, b, c\rangle$ are each absolute with respect to M. Since M is also a model of Axiom 5 it follows from Proposition 13.34 and properties of absoluteness that for $a, b \in M$

$$\begin{aligned}
\mathscr{F}_2(a, b) &= \{x \in a \mid (\exists\, y, z)[x = \langle y, z\rangle \wedge y \in z]\}\\
&= \{x \in a \mid (\exists\, y, z \in M)[x = \langle y, z\rangle \wedge y \in z]^M\} \in M.\\
\mathscr{F}_3(a, b) &= \{x \in a \mid x \notin b\} = \{x \in a \mid [x \notin b]^M\} \in M.\\
\mathscr{F}_4(a, b) &= \{x \in a \mid (\exists\, y, z)[x = \langle y, z\rangle \wedge y \in b]\}\\
&= \{x \in a \mid (\exists\, y, z \in M)[x = \langle y, z\rangle \wedge y \in b]^M\} \in M.\\
\mathscr{F}_5(a, b) &= \{x \in a \mid (\exists\, y)[\langle x, y\rangle \in b]\} = \{x \in a \mid (\exists\, y \in M)[\langle x, y\rangle \in b]^M\} \in M.\\
\mathscr{F}_6(a, b) &= \{x \in a \mid (\exists\, y, z)[x = \langle y, z\rangle \wedge \langle z, y\rangle \in b]\}\\
&= \{x \in a \mid (\exists\, y, z \in M)[x = \langle y, z\rangle \wedge \langle z, y\rangle \in b]\} \in M.\\
\mathscr{F}_7(a, b) &= \{x \in a \mid (\exists\, y, z, w)[x = \langle y, z, w\rangle \wedge \langle w, y, z\rangle \in b]\}\\
&= \{x \in a \mid (\exists\, y, z, w \in M)[x = \langle y, z, w\rangle \wedge \langle w, y, z\rangle \in b]^M\} \in M.\\
\mathscr{F}_8(a, b) &= \{x \in a \mid (\exists\, y, z, w)[x = \langle y, z, w\rangle \wedge \langle y, w, z\rangle \in b]\}\\
&= \{x \in a \mid (\exists\, y, z, w \in M)[x = \langle y, z, w\rangle \wedge \langle y, w, z\rangle \in b]^M\} \in M. \quad \square
\end{aligned}$$

Remark. An examination of the foregoing proof reveals that the full strength of the hypothesis that M is a standard transitive model of ZF was not used. All that is required is that M be a standard transitive model of Axiom 2 and of seven instances of Axiom 5. In view of this it is not reasonable to expect the condition that M be closed under the eight fundamental operations to be also sufficient for a nonempty transitive class M to be a standard transitive model of ZF. Surprisingly in addition to closure under the eight fundamental operations we need only the added condition that every subset of M have an extension in M.

Definition 14.4. M is almost universal iff

$$(\forall x)[x \subseteq M \to (\exists y \in M)[x \subseteq y]].$$

Remark. Note that if M is almost universal then M is not empty.

To prove that if M is transitive, almost universal, and closed under the eight fundamental operations then M is a standard transitive model of ZF we need a few preliminary results.

Proposition 14.5. *If M is transitive, almost universal, and closed under the eight fundamental operations and $a, b \in M$, then*

(1) $\{a, b\} \in M$, (3) $a \times b \in M$, (5) $a \cap b \in M$,
(2) $\langle a, b\rangle \in M$, (4) $a - b \in M$, (6) $a \cup b \in M$.

PROOF.

(1) $\{a, b\} = \mathscr{F}_1(a, b) \in M$.

(2) $\langle a, b\rangle = \{\{a\}, \{a, b\}\} \in M$.

(3) Since M is transitive $a \subseteq M$ and $b \subseteq M$. Therefore $a \times b \subseteq M$ and $b \times a \subseteq M$. But M is also almost universal. Therefore

$$(\exists x, y \in M)[a \times b \subseteq x \wedge b \times a \subseteq y].$$

Then

$$\begin{aligned} a \times b &= [x \cap (a \times V)] \cap [y \cap (b \times V)]^{-1} \\ &= \mathscr{F}_6(\mathscr{F}_4(x, a), \mathscr{F}_4(y, b)) \in M. \end{aligned}$$

(4) $a - b = \mathscr{F}_3(a, b) \in M$.

(5) $a \cap b = a - (a - b) \in M$.

(6) Since M is transitive $a \cup b \subseteq M$. Also M is almost universal. Therefore

$$(\exists x \in M)[a \cup b \subseteq x].$$

Then $a \cup b = x - [(x - a) - b] \in M$. □

Proposition 14.6. *If M is transitive, almost universal, and closed under the eight fundamental operations then*

$$(\forall x_1, \ldots, x_n \in M)[x_1 \times x_2 \times \cdots \times x_n \in M].$$

PROOF. Obvious from Proposition 14.5 (3) by induction. □

Proposition 14.7. *If M is transitive, almost universal, and closed under the eight fundamental operations and $a \in M$, then*

(1) $a^n \in M, n \geqq 1,$ (3) $\mathscr{D}(a) \in M,$ (5) $\mathrm{Cnv}_2(a) \in M,$

(2) $a^{-1} \in M,$ (4) $\mathscr{W}(a) \in M,$ (6) $\mathrm{Cnv}_3(a) \in M.$

PROOF. (1) Obvious from Proposition 14.6.

(2) Since M is transitive $\langle b, c\rangle \in a$ and $a \in M$ imply $b, c \in M$. This in turn implies that $\langle c, b\rangle \in M$. Therefore, $a^{-1} \subseteqq M$. But since M is almost universal $(\exists\, x \in M)[a^{-1} \subseteqq x]$ and hence

$$a^{-1} = x \cap a^{-1} = \mathscr{F}_6(x, a) \in M.$$

(3) If $b \in \mathscr{D}(a)$ then $(\exists\, y)[\langle b, y\rangle \in a]$. Again from transitivity it follows that $\mathscr{D}(a) \subseteqq M$ and hence $(\exists\, x \in M)[\mathscr{D}(a) \subseteqq x]$. Therefore

$$\mathscr{D}(a) = x \cap \mathscr{D}(a) = \mathscr{F}_5(x, a) \in M.$$

(4) $\mathscr{W}(a) = \mathscr{D}(a^{-1}) \in M.$

(5) Since M is transitive $\mathrm{Cnv}_2(a) \subseteqq M$. Therefore

$$(\exists\, x \in M)[\mathrm{Cnv}_2(a) \subseteqq x].$$

Then

$$\mathrm{Cnv}_2(a) = x \cap \mathrm{Cnv}_2(a) = \mathscr{F}_7(x, a) \in M.$$

(6) The proof is left to the reader. □

Proposition 14.8. *If M is transitive, almost universal, and closed under the eight fundamental operations, if (i_1, i_2, i_3) is a permutation of 1, 2, 3 and if $a \in M$ then*

$$\{\langle x_1, x_2, x_3\rangle \mid \langle x_{i_1}, x_{i_2}, x_{i_3}\rangle \in a\} \in M.$$

PROOF.

$$\{\langle x_1, x_2, x_3\rangle \mid \langle x_1, x_3, x_2\rangle \in a\} = \mathrm{Cnv}_3(a) \in M.$$

$$\{\langle x_1, x_2, x_3\rangle \mid \langle x_3, x_1, x_2\rangle \in a\} = \mathrm{Cnv}_2(a) \in M.$$

$$\{\langle x_1, x_2, x_3\rangle \mid \langle x_3, x_2, x_1\rangle \in a\} = \mathrm{Cnv}_2(\mathrm{Cnv}_3(a)) \in M.$$

$$\{\langle x_1, x_2, x_3\rangle \mid \langle x_1, x_2, x_3\rangle \in a\} = \mathrm{Cnv}_3(\mathrm{Cnv}_3(a)) \in M.$$

$$\{\langle x_1, x_2, x_3\rangle \mid \langle x_2, x_1, x_3\rangle \in a\} = \mathrm{Cnv}_3(\mathrm{Cnv}_2(a)) \in M.$$

$$\{\langle x_1, x_2, x_3\rangle \mid \langle x_2, x_3, x_1\rangle \in a\} = \mathrm{Cnv}_2(\mathrm{Cnv}_2(a)) \in M.$$ □

Proposition 14.9. *If M is transitive, almost universal, and closed under the eight fundamental operations and $a, b \in M$, then*

(1) $\{\langle x, y, z\rangle \mid \langle x, y\rangle \in a \wedge z \in b\} \in M,$

(2) $\{\langle x, z, y\rangle \mid \langle x, y\rangle \in a \wedge z \in b\} \in M,$

(3) $\{\langle z, x, y\rangle \mid \langle x, y\rangle \in a \wedge z \in b\} \in M.$

PROOF. Obvious from Proposition 14.8 and the fact that $a \times b \in M$. □

Proposition 14.10. *If M is transitive, almost universal, and closed under the eight fundamental operations and if $\varphi(a_1, \ldots, a_m)$ is a* wff *all of whose free variables are among $b_1, \ldots, b_n$, $a_1, \ldots, a_m$ if $c_1, \ldots, c_m \in M$ and if $b_1, \ldots, b_n \in M$, then*

$$a \triangleq \{\langle x_1, \ldots, x_m\rangle | x_1 \in c_1 \wedge \cdots \wedge x_m \in c_m \wedge \varphi^M(x_1, \ldots, x_m)\} \in M.$$

PROOF. (By induction on k the number of logical symbols in $\varphi(a_1, \ldots, a_m)$). If $k = 0$ then $\varphi(a_1, \ldots, a_m)$ either (1) contains none of the variables $a_1, \ldots, a_m$ or it is of the form (2) $a_i \in a_j$ or (3) $a_i \in b_j$ or (4) $b_i \in a_j$.

Case 1. If $\varphi(a_1, \ldots, a_m)$ contains none of the variables $a_1, \ldots, a_m$ then $a = c_1 \times c_2 \times \cdots \times c_m$ or $a = 0$ according as $\varphi^M(a_1, \ldots, a_m)$ holds or does not hold. In either case $a \in M$.

Case 2. If $\varphi(a_1, \ldots, a_m)$ is $a_i \in a_j$ then $i < j$ or $i = j$ or $j < i$. If $i < j$ then since $(c_i \times c_j) \cap E \in M$, and $c_1 \times \cdots \times c_{i-1} \in M$ we have from Proposition 14.9 (3) that

$$\{\langle x_1, \ldots, x_i, x_j\rangle | \langle x_i, x_j\rangle \in [(c_i \times c_j) \cap E] \wedge \langle x_1, \ldots, x_{i-1}\rangle \in c_1 \times \cdots \times c_{i-1}\} \in M.$$

From this we obtain after $j - (i + 1)$ applications of Proposition 14.9 (2)

$$\{\langle x_1, \ldots, x_j\rangle | x_1 \in c_1 \wedge \cdots \wedge x_j \in c_j \wedge x_i \in x_j\} \in M.$$

With $m - (j + 1)$ applications of Proposition 14.9.1 we have

$$\{\langle x_1, \ldots, x_m\rangle | x_1 \in c_1 \wedge \cdots \wedge x_m \in c_m \wedge x_i \in x_j\} \in M.$$

If $i = j$ then $a = 0 \in M$. If $j < i$ then $[(c_i \times c_j) \cap E]^{-1} \in M \wedge c_1 \times \cdots \times c_{j-1} \in M$. Then

$$\{\langle x_1, \ldots, x_j, x_i\rangle | \langle x_j, x_i\rangle \in [(c_i \times c_j) \cap E]^{-1} \wedge \langle x_1, \ldots, x_{j-1}\rangle \in c_1 \times \cdots \times c_{j-1}\} \in M.$$

We then proceed as before.

Case 3. If $\varphi(a_1, \ldots, a_m)$ is $a_i \in b_j$ then

$$\{\langle x_1, \ldots, x_m\rangle | x_1 \in c_1 \wedge \cdots \wedge x_m \in c_m \wedge x_i \in b_j\} = (c_1 \times \cdots \times c_m) \cap (c_1 \times \cdots \times c_{i-1} \times b_j \times c_{j+1} \times \cdots \times c_m) \in M.$$

Case 4. If $\varphi(a_1, \ldots, a_m)$ is $b_j \in a_i$ then

$$\{x_i \in c_i | b_j \in x_i\} = \mathscr{W}((\{b_j\} \times c_i) \cap E) \in M.$$

Then

$$\{\langle x_1, \ldots, x_m\rangle | x_1 \in c_1 \wedge \cdots \wedge x_m \in c_m \wedge b_j \in x_i\} = c_1 \times \cdots \times c_{i-1} \times \{x_i \in c_i | b_j \in x_i\} \times c_{i+1} \times \cdots \times c_m \in M.$$

If $k > 0$ then $\varphi(a_1, \ldots, a_m)$ is of the form (1) $\neg\psi(a_1, \ldots, a_m)$ or (2) $\psi(a_1, \ldots, a_m) \wedge \eta(a_1, \ldots, a_m)$ or (3) $(\exists x)\psi(a_1, \ldots, a_m, x)$.

Case 1. If $\varphi(a_1, \ldots, a_m)$ is $\neg\psi(a_1, \ldots, a_m)$ then as our induction hypothesis we have

$$\{\langle x_1, \ldots, x_m\rangle \mid x_1 \in c_1 \wedge \cdots \wedge x_m \in c_m \wedge \psi^M(x_1, \ldots, x_m)\} \in M.$$

Then

$$\begin{aligned}&\{\langle x_1, \ldots, x_m\rangle \mid x_1 \in c_1 \wedge \cdots \wedge x_m \in c_m \wedge \neg\psi^M(x_1, \ldots, x_m)\}\\ &\quad = c_1 \times \cdots \times c_m - \{\langle x_1, \ldots, x_m\rangle \mid x_1 \in c_1 \wedge \cdots \wedge x_m \in c_m\\ &\quad \wedge \psi^M(x_1, \ldots, x_m)\} \in M.\end{aligned}$$

Case 2. If $\varphi(a_1, \ldots, a_m)$ is $\psi(a_1, \ldots, a_m) \wedge \eta(a_1, \ldots, a_m)$ then from the induction hypothesis

$$\begin{aligned}&\{\langle x_1, \ldots, x_m\rangle \mid x_1 \in c_1 \wedge \cdots \wedge x_m \in c_m \wedge \psi^M(x_1, \ldots, x_m) \wedge \eta^M(x_1, \ldots, x_m)\}\\ &\quad = \{\langle x_1, \ldots, x_m\rangle \mid x_1 \in c_1 \wedge \cdots \wedge x_m \in c_m \wedge \psi^M(x_1, \ldots, x_m)\}\\ &\quad \cap \{\langle x_1, \ldots, x_m\rangle \mid x_1 \in c_1 \wedge \cdots \wedge x_m \in c_m \wedge \eta^M(x_1, \ldots, x_m)\} \in M.\end{aligned}$$

Case 3. If $\varphi(a_1, \ldots, a_m)$ is $(\exists x)\psi(a_1, \ldots, a_m, x)$ and

$$\begin{aligned}F`\langle a_1, \ldots, a_m\rangle &= \{x \in M \mid \psi^M(a_1, \ldots, a_m, x) \wedge (\forall y \in M)[\psi^M(a_1, \ldots, a_m, y)\\ &\qquad \rightarrow \operatorname{rank}(x) \leqq \operatorname{rank}(y)\} \quad \text{if } a_i \in c_i \quad \text{for } 1 \leqq i \leqq m,\\ &= 0 \text{ otherwise}\end{aligned}$$

then $F``(c_1 \times \cdots \times c_m)$ is a set and $\cup F``(c_1 \times \cdots \times c_m) \subseteq M$. Since M is almost universal

$$(\exists x \in M)[\cup F``(c_1 \times \cdots \times c_m) \subseteq x]$$

then $(\exists x \in M)\psi^M(a_1, \ldots, a_m, x) \leftrightarrow (\exists x \in c)\psi^M(a_1, \ldots, a_m, x)$. But by our induction hypothesis

$$\{\langle x_1, \ldots, x_m, x\rangle \mid x_1 \in c_1 \wedge \cdots \wedge x_m \in c_m \wedge x \in c \wedge \psi^M(x_1, \ldots, x_m, x)\} \in M.$$

Then

$$\begin{aligned}&\{\langle x_1, \ldots, x_m\rangle \mid x_1 \in c_1 \wedge \cdots \wedge x_m \in c_m \wedge (\exists x \in M)\psi^M(x_1, \ldots, x_m, x)\}\\ &\quad = \mathscr{D}\{\langle x_1, \ldots, x_m, x\rangle \mid x_1 \in c_1 \wedge \cdots \wedge x_m \in c_m \wedge x \in c\\ &\quad \wedge \psi^M(x_1, \ldots, x_m, x)\} \in M.\end{aligned}$$

□

Theorem 14.11. *If M is transitive, almost universal, and closed under the eight fundamental operations then M is a standard transitive model of* ZF *and* On $\subseteq M$.

Proof. Since M is almost universal $M \neq 0$. Therefore, M is a standard transitive model of the Axiom of Extentionality (Axiom 1) and the Axiom of Regularity (Axiom 6). By Proposition 14.5 $(\forall x, y \in M)[\{x, y\} \in M]$. Therefore M is a model of the Axiom of Pairing (Axiom 2).

Since M is transitive $a \in M$ implies that $\cup(a) \subseteq M$. Hence

$$(\exists\, y \in M)[\cup(a) \subseteq y].$$

Since by Proposition 14.5, $b \times a \in M$ and since M is closed under the eight fundamental operations

$$(b \times a) \cap E = \mathscr{F}_2(b \times a, a) \in M.$$

Then from Proposition 14.7

$$\cup(a) = \mathscr{D}((b \times a) \cap E) \in M.$$

Thus M is a model of the Axiom of Unions (Axiom 3).

Since $a \in M$ implies $\mathscr{P}(a) \cap M \subset M]$, $(\exists\, y \in M)[\mathscr{P}(a) \cap M \subseteq y]$. Then by Proposition 14.10

$$\mathscr{P}(a) \cap M = \{x | x \in y \wedge [x \subseteq a]^M\} \in M.$$

Therefore M is a model of the Axiom of Powers (Axiom 4).

If $(\forall\, x, y, z \in M)[\varphi^M(x, y) \wedge \varphi^M(x, z) \rightarrow y = z]$ and if $a \in M$, then

$$F \triangleq \{\langle x, y\rangle \in M^2 | x \in a \wedge \varphi^M(x, y)\}$$

is a function. Since $\mathscr{D}(F) \subseteq a$ both $\mathscr{D}(F) \wedge \mathscr{W}(F)$ are sets. Therefore since $F``a \subseteq M$, $(\exists\, z \in M)[F``a \subseteq z)$. Then

$$\begin{aligned}\{y \in M | (\exists\, x \in a)\varphi^M(x, y)\} &= \mathscr{W}(\{\langle x, y\rangle \in M^2 | x \in a \wedge \varphi^M(x, y)\}) \\ &= \mathscr{W}(\{\langle x, y\rangle | x \in a \wedge y \in z \wedge \varphi^M(x, y)\}) \in M.\end{aligned}$$

Thus M is a model of the Axiom Schema of Replacement (Axiom 5).

Since M is a standard transitive model of the Axiom of Pairing and the Axiom of Unions it follows from Proposition 13.37 that $\omega \subseteq M$. Since M is almost universal there exists an $a \in M$ such that

$$\omega \subseteq a.$$

Since $[x \in \omega]$ Abs M and M is a model of the Axiom Schema of Replacement it follows from Proposition 13.34 that

$$\omega = \{x \in a | x \in \omega\} = \{x \in a | [x \in \omega]^M\} \in M.$$

Therefore M is a model of the Axiom of Infinity (Axiom 7).

Thus M is a standard transitive model of ZF. That On $\subseteq M$ we prove by induction.

If $\alpha \subseteq M$ then since M is almost universal there is an $a \in M$ such that

$$\alpha \subseteq a.$$

Then

$$\{x \in a | \mathrm{Ord}(x)\} = \{x \in a | [\mathrm{Ord}(x)]^M\} \in M.$$

Therefore

$$\cup\{x \in a \mid \mathrm{Ord}(x)\} \in M.$$

But the union of a collection of ordinals is an ordinal, i.e.,

$$(\exists \beta \in M)(\forall \gamma \in a)[\gamma \leqq \beta].$$

Then $(\forall \gamma \in a)[\gamma \in \beta + 1]$. Since $\alpha \subseteq a$ it follows that $\alpha \leqq \beta + 1$. Since $\beta \in M$ it follows that $\beta + 1 = \beta \cup \{\beta\} \in M$. Hence $\alpha \in M$. □

Remark. With the proof of Proposition 14.11 we have achieved our major objective for this section. There is however an interesting theory of classes in which certain results of this section are generalized. We state the main results of this theory leaving the proofs as exercises.

Definition 14.12. A is M-constructible $\overset{\Delta}{\leftrightarrow} A \subseteq M \wedge (\forall x \in M)[x \cap A \in M]$.

Proposition 14.13. *If M is almost universal, and if a is M-constructible then $a \in M$.*

Remark. Proposition 14.13 tells us that if M is almost universal then every M-constructible set is an element in M. As an application of this proposition it can be proved that if in addition M is closed under the eight fundamental operations then $a \in M$ implies that $\cup(a)$ is M-constructible. It then follows that $\cup(a) \in M$ and hence M satisfies the Axiom of Unions.

Proposition 14.14. *If M is transitive then M is M-constructible.*

Proposition 14.15. *If M is transitive, almost universal, and closed under the eight fundamental operations and if A and B are M-constructible then*

(1) *$E \cap M$ is M-constructible.*

(2) *$A - B$ is M-constructible.*

(3) *$A \cap B$ is M-constructible.*

(4) *$A \cup B$ is M-constructible.*

Definition 14.16.

$$Q_4 \overset{\Delta}{=} \{\langle x, \langle x, y\rangle\rangle \mid x \in V \wedge y \in V\},$$

$$Q_5 \overset{\Delta}{=} \{\langle\langle x, y\rangle, x\rangle \mid x \in V \wedge y \in V\},$$

$$Q_6 \overset{\Delta}{=} \{\langle\langle x, y\rangle, \langle y, x\rangle\rangle \mid x \in V \wedge y \in V\},$$

$$Q_7 \overset{\Delta}{=} \{\langle\langle x, y, z\rangle, \langle y, z, x\rangle\rangle \mid x \in V \wedge y \in V \wedge z \in V\},$$

$$Q_8 \overset{\Delta}{=} \{\langle\langle x, y, z\rangle, \langle x, z, y\rangle\rangle \mid x \in V \wedge y \in V \wedge z \in V\}.$$

Proposition 14.17.

(1) $Q_4``A = A \times V$ *and* $\mathscr{F}_4(a, b) = a \cap Q_4``b$.

(2) $Q_5``A = \mathscr{D}(A)$, $\mathscr{U}n(Q_5)$ *and* $\mathscr{F}_5(a, b) = a \cap Q_5``b$.

(3) $Q_6``A = A^{-1}$, $\mathscr{U}n(Q_6)$ *and* $\mathscr{F}_6(a, b) = a \cap Q_6``b$.

(4) $Q_7``A = \mathrm{Cnv}_2(A)$, $\mathscr{U}n(Q_7)$ *and* $\mathscr{F}_7(a, b) = a \cap Q_7``b$.

(5) $Q_8``A = \mathrm{Cnv}_3(A)$, $\mathscr{U}n(Q_8)$ *and* $\mathscr{F}_8(a, b) = a \cap Q_8``b$.

Proposition 14.18. *If M is transitive and closed under the eight fundamental operation and if* $a \in M$ *then*

$$Q_n`a \in M, \qquad n = 5, 6, 7, 8.$$

Lemma. *If M is transitive, almost universal, and closed under the eight fundamental operations, if A is M-constructible, if* $a \in M$ *and if* G_n *is a function on a defined by*

$$G_n`b = \{y \in A \mid \langle y, b\rangle \in Q_n \wedge (\forall x \in A)[\langle x, b\rangle \in Q_n \rightarrow \mathrm{rank}(y) \leqq \mathrm{rank}(x)]\}$$

then

(1) $(\exists y \in M)[\cup(G_n``a) \subseteq y]$,

(2) $(\exists z \in M)[a \cap Q_n``z = a \cap Q_n``A]$.

Proposition 14.19. *If M is transitive, almost universal and closed under the eight fundamental operations and if A is M-constructible then*

(1) $M \cap Q_4``A$ *is M-constructible.*

(2) $Q_n``A$ *is M-constructible,* $n = 5, 6, 7, 8$.

Corollary 14.20. *If M is transitive, almost universal, and closed under the eight fundamental operations and if A is M-constructible then*

(1) $M \cap (A \times V)$ *is M-constructible,*

(2) $\mathscr{D}(A)$ *is M-constructible,*

(3) A^{-1} *is M-constructible,*

(4) $\mathrm{Cnv}_2(A)$ *is M-constructible,*

(5) $\mathrm{Cnv}_3(A)$ *is M-constructible.*

Proposition 14.21. *If M is transitive, almost universal, and closed under the eight fundamental operations and if A and B are each M-constructible then*

(1) $A \times B$ *is M-constructible,*

(2) A^n *is M-constructible,*

(3) $\mathscr{W}(A)$ *is M-constructible,*

(4) $A \restriction B$ *is M-constructible.*

(5) $A``B$ *is M-constructible.*

Proposition 14.22. *If M is transitive, almost universal, and closed under the eight fundamental operations and if A and B are M-constructible, then*

(1) $\{x \in M \mid a \in x\}$ *is M-constructible,*

(2) $\{\langle x, y, z\rangle \mid \langle x, y\rangle \in A \wedge z \in B\}$ *is M-constructible,*

(3) $\{\langle x, z, y\rangle \mid \langle x, y\rangle \in A \wedge z \in B\}$ *is M-constructible,*

(4) $\{\langle z, x, y\rangle \mid \langle x, y\rangle \in A \wedge z \in B\}$ *is M-constructible.*

Proposition 14.23. *If M is transitive, almost universal, and closed under the eight fundamental operations and if $\varphi(a_1, \ldots, a_m)$ is a* wff *all of whose free variables are among $b_1, \ldots, b_n, a_1, \ldots, a_m$ then*

$$b_1 \in M \wedge \cdots \wedge b_n \in M$$

implies

$$A \triangleq \{\langle x_1, \ldots, x_m\rangle \in M^m \mid \varphi^M(x_1, \ldots, x_m)\}$$

is M-constructible.

Proposition 14.24. *If M is transitive, almost universal, and closed under the eight fundamental operations and $a \in M$, then*

(1) $\cup(a)$ *is M-constructible,*

(2) $\mathscr{P}(a) \cap M$ *is M-constructible,*

(3) *If all of the free variables of $\varphi(c, d)$ are among $c, d, b_1, \ldots, b_n$, if $b_1 \in M \wedge \cdots \wedge b_n \in M$ and*

$$(\forall x, y, z \in M)[\varphi^M(x, y) \wedge \varphi^M(x, z) \to y = z]$$

then $\{y \in M \mid (\exists x \in a)\varphi^M(x, y)\}$ is M-constructible.

Remark. From Propositions 14.24 and 14.13 we have another proof of Theorem 14.11.

CHAPTER 15

The Gödel Model

In Chapter 7 we defined a relation R_0 on On^2. We proved that R_0 well orders On^2 and, with respect to R_0, initial segments of On^2 are sets. Consequently there is an order isomorphism J_0 such that

$$J_0 \ \mathrm{Isom}_{R_0, E}(\mathrm{On}^2, \mathrm{On}).$$

This isomorphism we illustrate with the following diagram:

$\alpha \backslash \beta$	0	1	2	$\cdots$	ω
0	0	1	4	$\cdots$	ω
1	2	3	5	$\cdots$	$\omega + 1$
2 $\vdots$	6 $\vdots$	7 $\vdots$	8 $\vdots$	$\cdots$	$\omega + 2$ $\vdots$
ω	$\omega 2$	$\omega 2 + 1$	$\omega 2 + 2$	$\cdots$	$\omega 3$

Here the element in the αth row and βth column is $J_0`\langle\alpha, \beta\rangle$. From the diagram it is apparent that the entry in the $\alpha$th row and $\beta$th column, i.e., $J_0`\langle\alpha, \beta\rangle$ is at least as large as the maximum of α and β. It is also easily proved that the cardinality of $J_0`\langle\alpha, \beta\rangle$ does not exceed the maximum of $\bar{\bar{\alpha}}$ and $\bar{\bar{\beta}}$ for α or β infinite.

Proposition 15.1.

(1) $\max(\alpha, \beta) \leqq J_0`\langle\alpha, \beta\rangle$.

(2) $\alpha < \aleph_\gamma \wedge \beta < \aleph_\gamma \rightarrow J_0`\langle\alpha, \beta\rangle < \aleph_\gamma$.

PROOF. (1) If $(\forall \alpha)[F`\alpha \triangleq J_0`\langle 0, \alpha\rangle]$ then F is a strictly monotonic ordinal function and hence $\alpha \leqq F`\alpha$. In particular if $\gamma = \max(\alpha, \beta)$ then

$$\max(\alpha, \beta) = \gamma \leqq F`\gamma = J_0`\langle 0, \gamma\rangle.$$

But

$$\langle 0, \gamma\rangle R_0\langle\alpha, \beta\rangle \vee \langle 0, \gamma\rangle = \langle\alpha, \beta\rangle.$$

Therefore

$$J_0`\langle 0, \gamma\rangle \leqq J_0`\langle\alpha, \beta\rangle$$

i.e.,

$$\max(\alpha, \beta) \leqq J_0`\langle\alpha, \beta\rangle.$$

(2) If $J_0`\langle\alpha, \beta\rangle < \aleph_0$ then $J_0`\langle\alpha, \beta\rangle < \aleph_\gamma$. If $J_0`\langle\alpha, \beta\rangle \geqq \aleph_0$ then since order isomorphisms map initial segments onto initial segments

$$J_0`\langle\alpha, \beta\rangle \simeq (R_0^{-1})``\{\langle\alpha, \beta\rangle\}.$$

But

$$\begin{aligned}\langle\gamma, \delta\rangle \in (R_0^{-1})``\{\langle\alpha, \beta\rangle\} &\rightarrow \langle\gamma, \delta\rangle R_0\langle\alpha, \beta\rangle\\ &\rightarrow \max(\gamma, \delta) \leqq \max(\alpha, \beta)\\ &\rightarrow [\gamma < \max(\alpha, \beta) + 1] \wedge [\delta < \max(\alpha, \beta) + 1]\\ &\rightarrow \langle\gamma, \delta\rangle \in [\max(\alpha, \beta) + 1] \times [\max(\alpha, \beta) + 1].\end{aligned}$$

Thus

$$(R_0^{-1})``\{\langle\alpha, \beta\rangle\} \subseteq [\max(\alpha, \beta) + 1] \times [\max(\alpha, \beta) + 1].$$

Consequently $J_0`\langle\alpha, \beta\rangle$ is equivalent to a subset of

$$[\max(\alpha, \beta) + 1] \times [\max(\alpha, \beta) + 1].$$

From this we see that if $\max(\alpha, \beta)$ were finite then $J_0`\langle\alpha, \beta\rangle$ would also be finite. Since $J_0`\langle\alpha, \beta\rangle \geqq \aleph_0$ it then follows that $\max(\alpha, \beta) \geqq \omega$. Therefore by Proposition 10.41

$$\overline{\overline{J_0`\langle\alpha, \beta\rangle}} \leqq \overline{\overline{[\max(\alpha, \beta) + 1] \times [\max(\alpha, \beta) + 1]}} = \overline{\overline{\max(\alpha, \beta) + 1}} < \aleph_\gamma.$$

Hence

$$J_0`\langle\alpha, \beta\rangle < \aleph_\gamma. \qquad \square$$

Remark. From the relation R_0 we can define a relation S on $\mathrm{On}^2 \times 9$ that will be used to define the Gödel model. This relation S well orders $\mathrm{On}^2 \times 9$

and is well founded on $\mathrm{On}^2 \times 9$. Consequently the ordering is order isomorphic to On. Indeed for our purposes this order isomorphism is of greater interest than S. We therefore choose to define it directly from J_0.

Definition 15.2. $J`\langle\alpha, \beta, m\rangle \triangleq 9 \cdot J_0`\langle\alpha, \beta\rangle + m, m < 9$.

Proposition 15.3. $J: \mathrm{On}^2 \times 9 \xrightarrow[\text{onto}]{1-1} \mathrm{On}$.

PROOF. Clearly α, β and m uniquely determine $9 \cdot J_0`\langle\alpha, \beta\rangle + m$. Therefore J is a function on $\mathrm{On}^2 \times 9$. If

$$J`\langle\alpha, \beta, m\rangle = J`\langle\gamma, \delta, n\rangle$$

then

$$9 \cdot J_0`\langle\alpha, \beta\rangle + m = 9 \cdot J_0`\langle\gamma, \delta\rangle + n.$$

From the uniqueness property in the division theorem for ordinals (Proposition 8.27) it then follows that

$$J_0`\langle\alpha, \beta\rangle = J_0`\langle\gamma, \delta\rangle \wedge m = n.$$

But J_0 is one-to-one. Hence $\alpha = \gamma \wedge \beta = \delta$. Therefore

$$\langle\alpha, \beta, m\rangle = \langle\gamma, \delta, n\rangle$$

and hence J is one-to-one.

Again from Proposition 8.27

$$(\forall \gamma)(\exists \delta)(\exists m < 9)[\gamma = 9 \cdot \delta + m].$$

Since $\delta \in \mathrm{On}$ and J_0 is onto

$$(\exists \alpha)(\exists \beta)[\delta = J_0`\langle\alpha, \beta\rangle].$$

Hence $J`\langle\alpha, \beta, m\rangle = 9 \cdot J_0`\langle\alpha, \beta\rangle + m = \gamma$, i.e., J is onto. ☐

Definition 15.4.

$$S \triangleq \{\langle\langle\alpha, \beta, m\rangle, \langle\gamma, \delta, n\rangle\rangle \mid m < 9 \wedge n < 9 \wedge [\langle\alpha, \beta\rangle R_0 \langle\gamma, \delta\rangle \vee [\langle\alpha, \beta\rangle = \langle\gamma, \delta\rangle \wedge m < n]]\}.$$

Proposition 15.5. $J\ \mathrm{Isom}_{S,E}(\mathrm{On}^2 \times 9, \mathrm{On})$.

Proposition 15.6.

(1) S We $(\mathrm{On}^2 \times 9)$.

(2) S Wfr $(\mathrm{On}^2 \times 9)$.

The proofs are left to the reader.

Definition 15.7.

$$K_1 \triangleq \{\langle\gamma, \alpha\rangle \mid (\exists\, n < 9)(\exists\, \beta)[\gamma = J`\langle\alpha, \beta, n\rangle]\}.$$

$$K_2 \triangleq \{\langle\gamma, \beta\rangle \mid (\exists\, n < 9)(\exists\, \alpha)[\gamma = J`\langle\alpha, \beta, n\rangle]\}.$$

$$K_3 \triangleq \{\langle\gamma, n\rangle \mid n < 9 \wedge (\exists\, \alpha)(\exists\, \beta)[\gamma = J`\langle\alpha, \beta, n\rangle]\}.$$

Remark. J maps $\mathrm{On}^2 \times 9$ one-to-one onto On. Therefore for each γ in On there is one and only one ordered triple $\langle\alpha, \beta, m\rangle$ in $\mathrm{On}^2 \times 9$ such that $\gamma = J`\langle\alpha, \beta, m\rangle$ that is $\gamma$ determines an $\alpha$, $\beta$, and $m$ such that $\gamma = J`\langle\alpha, \beta, m\rangle$. The functions K_1, K_2, and K_3 are so defined that $K_1`\gamma$, $K_2`\gamma$, and $K_3`\gamma$ are respectively the first, second, and third components of the ordered triple in $\mathrm{On}^2 \times 9$ corresponding to γ under J.

Proposition 15.8. $\gamma = J`\langle K_1`\gamma, K_2`\gamma, K_3`\gamma\rangle$.

Corollary 15.9. *If* $m < 9$ *then*

(1) $K_1`J`\langle\alpha, \beta, m\rangle = \alpha$,

(2) $K_2`J`\langle\alpha, \beta, m\rangle = \beta$,

(3) $K_3`J`\langle\alpha, \beta, m\rangle = m$.

Details are left to the reader.

Proposition 15.10.

(1) $K_1`\gamma \leqq \gamma \wedge K_2`\gamma \leqq \gamma$.

(2) $K_3`\gamma \neq 0 \rightarrow K_1`\gamma < \gamma \wedge K_2`\gamma < \gamma$.

PROOF. Since $\gamma = J`\langle K_1`\gamma, K_2`\gamma, K_3`\gamma\rangle = 9 \cdot J_0`\langle K_1`\gamma, K_2`\gamma\rangle + K_3`\gamma$, it follows from properties of ordinal arithmetic (Corollary 8.5 and Proposition 8.21)

$$J_0`\langle K_1`\gamma, K_2`\gamma\rangle \leqq \gamma.$$

But by Proposition 15.1

$$\max(K_1`\gamma, K_2`\gamma) \leqq J_0`\langle K_1`\gamma, K_2`\gamma\rangle.$$

Thus

$$K_1`\gamma \leqq \gamma \quad \text{and} \quad K_2`\gamma \leqq \gamma.$$

If in addition $K_3`\gamma \neq 0$ then

$$J_0`\langle K_1`\gamma, K_2`\gamma\rangle < \gamma$$

and hence

$$K_1`\gamma < \gamma \quad \text{and} \quad K_2`\gamma < \gamma. \qquad \square$$

Proposition 15.11. $m < 9 \wedge \alpha < \aleph_\gamma \wedge \beta < \aleph_\gamma \rightarrow J`\langle\alpha, \beta, m\rangle < \aleph_\gamma$.

PROOF. By Proposition 15.1 we have

$$J_0`\langle\alpha, \beta\rangle < \aleph_\gamma.$$

If $J_0`\langle\alpha, \beta\rangle < \aleph_0$ then

$$J`\langle\alpha, \beta, m\rangle = 9 \cdot J_0`\langle\alpha, \beta\rangle + m < \aleph_0 \leqq \aleph_\gamma.$$

If $J_0`\langle\alpha, \beta\rangle \geqq \aleph_0$ then since

$$9 \cdot J_0`\langle\alpha, \beta\rangle \simeq 9 \times J_0`\langle\alpha, \beta\rangle$$

we have that

$$\overline{\overline{9 \cdot J_0`\langle\alpha, \beta\rangle}} = \overline{\overline{J_0`\langle\alpha, \beta\rangle}} < \aleph_\gamma$$

and hence

$$J`\langle\alpha, \beta, m\rangle = 9 \cdot J_0`\langle\alpha, \beta\rangle + m < \aleph_\gamma. \qquad \square$$

Proposition 15.12. $J`\langle 0, \aleph_\gamma, 0\rangle = \aleph_\gamma$.

PROOF. If it were the case that $K_1`\aleph_\gamma < \aleph_\gamma$ and $K_2`\aleph_\gamma < \aleph_\gamma$ then by Proposition 15.8 it would follow that

$$\aleph_\gamma = J`\langle K_1`\aleph_\gamma, K_2`\aleph_\gamma, K_3`\aleph_\gamma\rangle < \aleph_\gamma.$$

From this contradiction and Proposition 15.1 we conclude that

$$\aleph_\gamma \leqq \max(K_1`\aleph_\gamma, K_2`\aleph_\gamma) \leqq J_0`\langle K_1`\aleph_\gamma, K_2`\aleph_\gamma\rangle.$$

Furthermore it follows that

$$\langle 0, \aleph_\gamma\rangle R_0 \langle K_1`\aleph_\gamma, K_2`\aleph_\gamma\rangle \quad \text{or} \quad \langle 0, \aleph_\gamma\rangle = \langle K_1`\aleph_\gamma, K_2`\aleph_\gamma\rangle$$

and hence

$$\aleph_\gamma \leqq J_0`\langle 0, \aleph_\gamma\rangle \leqq J_0`\langle K_1`\aleph_\gamma, K_2`\aleph_\gamma\rangle.$$

Therefore

$$\aleph_\gamma \leqq 9 \cdot J_0`\langle 0, \aleph_\gamma\rangle + 0 \leqq 9 \cdot J_0`\langle K_1`\aleph_\gamma, K_2`\aleph_\gamma\rangle + K_3`\aleph_\gamma$$

i.e.,

$$\aleph_\gamma \leqq J`\langle 0, \aleph_\gamma, 0\rangle \leqq J`\langle K_1`\aleph_\gamma, K_2`\aleph_\gamma, K_3`\aleph_\gamma\rangle = \aleph_\gamma.$$

Thus

$$J`\langle 0, \aleph_\gamma, 0\rangle = \aleph_\gamma. \qquad \square$$

Remark. We are now ready to define the Gödel model L. This is a standard transitive model that we define as the range of a special function F that is in turn defined by transfinite recursion in the following way.

Definition 15.13.

$$G`x \triangleq \mathscr{W}(x) \qquad \text{if } K_3`\mathscr{D}(x) = 0$$
$$\triangleq \mathscr{F}_n(x`K_1`\mathscr{D}(x), x`K_2`\mathscr{D}(x)) \quad \text{if } K_3`\mathscr{D}(x) = n \neq 0.$$

$F \mathscr{F}\!n \text{ On} \wedge (\forall \alpha)[F`\alpha \triangleq G`(F \upharpoonright \alpha)].$

Proposition 15.14.

$$F`\alpha = F``\alpha \qquad \text{if } K_3`\alpha = 0,$$
$$F`\alpha = \mathscr{F}_n`(F`K_1`\alpha, F`K_2`\alpha) \quad \text{if } K_3`\alpha = n \neq 0.$$

Proof. Since $\mathscr{D}(F \upharpoonright \alpha) = \alpha$ we have

$$F`\alpha = G`(F \upharpoonright \alpha) = \mathscr{W}(F \upharpoonright \alpha) = F``\alpha \qquad \text{if } K_3`\alpha = 0,$$
$$F`\alpha = G`(F \upharpoonright \alpha) = \mathscr{F}_n((F \upharpoonright \alpha)`K_1`\alpha, (F \upharpoonright \alpha)`K_2`\alpha) \quad \text{if } K_3`\alpha = n \neq 0.$$

But by Proposition 15.10

$$K_3`\alpha \neq 0 \rightarrow K_1`\alpha < \alpha \wedge K_2`\alpha < \alpha.$$

Thus

$$(F \upharpoonright \alpha)`K_1`\alpha = F`K_1`\alpha \quad \text{and} \quad (F \upharpoonright \alpha)`K_2`\alpha = F`K_2`\alpha.$$

Consequently

$$F`\alpha = \mathscr{F}_n`(F`K_1`\alpha, F`K_2`\alpha) \quad \text{if } K_3`\alpha = n \neq 0. \qquad \square$$

Examples

$J`\langle 0, 0, 0\rangle = 0 \quad K_3`0 = 0 \quad F`0 = F``0 = 0.$

$J`\langle 0, 0, 1\rangle = 1 \quad K_3`1 = 1 \quad F`1 = \mathscr{F}_1(F`0, F`0) = \{0\} = 1.$

$J`\langle 0, 0, 2\rangle = 2 \quad K_3`2 = 2 \quad F`2 = \mathscr{F}_2(F`0, F`0) = F`0 \cap E = 0.$

$J`\langle 0, 0, 3\rangle = 3 \quad K_3`3 = 3 \quad F`3 = \mathscr{F}_3(F`0, F`0) = F`0 - F`0 = 0.$

$J`\langle 0, 0, 4\rangle = 4 \quad K_3`4 = 4 \quad F`4 = \mathscr{F}_4(F`0, F`0) = F`0 \upharpoonright F`0 = 0.$

$J`\langle 0, 0, 5\rangle = 5 \quad K_3`5 = 5 \quad F`5 = \mathscr{F}_5(F`0, F`0) = F`0 \cap \mathscr{D}(F`0) = 0.$

$J`\langle 0, 0, 6\rangle = 6 \quad K_3`6 = 6 \quad F`6 = \mathscr{F}_6(F`0, F`0) = F`0 \cap (F`0)^{-1} = 0.$

$J`\langle 0, 0, 7\rangle = 7 \quad K_3`7 = 7 \quad F`7 = \mathscr{F}_7(F`0, F`0) = F`0 \cap \text{Cnv}_2(F`0) = 0.$

$J`\langle 0, 0, 8\rangle = 8 \quad K_3`8 = 8 \quad F`8 = \mathscr{F}_8(F`0, F`0) = F`0 \cap \text{Cnv}_3(F`0) = 0.$

$J`\langle 0, 1, 0\rangle = 9 \quad K_3`9 = 0 \quad F`9 = F``9 = \{0, 1\} = 2.$

Definition 15.15. $L \triangleq F``\text{On}.$

A set a is constructible iff $a \in L$.

Remark. We will prove that L is a model of ZF by proving that L is transitive, almost universal and closed under the eight fundamental operations. Those classes that are L-constructible in the sense of Definition 14.12 we will refer to simply as *constructible classes*. The elements of L we will call constructible sets. Indeed if $a = F`\alpha$ we refer to a as the set constructed at the αth stage. From the foregoing example we see that 0 is the set constructed at the 0th, 2nd, 3rd, 4th, 5th, 6th, 7th, and 8th stages. Also 1 is constructed at the first stage and 2 is constructed at the 9th stage. Thus the constructible sets are those sets that can be "built" up from the empty set by a finite or transfinite number of applications of the eight fundamental operations.

Closely related to the notion of constructibility is the notion of relative constructibility. There are many ways to generalize the notion of constructibility. One approach is to introduce an arbitrary set a of natural numbers at the $(\omega + 1)$th stage. Since we wish to construct models of ZF and every such model must contain ω and all of its elements we modify the first $(\omega + 1)$ stages to introduce ω and its elements in a most direct and obvious way. A set is then constructible relative to a iff it can be built up from ω and its elements and from a by a finite or transfinite number of applications of the eight fundamental operations.

Although relative constructibility will not be needed until later we introduce it here because the definition and theorems of interest so closely parallel those for constructibility.

Definition 15.16. If $a \subseteqq \omega$, then

$$\begin{aligned} G_a`x &\triangleq \mathscr{W}(x) \quad \text{if } \mathscr{D}(x) < \omega + 1 \vee K_3`\mathscr{D}(x) = 0 \\ &\triangleq a \qquad\quad \text{if } \mathscr{D}(x) = \omega + 1 \\ &\triangleq \mathscr{F}_n(x`K_1`\mathscr{D}(x), x`K_2`\mathscr{D}(x)) \quad \text{if } \mathscr{D}(x) > \omega + 1 \wedge K_3`\mathscr{D}(x) = n \neq 0. \end{aligned}$$

$$F_a \mathscr{F}\!n \,\mathrm{On} \wedge (\forall \alpha)[F_a`\alpha \triangleq G_a(F_a \restriction \alpha)]$$

$$L_a \triangleq F_a`` \mathrm{On}.$$

Proposition 15.17.

$$\begin{aligned} F_a`\alpha &= \alpha, \alpha \leqq \omega \\ &= a, \alpha = \omega + 1 \\ &= F_a``\alpha, \alpha > \omega + 1 \wedge K_3`\alpha = 0 \\ &= \mathscr{F}_n(F_a`K_1`\alpha, F_a`K_2`\alpha), \alpha > \omega + 1 \wedge K_3`\alpha = n \neq 0. \end{aligned}$$

The proof is left to the reader.

Definition 15.18.

(1) $\mathrm{Od}`x \triangleq \mu_\alpha(x = F`\alpha)$.

(2) $\mathrm{Od}_a`x \triangleq \mu_\alpha(x = F_a`\alpha), a \subseteqq \omega$.

Remark. The symbol $\mathrm{Od}`x$ is read "the order of $x$." If $x$ is constructible then $\mathrm{Od}`x$ is the smallest ordinal α for which $x = F`\alpha$, i.e., $\mathrm{Od}`x$ is the first stage at which x is constructed.

Proposition 15.19.

(1) $x \in L \leftrightarrow x = F`\mathrm{Od}`x$.

(2) $x \in L_a \leftrightarrow x = F_a`\mathrm{Od}_a`x$.

PROOF. Definition 15.18. □

Remark. We wish to prove that L is transitive. For this we prove that the set constructed at the αth stage is constructed only from sets that were constructed at earlier stages.

Proposition 15.20.

(1) $(\forall \alpha)[F`\alpha \subseteq F``\alpha]$.

(2) $(\forall \alpha)[F_a`\alpha \subseteq F_a``\alpha]$.

PROOF. (1) (By transfinite induction). If $\beta = K_1`\alpha$, $\gamma = K_2`\alpha$ and $n = K_3`\alpha$ then

$$\alpha = J`\langle \beta, \gamma, n \rangle.$$

If $n = 0$ then $F`\alpha = F``\alpha$ and hence $F`\alpha \subseteq F``\alpha$. If $n \neq 0$ then by Proposition 15.10, $\beta < \alpha$, $\gamma < \alpha$ and hence

$$F`\beta \in F``\alpha \wedge F`\gamma \in F``\alpha.$$

If $n = 1$ then

$$F`\alpha = \mathscr{F}_1(F`\beta, F`\gamma) = \{F`\beta, F`\gamma\} \subseteq F``\alpha.$$

If $n > 1$ then

$$F`\alpha = \mathscr{F}_n(F`\beta, F`\gamma) \subseteq F`\beta.$$

From the induction hypothesis and the fact that $\beta < \alpha$ we have

$$F`\beta \subseteq F``\beta \subseteq F``\alpha.$$

Therefore

$$F`\alpha \subseteq F``\alpha.$$

(2) The proof is left to the reader. □

Proposition 15.21.

(1) $\mathrm{Tr}(F``\alpha)$.

(2) $\mathrm{Tr}(F_a``\alpha)$.

Proof. (1) If $x \in F``\alpha$, then $(\exists\, \beta < \alpha)[x = F`\beta]$. But from Proposition 15.20 and the fact that $\beta < \alpha$ we have

$$x = F`\beta \subseteq F``\beta \subseteq F``\alpha.$$

(2) The proof is left to the reader. □

Proposition 15.22.

(1) $\mathrm{Tr}(L)$.

(2) $\mathrm{Tr}(L_a)$.

Proof. (1) If $x \in L$, then $(\exists\, \alpha)[x = F`\alpha]$. Therefore

$$x = F`\alpha \in F``(\alpha + 1).$$

Since $F``(\alpha + 1)$ is transitive

$$x \subseteq F``(\alpha + 1) \subseteq L.$$

(2) The proof is left to the reader. □

Proposition 15.23.

(1) $x \in L \wedge y \in L \wedge x \in y \rightarrow \mathrm{Od}`x < \mathrm{Od}`y$.

(2) $x \in L_a \wedge y \in L_a \wedge x \in y \rightarrow \mathrm{Od}`_a x < \mathrm{Od}`_a y$.

Proof.

(1)
$$\begin{aligned} x \in y \wedge y \in L &\rightarrow x \in F`\mathrm{Od}`y \\ &\rightarrow x \in F``\mathrm{Od}`y \\ &\rightarrow (\exists\, \beta < \mathrm{Od}`y)[x = F`\beta] \\ &\rightarrow \mathrm{Od}`x < \mathrm{Od}`y. \end{aligned}$$

(2) The proof is left to the reader. □

Proposition 15.24.

$$(\forall\, x \in L_a)(\exists\, \alpha > \omega)[x = F`_a\alpha].$$

The proof is left to the reader. □

Proposition 15.25.

(1) $(\forall\, x, y \in L)[\mathscr{F}_n(x, y) \in L]$, $n = 1, \ldots, 8$.

(2) $(\forall\, x, y \in L_a)[\mathscr{F}_n(x, y) \in L_a]$, $n = 1, \ldots, 8$.

Proof. (1) If $\alpha = \mathrm{Od}`x$ and $\beta = \mathrm{Od}`y$ then $x = F`\alpha$ and $y = F`\beta$. Let $\gamma = J`\langle \alpha, \beta, n\rangle$. Then

$$\mathscr{F}_n(x, y) = \mathscr{F}_n(F`\alpha, F`\beta) = \mathscr{F}_n(F`K`_1\gamma, F`K`_2\gamma) = F`\gamma \in L.$$

(2) The proof is left to the reader. □

Proposition 15.26.

(1) $b \subseteq L \to (\exists x \in L)[b \subseteq x]$.

(2) $b \subseteq L_a \to (\exists x \in L_a)[b \subseteq x]$.

PROOF. (1) Since Od is a function from V into On, Od“b is a set of ordinals. Therefore

$$(\exists \alpha)[\text{Od}``b \subseteq \alpha].$$

Let

$$\beta = J`\langle 0, \alpha, 0\rangle.$$

Then $K_3`\beta = 0$ and hence

$$F`\beta = F``\beta.$$

Furthermore $\alpha \leqq \beta$. Therefore if $x = F`\beta$

$$y \in b \to y \in L \wedge \text{Od}`y < \alpha \leqq \beta,$$

and hence

$$(\exists \gamma < \beta)[y = F`\gamma].$$

Then

$$y = F`\gamma \in F``\beta = F`\beta = x$$

that is

$$b \subseteq x \wedge x \in L.$$

(2) The proof is left to the reader. □

Theorem 15.27. (1) *L is a standard transitive model of* ZF *and* On $\subseteq L$.

(2) *L_a is a standard transitive model of* ZF *and* On $\subseteq L_a$.

PROOF. Propositions 15.22, 15.25, 15.26, and Theorem 14.11. □

Remark. We have now shown that L is a model of ZF and for each $a \subseteq \omega$, L_a is a model of ZF; but are these models different? It is not difficult to show that if a is constructible then $L_a = L$. Do there exist nonconstructible sets? From Cohen's work we know that this question is undecidable in ZF.

The assumption that every set is constructible is called the Axiom of Constructibility.

Axiom of Constructibility

$$V = L.$$

Gödel's program for proving the consistency of GCH and AC consists of proving that the Axiom of Constructibility implies GCH and AC. It is then sufficient to establish the consistency of the Axiom of Constructibility with

ZF. This is done by proving that L is a model of $V = L$. To prove this we must prove in ZF that

$$[V = L]^L$$

that is, since $L = \{x \mid (\exists\, \alpha)[x = F`\alpha]\}$ we must prove that

$$L = \{x \in L \mid (\exists\, \alpha \in L)[x = F`\alpha]^L\}.$$

Since $\mathrm{On} \subseteqq L$ it is sufficient to prove that $x = F`\alpha$ is absolute with respect to $L$. This we will do by proving that $x = F`\alpha$ is absolute with respect to every standard transitive model of ZF. We need the following lemmas in which M is a standard transitive model of ZF and G is as given in Definition 15.13.

Lemma 1. $\langle \alpha, \beta, m\rangle S \langle \gamma, \delta, n\rangle$ Abs M.

PROOF.

$$\langle \alpha, \beta, m\rangle S \langle \gamma, \delta, n\rangle \leftrightarrow \langle \alpha, \beta\rangle R_0 \langle \gamma, \delta\rangle \vee [\langle \alpha, \beta\rangle = \langle \gamma, \delta\rangle \wedge m < n]. \quad \square$$

Lemma 2. f $\mathrm{Isom}_{S,E}(\beta^2 \times 9, \alpha)$ Abs M.

PROOF. Proposition 13.30. □

Lemma 3.

(1) $\beta = J`\langle \gamma, \delta, m\rangle$ Abs M.

(2) $\beta = J_0`\langle \gamma, \delta\rangle$ Abs M.

PROOF. (1) From properties of order isomorphisms (Proposition 7.53)

$$(\exists !\, f)(\exists !\, \alpha)[f\ \mathrm{Isom}_{S,E}(\mu \times \mu \times 9, \alpha)].$$

Therefore, from the definition of J

$$\begin{aligned}\beta = J`\langle \gamma, \delta, m\rangle &\leftrightarrow (\exists\, f)(\exists\, \alpha)[m < 9 \wedge f\ \mathrm{Isom}_{S,E}(\max(\gamma, \delta) \\ &\qquad \times \max(\gamma, \delta) \times 9, \alpha) \wedge f`\langle \gamma, \delta, m\rangle = \beta] \\ &\leftrightarrow (\forall\, f)(\forall\, \alpha)[m < 9 \wedge f\ \mathrm{Isom}_{S,E}(\max(\gamma, \delta) \\ &\qquad \times \max(\gamma, \delta) \times 9, \alpha) \to f`\langle \gamma, \delta, m\rangle = \beta].\end{aligned}$$

From Theorem 13.8 it then follows that

$$\beta = J`\langle \gamma, \delta, m\rangle \text{ Abs } M.$$

(2) The proof is left to the reader. □

Lemma 4.

(1) $K_1`\alpha = \beta$ Abs M.

(2) $K_2`\alpha = \beta$ Abs M.

(3) $K_3`\alpha = \beta$ Abs M.

PROOF.

$$K_1`\alpha = \beta \leftrightarrow (\exists\, m)(\exists\, \gamma)[m < 9 \wedge J`\langle \beta, \gamma, m\rangle = \alpha].$$

$$K_2`\alpha = \beta \leftrightarrow (\exists\, m)(\exists\, \gamma)[m < 9 \wedge J`\langle \gamma, \beta, m\rangle = \alpha].$$

$$K_3`\alpha = \beta \leftrightarrow \beta < 9 \wedge (\exists\, \gamma)(\exists\, \delta)[J`\langle \gamma, \delta, m\rangle = \alpha].$$

Since $J`\langle \beta, \gamma, m\rangle = \alpha$ is absolute with respect to $M$ and $\max(\beta, \gamma) \leqq J`\langle \beta, \gamma, m\rangle = \alpha$ it follows that $\alpha \in M$ implies $\beta, \gamma \in M$. The results then follow from Proposition 13.5. □

Lemma 5. $b = \mathscr{F}_n(c, d)$ Abs M, $\quad n = 1, \ldots, 8.$

PROOF.

$$b = \mathscr{F}_1(c, d) \leftrightarrow b = \{c, d\}.$$

$$\begin{aligned} b = \mathscr{F}_2(c, d) &\leftrightarrow b = c \cap E \\ &\leftrightarrow (\forall\, x)[x \in b \leftrightarrow x \in c \wedge (\exists\, y)(\exists\, z)[x = \langle y, z\rangle \wedge y \in z]]. \end{aligned}$$

$$\begin{aligned} b = \mathscr{F}_3(c, d) &\leftrightarrow b = c - d \\ &\leftrightarrow (\forall\, x)[x \in b \leftrightarrow x \in c \wedge x \notin d]. \end{aligned}$$

$$\begin{aligned} b = \mathscr{F}_4(c, d) &\leftrightarrow b = c \restriction d \\ &\leftrightarrow (\forall\, x)[x \in b \leftrightarrow x \in c \wedge (\exists\, y)(\exists\, z)[x = \langle y, z\rangle \wedge y \in d]]. \end{aligned}$$

$$\begin{aligned} b = \mathscr{F}_5(c, d) &\leftrightarrow b = c \cap \mathscr{D}(d) \\ &\leftrightarrow (\forall\, x)[x \in b \leftrightarrow x \in c \wedge (\exists\, y)[y = \mathscr{D}(d) \wedge x \in y]]. \end{aligned}$$

$$\begin{aligned} b = \mathscr{F}_6(c, d) &\leftrightarrow b = c \cap d^{-1} \\ &\leftrightarrow (\forall\, x)[x \in b \leftrightarrow x \in c \wedge (\exists\, y)[y = d^{-1} \wedge x \in y]]. \end{aligned}$$

$$\begin{aligned} b = \mathscr{F}_7(c, d) &\leftrightarrow b = c \cap \mathrm{Cnv}_2(d) \\ &\leftrightarrow (\forall\, x)[x \in b \leftrightarrow x \in c \\ &\qquad \wedge (\exists\, u)(\exists\, v)(\exists\, w)[x = \langle u, v, w\rangle \wedge \langle w, u, v\rangle \in d]]. \end{aligned}$$

$$\begin{aligned} b = \mathscr{F}_8(c, d) &\leftrightarrow b = c \cap \mathrm{Cnv}_3(d) \\ &\leftrightarrow (\forall\, x)[x \in b \leftrightarrow x \in c \\ &\qquad \wedge (\exists\, u)(\exists\, v)(\exists\, w)[x = \langle u, v, w\rangle \wedge \langle u, w, v\rangle \in d]]. \end{aligned}$$ □

Lemma 6. $b = G`(f \restriction \beta)$ Abs M.

PROOF.

$$\begin{aligned} b = G`(f \restriction \beta) \leftrightarrow\ & [K_3`\beta = 0 \wedge b = f``\beta] \vee \\ & [K_3`\beta = 1 \wedge b = \mathscr{F}_1(f`K_1`\beta, f`K_2`\beta)] \vee \cdots \vee \\ & [K_3`\beta = 8 \wedge b = \mathscr{F}_8(f`K_1`\beta, f`K_2`\beta)]. \end{aligned}$$ □

Proposition 15.28.

(1) $b = F`\alpha$ Abs M.

(2) $b = F`_a\alpha$ Abs M.

PROOF. (1) From the definition of F and Corollary 7.42

$$(\exists!\, f)[f\ \mathscr{F}n(\alpha + 1) \wedge (\forall \beta \leqq \alpha)[f`\beta = G`(f \restriction \beta)]].$$

Therefore

$$b = F`\alpha \leftrightarrow (\exists f)[f\ \mathscr{F}n(\alpha + 1) \wedge (\forall \beta \leqq \alpha)[f`\beta = G`(f \restriction \beta)] \wedge \langle \alpha, b\rangle \in f]$$
$$\leftrightarrow (\forall f)[f\ \mathscr{F}n(\alpha + 1) \wedge (\forall \beta \leqq \alpha)[f`\beta = G`(f \restriction \beta)] \rightarrow \langle \alpha, b\rangle \in f].$$

From the preceding lemmas and Theorem 13.8 it then follows that $b = F`\alpha$ Abs M.

(2) The proof is left to the reader. □

Proposition 15.29.

(1) $\text{On} \subseteqq M \rightarrow L \subseteqq M$.

(2) $\text{On} \subseteqq M \wedge a \in M \rightarrow L_a \subseteqq M$.

PROOF. (1) Since $(\forall \alpha)(\exists x)[x = F`\alpha]$, it follows that

$$(\forall \alpha \in M)(\exists x \in M)[x = F`\alpha]^M.$$

But since $\text{On} \subseteqq M$ and $x = F`\alpha$ Abs M

$$(\forall \alpha)(\exists x \in M)[x = F`\alpha].$$

Therefore

$$L \subseteqq M.$$

(2) The proof is left to the reader. □

Remark. From Proposition 15.29 we see that L is the smallest of all the standard transitive models that contain On. In particular if $a \subseteqq \omega$ then $L \subseteqq L_a$. Furthermore if a is constructible, i.e., if $a \in L$ then $L_a \subseteqq L$, i.e., $L = L_a$.

Theorem 15.30.

(1) *L is a model of $V = L$.*

(2) *L_a is a model of $V = L_a$.*

PROOF. (1) $V = L \leftrightarrow (\forall x)(\exists \alpha)[x = F`\alpha]$.

From the definition of L

$$(\forall x \in L)(\exists \alpha)[x = F`\alpha].$$

Since $\mathrm{On} \subseteq L$ and $x = F`\alpha$ Abs L it follows that

$$(\forall x \in L)(\exists \alpha \in L)[x = F`\alpha]^L$$

i.e.,

$$[V = L]^L.$$

(2) The proof is left to the reader. □

Definition 15.31.

(1) $\mathrm{As} \triangleq \{\langle x, y\rangle \in L^2 \mid y \in x \wedge (\forall z \in x)[\mathrm{Od}`y \leq \mathrm{Od}`z]\}$.

(2) $\mathrm{As}_a \triangleq \{\langle x, y\rangle \in L_a^2 \mid y \in x \wedge (\forall z \in x)[\mathrm{Od}_a`y \leq \mathrm{Od}_a`z]\}$.

Proposition 15.32.

(1) $(\forall x \in L)[x \neq 0 \rightarrow \mathrm{As}`x \in x]$.

(2) $(\forall x \in L_a)[x \neq 0 \rightarrow \mathrm{As}_a`x \in x]$.

PROOF. (1) Since L is transitive $L - \{0\} \subseteq \mathscr{D}(\mathrm{As})$. Furthermore As is single valued. Therefore

$$x \in L \wedge x \neq 0 \rightarrow \mathrm{As}`x \in x.$$

(2) The proof is left to the reader. □

Theorem 15.33.

(1) $V = L \rightarrow \mathrm{AC}$.

(2) $V = L_a \rightarrow \mathrm{AC}$.

PROOF. Obvious from Proposition 15.32. □

Theorem 15.34.

(1) *L is a model of* AC.

(2) *L_a is a model of* AC.

PROOF. Propositions 15.30 and 15.33. □

Remark. In Proposition 15.32 we have a result that is in fact stronger than the strong form of AC. The strong form of AC asserts the existence of a universal choice function. In the proof of Proposition 15.32 we have exhibited such a function.

We turn now to a proof that the Axiom of Constructibility implies GCH:[1]

$$(\forall \alpha)[\overline{\overline{2^{\aleph_\alpha}}} = \aleph_{\alpha+1}].$$

The key to the proof lies in proving two results. First, we prove that the cardinality of $F``\aleph_\alpha$ is $\aleph_\alpha$. From this we deduce GCH by proving that if $V = L$ then every subset of $F``\aleph_\alpha$ is constructed before the $\aleph_{\alpha+1}$th stage.

[1] A simpler proof is presented in Chapter 17. The remainder of this chapter can be omitted without loss of continuity.

Definition 15.35.

(1) $C`\alpha \triangleq \text{Od}`\text{As}`F`\alpha$.

(2) $C`_a\alpha \triangleq \text{Od}`_a\text{As}`_aF`_a\alpha$.

Proposition 15.36.

(1) $C`\alpha \leqq \alpha$.

(2) $C`_a\alpha \leqq \alpha$.

Proof. (1) If $F`\alpha = 0$ then $\text{As}`F`\alpha = 0$ and $\text{Od}`\text{As}`F`\alpha = 0$, i.e., $C`\alpha = 0 \leqq \alpha$. If $F`\alpha \neq 0$ then $\text{As}`F`\alpha \in F`\alpha$. Therefore

$$C`\alpha = \text{Od}`\text{As}`F`\alpha < \text{Od}`F`\alpha \leqq \alpha.$$

(2) The proof is left to the reader. □

Proposition 15.37.

(1) $\overline{\overline{F``\aleph_\alpha}} = \aleph_\alpha$.

(2) $\overline{\overline{F`_a`\aleph_\alpha}} = \aleph_\alpha$.

Proof. (1) Since F is a function it follows that

$$\overline{\overline{F``\aleph_\alpha}} \leqq \aleph_\alpha.$$

Furthermore

$$F`J`\langle 0, \beta, 0\rangle = F``J`\langle 0, \beta, 0\rangle.$$

Therefore since $\gamma < \beta$ implies $J`\langle 0, \gamma, 0\rangle < J`\langle 0, \beta, 0\rangle$ it follows that if $\gamma < \beta$ then

$$F`J`\langle 0, \gamma, 0\rangle \in F`J`\langle 0, \beta, 0\rangle$$

that is

$$\gamma \neq \beta \rightarrow F`J`\langle 0, \gamma, 0\rangle \neq F`J`\langle 0, \beta, 0\rangle.$$

Thus if

$$H`\beta \triangleq F`J`\langle 0, \beta, 0\rangle, \qquad \beta \in \text{On}$$

then $H: \text{On} \xrightarrow{1-1} L$. Since

$$\beta < \aleph_\alpha \rightarrow J`\langle 0, \beta, 0\rangle < \aleph_\alpha$$

it follows that

$$H``\aleph_\alpha \subseteq F``\aleph_\alpha.$$

Since H is one-to-one

$$\aleph_\alpha = \overline{\overline{H``\aleph_\alpha}} \leqq \overline{\overline{F``\aleph_\alpha}}.$$

Therefore $\overline{\overline{F``\aleph_\alpha}} = \aleph_\alpha$.

(2) The proof is left to the reader. □

Proposition 15.38.

(1) $(\forall \alpha)[\mathscr{P}(F``\aleph_\alpha) \subseteq F``\aleph_{\alpha+1} \to \overline{\overline{2^{\aleph_\alpha}}} = \aleph_{\alpha+1}]$.

(2) $(\forall \alpha)[\mathscr{P}(F_a``\aleph_\alpha) \subseteq F_a``\aleph_{\alpha+1} \to \overline{\overline{2^{\aleph_\alpha}}} = \aleph_{\alpha+1}]$.

PROOF. (1) If $\mathscr{P}(F``\aleph_\alpha) \subseteq F``\aleph_{\alpha+1}$ then from Proposition 15.37

$$\overline{\overline{2^{\aleph_\alpha}}} = \overline{\overline{\mathscr{P}(\aleph_\alpha)}} = \overline{\overline{\mathscr{P}(F``\aleph_\alpha)}} \leq \overline{\overline{F``\aleph_{\alpha+1}}} = \aleph_{\alpha+1}.$$

Since by Cantor's Theorem $\overline{\overline{2^{\aleph_\alpha}}} > \aleph_\alpha$ we have

$$\overline{\overline{2^{\aleph_\alpha}}} = \aleph_{\alpha+1}.$$

(2) The proof is left to the reader. □

Remark. It now remains to be proved that if $V = L$ then each subset of $F``\aleph_\alpha$ is constructed before the $\aleph_{\alpha+1}$th stage. This we do in the following sequence of propositions.

Proposition 15.39. (1) *If* $V = L$ *and* $(\forall x)(\forall \gamma)(\forall f)[9 \subseteq x \subseteq \mathrm{On} \wedge C``x \subseteq x \wedge K_1``x \subseteq x \wedge K_2``x \subseteq x \wedge J``x^3 \subseteq x \wedge f \ \mathrm{Isom}_{E,E}(x, \gamma) \to (\forall \alpha \in x)(\forall \beta \in x)[F`\alpha \in F`\beta \leftrightarrow F`f`\alpha \in F`f`\beta]]$ *then* $(\forall \alpha)[\mathscr{P}(F``\aleph_\alpha) \subseteq F``\aleph_{\alpha+1}]$.

(2) *If* $V = L_a$ *and* $(\forall x)(\forall \gamma)(\forall f)[\omega + 2 \subseteq x \subseteq \mathrm{On} \wedge C_a``x \subseteq x \wedge K_1``x \subseteq x \wedge K_2``x \subseteq x \wedge J``x^3 \subseteq x \wedge f \ \mathrm{Isom}_{E,E}(x, \gamma) \to (\forall \alpha \in x)(\forall \beta \in x)[F_a`\alpha \in F_a`\beta \leftrightarrow F_a`f`\alpha \in F_a`f`\beta]]$ *then* $(\forall \alpha)[\mathscr{P}(F_a``\aleph_\alpha) \subseteq F_a``\aleph_{\alpha+1}]$.

PROOF. (1) If $x \subseteq F``\aleph_\alpha$ then from the Axiom of Constructibility

$$(\exists \delta)[x = F`\delta].$$

Since C, K_1, K_2 and J are each single valued and $\aleph_\alpha \cup \{\delta\}$ is infinite it follows from Proposition 11.32 that there exists a set b such that

$$[C``b \subseteq b \wedge K_1``b \subseteq b \wedge K_2``b \subseteq b \wedge J``b^3 \subseteq b \wedge$$
$$\aleph_\alpha \cup \{\delta\} \subseteq b \subseteq \mathrm{On} \wedge \overline{\overline{b}} = \aleph_\alpha].$$

Since $V = L$ implies AC, $(\exists \gamma)(\exists f) \ \mathrm{Isom}_{E,E}(b, \gamma)$. By hypothesis

$$f \ \mathrm{Isom}_{E,E}(b, \gamma) \to (\forall \alpha \in b)(\forall \beta \in b)[F`\alpha \in F`\beta \leftrightarrow F`f`\alpha \in F`f`\beta].$$

In particular, since $\aleph_\alpha \subseteq b$ and f is order preserving

$$\beta < \aleph_\alpha \to f`\beta = \beta.$$

Thus, since $\delta \in b$, if $\beta < \aleph_\alpha$

$$[F`\beta \in F`\delta \leftrightarrow F`\beta \in F`f`\delta].$$

Consequently

$$F`\delta \cap F``\aleph_\alpha = F`f`\delta \cap F``\aleph_\alpha.$$

But since $\aleph_\alpha = J`\langle 0, \aleph_\alpha, 0\rangle$,

$$F``\aleph_\alpha = F`\aleph_\alpha.$$

Furthermore, since $F`\delta = x \subseteq F``\aleph_\alpha$,

$$F`\delta \cap F``\aleph_\alpha = F`\delta = x.$$

Then

$$\begin{aligned} x &= F`f`\delta \cap F`\aleph_\alpha \\ &= F`f`\delta - [F`f`\delta - F`\aleph_\alpha] \\ &= F`J`\langle f`\delta, J`\langle f`\delta, \aleph_\alpha, 3\rangle, 3\rangle. \end{aligned}$$

But since $\delta \in b \wedge \bar{\bar{b}} = \aleph_\alpha$ and since f is an order isomorphism, $f`\delta < \aleph_{\alpha+1}$. Therefore by Proposition 15.1

$$J`\langle f`\delta, \aleph_\alpha, 3\rangle < \aleph_{\alpha+1}$$

and hence by a second application of Proposition 15.1

$$J`\langle f`\delta, J`\langle f`\delta, \aleph_\alpha, 3\rangle, 3\rangle < \aleph_{\alpha+1}.$$

Consequently

$$x \in F``\aleph_{\alpha+1}.$$

(2) The proof is left to the reader. □

Remark. In somewhat over simplified terms Proposition 15.39 states that if every set is constructible and if certain ordinal isomorphisms preserve the order of constructibility then each subset of $F``\aleph_\alpha$ is constructed before the $\aleph_{\alpha+1}$th stage.

Since for any set of ordinals b

$$(\exists!\, \gamma)(\exists!\, f)[f \text{ Isom}_{E,E}(b, \gamma)]$$

it remains to be proved that for appropriately chosen sets b, namely those closed under C, K_1, K_2, K_3, and J, f does preserve the order of constructibility.

Proposition 15.40. *If* $9 \subseteq b \subseteq \text{On} \wedge K_1``b \subseteq b \wedge K_2``b \subseteq b \wedge J``b^3 \subseteq b \wedge f \text{ Isom}_{E,E}(b, \eta)$ *then*

(1) $m < 9 \wedge \alpha \in b \wedge \beta \in b \rightarrow J`\langle f`\alpha, f`\beta, m\rangle = f`J`\langle\alpha, \beta, m\rangle$

and

(2) $J``\eta^3 \subseteq \eta$.

Proof. (1) Since J is order preserving

$$J`\langle f`\alpha, f`\beta, m\rangle < J`\langle f`\gamma, f`\delta, n\rangle \leftrightarrow \langle f`\alpha, f`\beta, m\rangle S \langle f`\gamma, f`\delta, n\rangle.$$

But f is also order preserving. Therefore

$$\langle f`\alpha, f`\beta, m\rangle S \langle f`\gamma, f`\delta, n\rangle \leftrightarrow \langle\alpha, \beta, m\rangle S \langle\gamma, \delta, n\rangle.$$

If there exists an ordered triple $\langle\alpha, \beta, m\rangle$ such that

$$J`\langle f`\alpha, f`\beta, m\rangle \neq f`J`\langle\alpha, \beta, m\rangle$$

then there is an S-minimal such element. We will show that the assumption that $\langle \alpha, \beta, m \rangle$ is such an ordered triple leads to a contradiction.

If $J`\langle f`\alpha, f`\beta, m \rangle < f`J`\langle \alpha, \beta, m \rangle$ then since b is closed w.r.t. J and since α, β, and m are in b it follows that

$$J`\langle \alpha, \beta, m \rangle \in b$$

and hence

$$f`J`\langle \alpha, \beta, m \rangle \in \eta.$$

Since

$$J`\langle f`\alpha, f`\beta, m \rangle < f`J`\langle \alpha, \beta, m \rangle$$

it follows that

$$(\exists\, v \in b)[f`v = J`\langle f`\alpha, f`\beta, m \rangle].$$

If

$$\gamma = K_1`v \wedge \delta = K_2`v \wedge n = K_3`v$$

then since $v \in b$ and b is closed w.r.t. K_1, K_2, K_3

$$\gamma \in b \wedge \delta \in b \wedge n \in b.$$

Therefore

$$v = J`\langle \gamma, \delta, n \rangle \in b$$

and

$$f`J`\langle \gamma, \delta, n \rangle = f`v = J`\langle f`\alpha, f`\beta, m \rangle.$$

Since by hypothesis

$$J`\langle f`\alpha, f`\beta, m \rangle < f`J`\langle \alpha, \beta, m \rangle$$

we have that

$$f`J`\langle \gamma, \delta, n \rangle < f`J`\langle \alpha, \beta, m \rangle.$$

From this it follows that

$$\langle \gamma, \delta, n \rangle S \langle \alpha, \beta, m \rangle.$$

But from the defining property of $\langle \alpha, \beta, m \rangle$

$$J`\langle f`\gamma, f`\delta, n \rangle = f`J`\langle \gamma, \delta, n \rangle = J`\langle f`\alpha, f`\beta, m \rangle.$$

Since both J and f are one-to-one this implies that

$$\langle \gamma, \delta, n \rangle = \langle \alpha, \beta, m \rangle$$

which is a contradiction.

If $f`J`\langle\alpha, \beta, m\rangle < J`\langle f`\alpha, f`\beta, m\rangle$ and if

$$\gamma = K_1` f`J`\langle\alpha, \beta, m\rangle \wedge \delta = K_2` f`J`\langle\alpha, \beta, m\rangle \wedge n = K_3` f`J`\langle\alpha, \beta, m\rangle$$

then

$$f`J`\langle\alpha, \beta, m\rangle = J`\langle\gamma, \delta, n\rangle < J`\langle f`\alpha, f`\beta, m\rangle$$

hence

$$\langle\gamma, \delta, n\rangle S \langle f`\alpha, f`\beta, m\rangle.$$

Therefore

$$\gamma \leqq \max(f`\alpha, f`\beta) \wedge \delta \leqq \max(f`\alpha, f`\beta).$$

Since $\alpha, \beta \in b$,

$$f`\alpha \in \eta \wedge f`\beta \in \eta$$

that is

$$\max(f`\alpha, f`\beta) < \eta.$$

Then $\gamma, \delta \in \eta$, and consequently

$$(\exists\, \gamma_0, \delta_0 \in b)[\gamma = f`\gamma_0 \wedge \delta = f`\delta_0].$$

Since

$$\langle\gamma, \delta, n\rangle S \langle f`\alpha, f`\beta, m\rangle,$$

$$\langle f`\gamma_0, f`\delta_0, n\rangle S \langle f`\alpha, f`\beta, m\rangle$$

and hence

$$\langle\gamma_0, \delta_0, n\rangle S \langle\alpha, \beta, m\rangle.$$

Again from the defining property of $\langle\alpha, \beta, m\rangle$

$$f`J`\langle\gamma_0, \delta_0, n\rangle = J`\langle f`\gamma_0, f`\delta_0, n\rangle = J`\langle\gamma, \delta, n\rangle = f`J`\langle\alpha, \beta, m\rangle.$$

Since f and J are one-to-one we conclude that

$$\langle\gamma_0, \delta_0, n\rangle = \langle\alpha, \beta, m\rangle$$

which is a contradiction.

(2) If $\gamma, \delta \in \eta$ then

$$(\exists\, \gamma_0 \in b)(\exists\, \delta_0 \in b)[\gamma = f`\gamma_0 \wedge \delta = f`\delta_0].$$

Therefore from (1)

$$J`\langle\gamma, \delta, m\rangle = J`\langle f`\gamma_0, f`\delta_0, m\rangle = f`J`\langle\gamma_0, \delta_0, m\rangle.$$

Since b is closed w.r.t. J and f maps b into η,

$$J`\langle\gamma, \delta, m\rangle \in \eta$$

i.e., η is closed w.r.t. J. □

Proposition 15.41. *If* $9 \subseteq b \subseteq \mathrm{On} \wedge 9 \subseteq c \subseteq \mathrm{On} \wedge K_1``b \subseteq b \wedge K_2``b \subseteq b \wedge J``b^3 \subseteq b \wedge K_1``c \subseteq c \wedge K_2``c \subseteq c \wedge J``c^3 \subseteq c \wedge f \; \mathrm{Isom}_{E,E}(b, c)$ *then*

(1) $\alpha \in b \wedge \beta \in b \wedge m < 9 \to J`\langle f`\alpha, f`\beta, m\rangle = f`J`\langle \alpha, \beta, m\rangle$

and

(2) $\alpha \in b \to K_1` f`\alpha = f`K_1`\alpha \wedge K_2` f`\alpha = f`K_2`\alpha \wedge K_3`\alpha = K_3` f`\alpha.$

PROOF. (1) Since $f \; \mathrm{Isom}_{E,E}(b, c)$ it follows that for some η, f_1, and f_2 we have

$$f_1 \; \mathrm{Isom}_{E,E}(b, \eta) \wedge f_2 \; \mathrm{Isom}_{E,E}(c, \eta) \wedge f_2 \circ f = f_1.$$

Since b is closed w.r.t. J

$$\alpha \in b \wedge \beta \in b \wedge m < 9 \to J`\langle \alpha, \beta, m\rangle \in b$$

and hence

$$f`J`\langle \alpha, \beta, m\rangle \in c.$$

If

$$\gamma = K_1` f`J`\langle \alpha, \beta, m\rangle \wedge \delta = K_2` f`J`\langle \alpha, \beta, m\rangle \wedge n = K_3` f`J`\langle \alpha, \beta, m\rangle$$

then since c is closed w.r.t. K_1, and K_2

$$\gamma \in c \wedge \delta \in c \wedge n < 9$$

and

$$f`J`\langle \alpha, \beta, m\rangle = J`\langle \gamma, \delta, n\rangle.$$

From Proposition 15.40 we then have that

$$f_1`J`\langle \alpha, \beta, m\rangle = (f_2 \circ f)`J`\langle \alpha, \beta, m\rangle = f_2`J`\langle \gamma, \delta, n\rangle = J`\langle f_2`\gamma, f_2`\delta, n\rangle.$$

On the other hand we also have from Proposition 15.40 that

$$f_1`J`\langle \alpha, \beta, m\rangle = J`\langle f_1`\alpha, f_1`\beta, m\rangle.$$

Therefore since J is one-to-one

$$f_1`\alpha = f_2`\gamma \wedge f_1`\beta = f_2`\delta \wedge m = n$$

that is

$$\gamma = f`\alpha \wedge \delta = f`\beta \wedge m = n.$$

Therefore

$$f`J`\langle \alpha, \beta, m\rangle = J`\langle \gamma, \delta, n\rangle = J`\langle f`\alpha, f`\beta, m\rangle.$$

(2) Since b is closed w.r.t. K_1 and K_2 we have that

$$\alpha \in b \to K_1`\alpha \in b \wedge K_2`\alpha \in b.$$

Since

$$\alpha = J`\langle K_1`\alpha, K_2`\alpha, K_3`\alpha\rangle$$

we have from (1)

$$f`\alpha = f`J`\langle K_1`\alpha, K_2`\alpha, K_3`\alpha\rangle = J`\langle f`K_1`\alpha, f`K_2`\alpha, K_3`\alpha\rangle.$$

Therefore

$$K_1`f`\alpha = f`K_1`\alpha \wedge K_2`f`\alpha = f`K_2`\alpha \wedge K_3`f`\alpha = K_3`\alpha. \qquad \square$$

Remark. For our next theorem we need several results that we prove as lemmas.

Lemma 1. *If* $9 \subseteqq b \subseteqq \mathrm{On}$ *and b is closed* w.r.t. J, *then $F``b$ is closed* w.r.t. *the fundamental operations.*

PROOF. If $x, y \in F``b$ then

$$(\exists\, \alpha, \beta \in b)[x = F`\alpha \wedge y = F`\beta]$$

and hence

$$\mathscr{F}_n(x, y) = \mathscr{F}_n(F`\alpha, F`\beta) = F`J`\langle \alpha, \beta, n\rangle.$$

Since b is closed w.r.t. J it follows that

$$J`\langle \alpha, \beta, n\rangle \in b$$

and hence

$$\mathscr{F}_n(x, y) \in F``b. \qquad \square$$

Lemma 2. *If* $9 \subseteqq b \subseteqq \mathrm{On}$, *if b is closed* w.r.t. C *and* J, *and if* $x \in F``b$, *then*

$$\mathrm{Od}`x \in b.$$

PROOF. From Lemma 1

$$\begin{aligned} x \in F``b &\to \{x\} \in F``b \\ &\to (\exists\, \alpha \in b)[\{x\} = F`\alpha]. \end{aligned}$$

But

$$\mathrm{Od}`x = \mathrm{Od}`\mathrm{As}`\{x\} = \mathrm{Od}`\mathrm{As}`F`\alpha = C`\alpha.$$

Since b is closed w.r.t. C and $\alpha \in b$ we have that

$$C`\alpha \in b$$

i.e.,

$$\mathrm{Od}`x \in b. \qquad \square$$

Lemma 3. *If* $9 \subseteqq b \subseteqq \mathrm{On}$ *and b is closed* w.r.t. C *then*

$$x \in F``b \wedge x \neq 0 \to x \cap F``b \neq 0.$$

PROOF. If $x \in F``b$, then $(\exists \alpha \in b)[x = F`\alpha]$. Since b is closed w.r.t. C

$$\begin{aligned} \alpha \in b &\to C`\alpha \in b \\ &\to F`C`\alpha \in F``b. \end{aligned}$$

But

$$F`C`\alpha = F`\mathrm{Od}`\mathrm{As}`F`\alpha = \mathrm{As}`x.$$

If $x \neq 0$ then

$$\mathrm{As}`x \in x$$

i.e.,

$$F`C`\alpha \in x \cap F``b. \qquad \square$$

Lemma 4. *If* $9 \subseteq b \subseteq \mathrm{On}$ *and* b *is closed* w.r.t. C *and* J *then*

(1) $(\forall x, y)[\{x, y\} \in F``b \to x, y \in F``b]$,
(2) $(\forall x, y)[\langle x, y\rangle \in F``b \to x, y \in F``b]$,
(3) $(\forall x, y, z)[\langle x, y, z\rangle \in F``b \to x, y, z \in F``b]$.

PROOF. (1) Since $\langle x, y\rangle \neq 0$ we have from Lemma 3 that

$$\{x, y\} \cap F``b \neq 0.$$

Therefore

$$x \in F``b \vee y \in F``b.$$

If $x \in F``b$ and $x \neq y$ then from Lemma 1, $\{x\} \in F``b$ and

$$\{y\} = \{x, y\} - \{x\} \in F``b.$$

Since $\{y\} \in F``b$ and $\{y\} \neq 0$ it follows from Lemma 3 that $y \in F``b$. Similarly if $y \in F``b$, then $x \in F``b$.

(2)–(3) The proofs are left to the reader. $\square$

Lemma 5. *If* $9 \subseteq b \subseteq \mathrm{On}$ *and* b *is closed* w.r.t. C *and* J, *then*

$$(\forall x, y)[\langle x, y\rangle \in Q_n \wedge y \in F``b \to x \in F``b], \qquad n = 4, 6, 7, 8.$$

PROOF. If $\langle x, y\rangle \in Q_4$, then $(\exists z)[y = \langle x, z\rangle]$. But by Lemma 4

$$y \in F``b \to x \in F``b.$$

If $\langle x, y\rangle \in Q_6$, then $(\exists z, w)[x = \langle z, w\rangle \wedge y = \langle w, z\rangle]$. By Lemmas 1 and 4

$$y \in F``b \to x \in F``b.$$

If $\langle x, y\rangle \in Q_7$, then $(\exists z)(\exists w)(\exists u)[x = \langle u, w, z\rangle \wedge y = \langle w, z, u\rangle]$.
If $\langle x, y\rangle \in Q_8$, then $(\exists z)(\exists w)(\exists u)[x = \langle u, w, z\rangle \wedge y = \langle u, z, w\rangle]$.

In each case we see from Lemmas 1 and 4 that

$$y \in F``b \rightarrow x \in F``b. \qquad \square$$

Lemma 6. *If* $9 \subseteq b \subseteq \mathrm{On}$ *and if* b *is closed* w.r.t. C *and* J *then*

$$(\forall x, y)[x \in F``(b \cap \eta) \wedge y \in x \cap F``b \rightarrow y \in F``(b \cap \eta)].$$

PROOF. If $x \in F``(b \cap \eta)$, then $x \in F``b$ and $x \in F``\eta$. Therefore

$$\mathrm{Od}`x \in \eta$$

and, by Lemma 2,

$$\mathrm{Od}`x \in b.$$

Also

$$\begin{aligned} y \in x \cap F``b &\rightarrow y \in x \wedge y \in F``b \\ &\rightarrow \mathrm{Od}`y < \mathrm{Od}`x \wedge \mathrm{Od}`y \in b \\ &\rightarrow \mathrm{Od}`y \in b \cap \eta \\ &\rightarrow y \in F``(b \cap \eta). \end{aligned} \qquad \square$$

Lemma 7. *If* $9 \subseteq b \subseteq \mathrm{On}$ *and if* b *is closed* w.r.t. C *and* J, *then*

$$(\forall x)[x \in F`\eta \cap F``b \rightarrow x \in F``(b \cap \eta)].$$

PROOF. If $x \in F`\eta$, then $\mathrm{Od}`x < \mathrm{Od}`F`\eta \leqq \eta$. By Lemma 2

$$x \in F``b \rightarrow \mathrm{Od}`x \in b.$$

Then

$$x \in F``(b \cap \eta). \qquad \square$$

Lemma 8. *If* $9 \subseteq b \subseteq \mathrm{On}$ *and* b *is closed* w.r.t. C *and* J, *then*

(1) $(\forall x, y)[\{x, y\} \in F``(b \cap \eta) \rightarrow x \in F``(b \cap \eta) \wedge y \in F``(b \cap \eta)]$.
(2) $(\forall x, y)[\langle x, y\rangle \in F``(b \cap \eta) \rightarrow x \in F``(b \cap \eta) \wedge y \in F``(b \cap \eta)]$.
(3) $(\forall x, y, z)[\langle x, y, z\rangle \in F``(b \cap \eta) \rightarrow x, y, z \in F``(b \cap \eta)]$.

PROOF. (1) Since $F``(b \cap \eta) \subseteq F``b$, we have by Lemma 4.

$$\begin{aligned} \{x, y\} \in F``(b \cap \eta) &\rightarrow \{x, y\} \in F``b \\ &\rightarrow x \in F``b \wedge y \in F``b \end{aligned}$$

then

$$\{x, y\} \in F``(b \cap \eta) \wedge x \in \{x, y\} \cap F``b \wedge y \in \{x, y\} \cap F``b.$$

From Lemma 6

$$x \in F``(b \cap \eta) \wedge y \in F``(b \cap \eta).$$

(2)–(3) The proofs are left to the reader. $\square$

Lemma 9. *If* $9 \subseteq b \subseteq \mathrm{On} \wedge 9 \subseteq c \subseteq \mathrm{On}$, *if* b *and* c *are each closed with respect to* C *and* J, *iff* $\mathrm{Isom}_{E,E}(b, c)$ *and if* $H\ \mathrm{Isom}_{E,E}(F``(b \cap \eta), F``f``(b \cap \eta))$ *for* $\eta \in b$ *then*

(1) $(\forall x, y)[\{x, y\} \in F``(b \cap \eta) \to H`\{x, y\} = \{H`x, H`y\}]$,

(2) $(\forall x. y)[\langle x, y\rangle \in F``(b \cap \eta) \to H`\langle x, y\rangle = \langle H`x, H`y\rangle]$,

(3) $(\forall x, y, z)[\langle x, y, z\rangle \in F``(b \cap \eta) \to H`\langle x, y, z\rangle = \langle H`x, H`y, H`z\rangle]$,

(4) $(\forall x, y \in F``(b \cap \eta))[\langle x, y\rangle \in Q_n \leftrightarrow \langle H`x, H`y\rangle \in Q_n]$, $n = 4, 5, 6, 7, 8$.

Proof. (1) From Lemma 8

$$\{x, y\} \in F``(b \cap \eta) \to x, y \in F``(b \cap \eta).$$

Therefore since $H\ \mathrm{Isom}_{E,E}(F``(b \cap \eta), F``f``(b \cap \eta))$ and since $x, y \in \{x, y\}$

$$H`x, H`y \in H`\{x, y\}$$

i.e.,

$$\{H`x, H`y\} \subseteq H`\{x, y\}.$$

Either $H`\{x, y\} = \{H`x, H`y\}$ or $\exists z \in [H`\{x, y\} - \{H`x, H`y\}]$. In the latter case we note that $\eta \in b$ and hence

$$f``(b \cap \eta) = c \cap f`\eta.$$

Since $H`\{x, y\}, H`x, H`y \in F``c$ it follows from Lemma 1 that

$$H`\{x, y\} - \{H`x, H`y\} \in F``c.$$

From Lemma 3

$$\exists z \in (H`\{x, y\} - \{H`x, H`y\}) \cap F``c.$$

Therefore

$$H`\{x, y\} \in F``(c \cap f`\eta) \wedge z \in H`\{x, y\} \cap F``c$$

and hence by Lemma 6

$$z \in F``(c \cap f`\eta) = F``f``(b \cap \eta).$$

Consequently $(\exists w \in F``(b \cap \eta))[z = H`w]$. But since $z \in H`\{x, y\}$ and $H\ \mathrm{Isom}_{E,E}(F``(b \cap \eta), F``f``(b \cap \eta))$,

$$w \in \{x, y\}.$$

Thus $z = H`x \vee z = H`y$. This is a contradiction. Hence

$$H`\{x, y\} = \{H`x, H`y\}.$$

(2) From Lemma 8

$$\langle x, y\rangle \in F``(b \cap \eta) \to x, y, \{x\}, \{x, y\} \in F``(b \cap \eta).$$

Therefore from (1) above

$$H`\langle x, y\rangle = H`\{\{x\}, \{x, y\}\} = \{H`\{x\}, H`\{x, y\}\}$$
$$= \{\{H`x\}, \{H`x, H`y\}\} = \langle H`x, H`y\rangle.$$

(3) The proof is left to the reader.

(4) If $\langle x, y\rangle \in Q_4$, then $(\exists z)[y = \langle x, z\rangle]$.

But from Lemma 8

$$y \in F``(b \cap \eta) \to x, z \in F``(b \cap \eta),$$

Then from (2) above

$$y = \langle x, z\rangle \to H`y = \langle H`x, H`z\rangle$$
$$\to \langle H`x, H`y\rangle \in Q_4.$$

Conversely

$$\langle H`x, H`y\rangle \in Q_4 \to (\exists w)[H`y = \langle H`x, w\rangle].$$

Since $y \in F``(b \cap \eta)$, $H`y \in F``(c \cap f`\eta)$. Hence by Lemma 8, $w \in F``(c \cap f`\eta) = F``f``(b \cap \eta)$. Consequently $(\exists z \in F``(b \cap \eta))[w = H`z]$.

Since f^{-1} $\text{Isom}_{E,E}(c, b) \wedge H^{-1}$ $\text{Isom}_{E,E}(F``(c \cap f`\eta), \; F``f^{-1}(c \cap f`\eta))$ the foregoing argument gives

$$H`y = \langle H`x, H`z\rangle \to y = \langle x, z\rangle$$
$$\to \langle x, y\rangle \in Q_4.$$

The arguments for Q_5, Q_6, Q_7, and Q_8 are similar and are left to the reader. □

Proposition 15.42. (1) *If* $9 \subseteq b \subseteq \text{On} \wedge 9 \subseteq c \subseteq \text{On}$, *if* b *and* c *are each closed with respect to* K_1, K_2, C, *and* J, *and if* f $\text{Isom}_{E,E}(b, c)$, *then*

$$(\forall \alpha, \beta \in b)[[F`\alpha \in F`\beta \leftrightarrow F`f`\alpha \in F`f`\beta] \wedge [F`\alpha = F`\beta \leftrightarrow F`f`\alpha = F`f`\beta]].$$

(2) *If* $\omega + 2 \subseteq b \subseteq \text{On} \wedge \omega + 2 \subseteq c \subseteq \text{On}$, *if* b *and* c *are each closed with respect to* K_1, K_2, C_a *and* J, *and if* f $\text{Isom}_{E,E}(b, c)$, *then*

$$(\forall \alpha, \beta \in b)[[F_a`\alpha \in F_a`\beta \leftrightarrow F_a`f`\alpha \in F_a`f`\beta]$$
$$\wedge [F_a`\alpha = F_a`\beta \leftrightarrow F_a`f`\alpha = F_a`f`\beta]].$$

Proof. (1) By induction on $\max(\alpha, \beta)$. If $\eta = \max(\alpha, \beta)$ and if $\eta = \alpha = \beta$ then the result is true because

$$F`\alpha = F`\beta \wedge F`f`\alpha = F`f`\beta \wedge F`\alpha \notin F`\beta \wedge F`f`\alpha \notin F`f`\beta.$$

If $\alpha < \beta = \eta \vee \beta < \alpha = \eta$ then it is sufficient to prove that if $\gamma \in b \cap \eta$.

(i) $F`\gamma \in F`\eta \leftrightarrow F`f`\gamma \in F`f`\eta$,

(ii) $F`\eta \in F`\gamma \leftrightarrow F`f`\eta \in F`f`\gamma$,

(iii) $F`\gamma = F`\eta \leftrightarrow F`f`\gamma = F`f`\eta$.

If $H = \{\langle F`\gamma, F`f`\gamma\rangle \mid \gamma \in b \cap \eta\}$ then by the induction hypothesis

$$(\forall \gamma, \delta \in b \cap \eta)[[F`\gamma \in F`\delta \leftrightarrow F`f`\gamma \in F`f`\delta] \\ \wedge\ [F`\gamma = F`\delta \leftrightarrow F`f`\gamma = F`f`\delta]]$$

consequently $H\ \mathrm{Isom}_{E,E}(F``(b \cap \eta), F``f``(b \cap \eta))$. With the aid of Lemma 9 we will prove (i), i.e.,

$$(\forall \gamma \in b \cap \eta)[F`\gamma \in F`\eta \leftrightarrow F`f`\gamma \in F`f`\eta].$$

We argue by cases.

If $K`_3\eta = 0$ then $K`_3 f`\eta = 0$ (Proposition 15.41). Therefore $F`\eta = F``\eta$ and $F`f`\eta = F``f``\eta$. Consequently

$$(\forall \gamma \in b \cap \eta)[F`\gamma \in F`\eta].$$

But since f is E-order preserving

$$\gamma \in b \cap \eta \rightarrow f`\gamma \in f`\eta.$$

Hence

$$(\forall \gamma \in b \cap \eta)[F`f`\gamma \in F`f`\eta].$$

If $K`_3\eta \neq 0$ then $K`_1\eta < \eta \wedge K`_2\eta < \eta$. Since $\eta \in b$ and b is closed with respect to K_1 and K_2,

$$K`_1\eta \in b \cap \eta \wedge K`_2\eta \in b \cap \eta.$$

From Proposition 15.41

$$K`_1 f`\eta = f`K`_1\eta, \qquad K`_2 f`\eta = f`K`_2\eta, \qquad K`_3 f`\eta = K`_3\eta.$$

If $K`_3\eta = 1$

$$F`\eta = \{F`K`_1\eta, F`K`_2\eta\} \wedge F`f`\eta = \{F`f`K`_1\eta, F`f`K`_2\eta\}.$$

Then from the induction hypothesis if $\gamma \in b \cap \eta$

$$\begin{aligned} F`\gamma \in F`\eta &\leftrightarrow F`\gamma = F`K`_1\eta \vee F`\gamma = F`K`_2\eta \\ &\leftrightarrow F`f`\gamma = F`f`K`_1\eta \vee F`f`\gamma = F`f`K`_2\eta \\ &\leftrightarrow F`f`\gamma \in F`f`\eta. \end{aligned}$$

If $K`_3\eta = 2$

$$F`\eta = E \cap F`K`_1\eta \wedge F`f`\eta = E \cap F`f`K`_1\eta.$$

Again from the induction hypothesis if $\gamma \in b \cap \eta$

$$\begin{aligned} F`\gamma \in F`\eta &\leftrightarrow F`\gamma \in E \wedge F`\gamma \in F`K`_1\eta \\ &\leftrightarrow (\exists x, y)[F`\gamma = \langle x, y\rangle \wedge x \in y] \wedge F`f`\gamma \in F`f`K`_1\eta. \end{aligned}$$

From Lemma 8

$$\langle x, y\rangle = F`\gamma \in F``(b \cap \eta) \rightarrow x \in F``(b \cap \eta) \wedge y \in F``(b \cap \eta).$$

Then from Lemma 9

$$F`f`\gamma = H`F`\gamma = H`\langle x, y\rangle = \langle H`x, H`y\rangle.$$

Furthermore

$$x \in y \rightarrow H`x \in H`y.$$

Thus

$$F`f`\gamma \in E \cap F`f`K_1`\eta$$

i.e.,

$$(\forall \gamma \in b \cap \eta)[F`\gamma \in F`\eta \rightarrow F`f`\gamma \in F`f`\eta].$$

If $K_3`\eta = 3$

$$F`\eta = F`K_1`\eta - F`K_2`\eta \wedge F`f`\eta = F`f`K_1`\eta - F`f`K_2`\eta.$$

From the induction hypothesis if $\gamma \in b \cap \eta$

$$\begin{aligned} F`\gamma \in F`\eta &\leftrightarrow F`\gamma \in F`K_1`\eta \wedge F`\gamma \notin F`K_2`\eta \\ &\leftrightarrow F`f`\gamma \in F`f`K_1`\eta \wedge F`f`\gamma \notin F`f`K_2`\eta \\ &\leftrightarrow F`f`\gamma \in F`f`\eta. \end{aligned}$$

If $K_3`\eta = n$, $n = 4, 6, 7, 8$

$$F`\eta = F`K_1`\eta \cap Q_n``F`K_2`\eta \wedge F`f`\eta = F`f`K_1`\eta \cap Q_n``F`f`K_2`\eta.$$

Then if $\gamma \in b \cap \eta$

$$\begin{aligned} F`\gamma \in F`\eta &\leftrightarrow F`\gamma \in F`K_1`\eta \wedge F`\gamma \in Q_n``F`K_2`\eta \\ &\leftrightarrow F`f`\gamma \in F`f`K_1`\eta \wedge (\exists x \in F`K_2`\eta)[\langle x, F`\gamma\rangle \in Q_n]. \end{aligned}$$

But from Lemma 5

$$F`\gamma \in F``b \wedge \langle x, F`\gamma\rangle \in Q_n \rightarrow x \in F``b.$$

Then

$$F`K_2`\eta \in F``(b \cap \eta) \wedge x \in F`K_2`\eta \cap F``b.$$

Therefore by Lemma 6, $x \in F``(b \cap \eta)$. Thus

$$x \in F``(b \cap \eta) \wedge F`\gamma \in F``(b \cap \eta) \wedge \langle x, F`\gamma\rangle \in Q_n.$$

From Lemma 9

$$\langle H`x, H`F`\gamma\rangle \in Q_n.$$

But

$$H`F`\gamma = F`f`\gamma.$$

Therefore since $x \in F`K`_2\eta$

$$H`x \in H`F`K`_2\eta = F`f`K`_2\eta$$

i.e.

$$F`f`\gamma \in Q_n``F`f`K`_2\eta.$$

Thus

$$(\forall \gamma \in b \cap \eta)[F`\gamma \in F`\eta \rightarrow F`f`\gamma \in F`f`\eta].$$

If $K`_3\eta = 5$

$$F`\eta = F`K`_1\eta \cap Q_5``F`K`_2\eta \wedge F`f`\eta = F`f`K`_1\eta \cap Q_5``F`f`K`_2\eta.$$

Thus if $\gamma \in b \cap \eta$ then since $Q_5 = Q_4^{-1}$

$$F`\gamma \in F`\eta \leftrightarrow F`\gamma \in F`K`_1\eta \wedge F`\gamma \in Q_5``F`K`_2\eta$$
$$\leftrightarrow F`f`\gamma \in F`f`K`_1\eta \wedge (\exists x \in F`K`_2\eta)[\langle F`\gamma, x\rangle \in Q_4].$$

Then

$$F`K`_2\eta \cap Q_4``\{F`\gamma\} \neq 0.$$

Furthermore since $F`\gamma \in F``b \wedge F`K`_2\eta \in F``b$ we have from Lemma 1

$$F`K`_2\eta \cap Q_4``\{F`\gamma\} \in F``b.$$

Therefore by Lemma 3

$$(\exists y \in F``b)[y \in F`K`_2\eta \cap Q_4``\{F`\gamma\}].$$

Thus

$$y \in F``b \wedge \langle F`\gamma, y\rangle \in Q_4.$$

Then

$$F`K`_2\eta \in F``(b \cap \eta) \wedge y \in F`K`_2\eta \cap F``b$$

and hence by Lemma 6

$$y \in F``(b \cap \eta).$$

Then since $\langle F`\gamma, y\rangle \in Q_4$ we have from Lemma 9

$$\langle H`F`\gamma, H`y\rangle \in Q_4,$$
$$\langle H`y, H`F`\gamma\rangle \in Q_5.$$

But $H`F`\gamma = F`f`\gamma$. Therefore since $y \in F`K`_2\eta$

$$H`y \in H`F`K`_2\eta = F`f`K`_2\eta$$

i.e.,

$$F`f`\gamma \in Q_5``F`f`K`_2\eta.$$

Thus

$$(\forall \gamma \in b \cap \eta)[F`\gamma \in F`\eta \to F`f`\gamma \in F`f`\eta].$$

Having exhausted all cases we have proved

$$(\forall \gamma \in b \cap \eta)[F`\gamma \in F`\eta \to F`f`\gamma \in F`f`\eta].$$

The implication in the reverse direction follows from symmetry, i.e.,

$$f^{-1} \text{ Isom}_{E,E}(c, b) \wedge H^{-1} \text{ Isom}_{E,E}(F``(c \cap f`\eta), F``f^{-1}(c \cap f`\eta)).$$

Therefore, since

$$\gamma \in b \cap \eta \to f`\gamma \in c \cap f`\eta,$$

$$F`f`\gamma \in F`f`\eta \to F`\gamma \in F`\eta.$$

This completes the proof of (i).

From (i) we next prove (iii). If $F`\eta \neq F`\gamma$ then $F`\eta - F`\gamma \neq 0$ or $F`\gamma - F`\eta \neq 0$. Since $F`\gamma \in F``b \wedge F`\eta \in F``b$ we have from Lemma 1

$$F`\eta - F`\gamma \in F``b \wedge F`\gamma - F`\eta \in F``b.$$

If $F`\eta - F`\gamma \neq 0$ then by Lemma 3

$$(\exists x \in F``b)[x \in F`\eta - F`\gamma].$$

Thus

$$x \in F`\eta \cap F``b$$

and hence by Lemma 7

$$x \in F``(b \cap \eta)$$

i.e.,

$$(\exists \nu \in b \cap \eta)[x = F`\nu].$$

But $x \in F`\eta - F`\gamma$. Therefore

$$F`\nu \in F`\eta \wedge F`\nu \notin F`\gamma.$$

From (i)

$$F`f`\nu \in F`f`\eta \wedge F`f`\nu \notin F`f`\gamma.$$

Thus $F`f`\nu \in F`f`\eta - F`f`\gamma$ and hence $F`f`\gamma \neq F`f`\eta$.

If $F`\gamma - F`\eta \neq 0$ the argument is similar to the foregoing one and is left to the reader. We have then proved

$$(\forall \gamma \in b \cap \eta)[F`f`\eta = F`f`\gamma \to F`\eta = F`\gamma].$$

Again the implication in the reverse direction follows from symmetry.

We next prove (ii) from (i) and (iii). If $F`\eta \in F`\gamma$ and if $\nu = \text{Od}`F`\eta$ then

$$\nu = \text{Od}`F`\eta < \text{Od}`F`\gamma \leqq \gamma < \eta$$

i.e., $\nu < \eta$. Since $F`\eta \in F``b$ we have from Lemma 2

$$\nu = \mathrm{Od}`F`\eta \in b.$$

Then $\nu \in b \cap \eta$ and hence by the induction hypothesis

$$F`\nu \in F`\gamma \rightarrow F`f`\nu \in F`f`\gamma.$$

But since $F`\nu = F`\eta$ we have from (iii)

$$F`f`\nu = F`f`\eta.$$

Hence

$$F`\eta \in F`\gamma \rightarrow F`f`\eta \in F`f`\gamma.$$

Again the reverse implication follows from symmetry.

(2) The proof is left to the reader. □

Proposition 15.43. (1) *If* $9 \subseteq b \subseteq \mathrm{On}$, *if* b *is closed with respect to* C, K_1, K_2, *and* J, *and if* f $\mathrm{Isom}_{E,E}(b, \eta)$ *then*

$$(\forall \alpha, \beta \in b)[F`\alpha \in F`\beta \leftrightarrow F`f`\alpha \in F`f`\beta].$$

(2) *If* $9 \subseteq b \subseteq \mathrm{On}$, *if* b *is closed with respect to* C_a, K_1, K_2, *and* J, *and if* f $\mathrm{Isom}_{E,E}(b, \eta)$, *then*

$$(\forall \alpha, \beta \in b)[F_a`\alpha \in F_a`\beta \leftrightarrow F_a`f`\alpha \in F_a`f`\beta].$$

PROOF. (1) From Proposition 15.40, η is closed with respect to J. Since

$$K_1`\alpha \leqq \alpha \wedge K_2`\alpha \leqq \alpha \wedge C`\alpha \leqq \alpha$$

η is also closed with respect to C, K_1, and K_2. Therefore by Proposition 15.42

$$(\forall \alpha, \beta \in b)[F`\alpha \in F`\beta \leftrightarrow F`f`\alpha \in F`f`\beta].$$

(2) The proof is left to the reader. □

Theorem 15.44.

(1) $V = L \rightarrow \mathrm{GCH}$.

(2) $V = L_a \rightarrow \mathrm{GCH}$.

PROOF. Propositions 15.43, 15.39, and 15.38. □

Theorem 15.45.

(1) L *is a model of* GCH.

(2) L_a *is a model of* GCH.

PROOF. Theorems 15.44 and 15.30. □

Remark. We have now shown how to select from V a subclass L that is a model of ZF + AC + GCH + $V = L$. This process can be relativized to any standard transitive model M to produce a subclass of M that is also a model of ZF + AC + GCH + $V = L$. Recall that

$$L^M = \{x \in M \mid (\exists\, \alpha \in M)[x = F`\alpha]\}.$$

Proposition 15.46. *If M is a standard transitive model of* ZF *then*

(1) $(\forall\, \alpha \in M)[F`\alpha \in M]$,

(2) $a \subseteq \omega \wedge a \in M \rightarrow (\forall\, \alpha \in M)[F_a`\alpha \in M]$.

PROOF. (1) (By induction). Since M is a model of the Axiom Schema of Replacement it follows from the induction hypothesis that if $K_3`\alpha = 0$, then

$$F`\alpha = F``\alpha = \{x \mid (\exists\, \beta \in \alpha)[x = F`\beta]\} = \{x \in M \mid (\exists \beta \in \alpha)[x = F`\beta]^M\} \in M.$$

If $K_3`\alpha = i \neq 0$ then $K_1`\alpha < \alpha$, $K_2`\alpha < \alpha$ and from the induction hypothesis and the fact that M is closed under the eight fundamental operations

$$F`\alpha = \mathscr{F}_i(F`K_1`\alpha, F`K_2`\alpha) \in M.$$

(2) The proof is left to the reader. □

Proposition 15.47. *If M is a standard transitive model of* ZF *then*

(1) $L^M = \{x \mid (\exists\, \alpha \in M)[x = F`\alpha]\}$,

(2) $a \subseteq \omega \wedge a \in M \rightarrow L_a^M = \{x \mid (\exists\, \alpha \in M)[x = F_a`\alpha]\}$.

PROOF. Obvious from Proposition 15.46. □

Remark. That L^M is a standard transitive model of ZF + AC + GCH + $V = L$ is immediate from the following theorem.

Theorem 15.48. *If M is a standard transitive model of* ZF *and if φ is a* wff *of* ZF *then*

(1) $(\varphi^L)^M \leftrightarrow \varphi^{L^M}$

(2) $(\varphi^{L_a})^M \leftrightarrow \varphi^{L_a^M}$, *if $a \in M$.*

PROOF. (1) (By induction on the number of logical symbols in φ.) The formula φ must be of the form (1) $a \in b$, (2) $\neg \psi$, (3) $\psi \wedge \eta$, or (4) $(\forall\, x)\psi$. The arguments for cases (1)–(3) we leave to the reader. If φ is of the form $(\forall\, x)\psi$ then as our induction hypothesis

$$(\psi^L)^M \leftrightarrow \psi^{L^M}.$$

Then

$$\begin{aligned}
[[(\forall\, x)\psi]^L]^M &\leftrightarrow [(\forall\, x)[x \in L \rightarrow \psi^L]]^M \\
&\leftrightarrow [(\forall\, x)[(\exists\, \alpha)[x = F`\alpha] \rightarrow \psi^L]]^M \\
&\leftrightarrow (\forall\, x \in M)[(\exists\, \alpha \in M)[x = F`\alpha] \rightarrow (\psi^L)^M] \\
&\leftrightarrow (\forall\, x)[x \in L^M \rightarrow \psi^{L^M}] \\
&\leftrightarrow [(\forall\, x)\psi]^{L^M}.
\end{aligned}$$

(2) The proof is left to the reader. □

Theorem 15.49. *If M is a standard transitive model of* ZF *then*

(1) L^M *is a standard transitive model of* ZF + AC + GCH + $V = L$ *and* $\mathrm{On}^{L^M} = \mathrm{On}^M$.

(2) L_a^M *is a standard transitive model of* ZF + AC + GCH + $V = L_a$ *and* $\mathrm{On}^{L_a^M} = \mathrm{On}^M$, *if* $a \in M$.

Proof. (1) Since $0 \in M$ and $F`0 = 0$ it follows that $0 \in L^M$ and hence L^M is not empty. Furthermore if $y \in x \in L^M$ then

$$(\exists\, \alpha \in M)[y \in x = F`\alpha \subseteq F``\alpha].$$

Therefore $(\exists\, \beta < \alpha)[y = F`\beta]$. But $\alpha \in M$ and M is transitive. Thus $\beta \in M$ and $y \in L^M$, and hence L^M is transitive.

Since L is a model of ZF + AC + GCH + $V = L$ it follows that if φ is any axiom of ZF + AC + GCH + $V = L$ then φ^L is a theorem of ZF. Since M is a model of ZF every theorem of ZF relativized to M is a theorem of ZF. Therefore $(\varphi^L)^M$ is a theorem of ZF. Then by Theorem 15.48 φ^{L^M} is a theorem of ZF. Hence L^M is a model of φ.

Consequently L^M is a standard transitive model of ZF + AC + GCH + $V = L$.

By Theorem 15.27, $\mathrm{On}^L = \mathrm{On}$. Relativizing to M we have $\mathrm{On}^{L^M} = \mathrm{On}^M$. Details are left to the reader.

(2) The proof is left to the reader. □

CHAPTER 16

Silver Machines

Gödel's use of the Axiom of Constructibility to prove the relative consistency of ZFC, and ZFC + GCH, might suggest that he introduced constructibility simply as a means to an end. That, however, is not at all the case. Gödel held his discovery of constructible sets, and his proof that the class of constructible sets, L, is a model of ZFC, to be by itself, one of his major achievements. His confidence in the importance of the notion of constructibility was further vindicated when in 1967 Ronald Björn Jensen used $V = L$ to solve a problem in real analysis, the Souslin problem. In addition Jensen derived, from $V = L$, three principles that can be understood and used by people who are not specialists in set theory. Following Jensen, Saharon Shelah in 1974, used $V = L$ to settle a problem in group theory, the Whitehead problem.[1]

In view of these results it is natural to ask about the status of the Axiom of Constructibility, $V = L$. If one is of the opinion that the purpose of set theory is to axiomatize the largest possible part of Cantor's world of sets, then $V = L$ must be rejected because it is severely restrictive. But rejecting $V = L$ as an axiom does not diminish the importance of constructibility and the achievements of Jensen. Rather it directs our attention to L, which is the smallest natural universe of sets. To understand L is an important part of understanding set theory. With Jensen the first step was taken toward a better understanding of L.

In the late 1970s, at a date unknown to the authors because he did not publish his results, Jack Silver introduced a special technique for deriving consequences of $V = L$ and hence for deriving information about L. This technique involves the use of structures that Silver called machines and

[1] For a discussion of Jensen's solution of the Souslin problem and Shelah's solution of the Whitehead problem see Devlin, Keith J. *The Axiom of Constructibility: A Guide for the Mathematician.* Lecture Notes in Mathematics, Vol. 617, New York: Springer-Verlag, 1977.

which are now known as Silver machines. In this chapter we will study an elementary part of the theory of Silver machines.

A Silver machine is a special kind of structure that we call an *algebra*. But to define an algebra we need some preliminary notions and notation:

Definition 16.1. $A^{\underset{\smile}{\omega}} \triangleq \{f \mid (\exists n \in \omega)[f : n \to A]\}$.

Remark. If f is an element of $A^{\underset{\smile}{\omega}}$ with domain n, and if $f`i = a_i$ we will denote f by $\langle a_1, a_2, \ldots, a_n\rangle$. We will also use $\mathbf{a}$ as a variable on $A^{\underset{\smile}{\omega}}$.

Definition 16.2. A function f is a partial map from A to B iff $(\exists C \subseteqq A)$ $[f : C \to B]$.

Definition 16.3. A structure $\langle A, F_x\rangle_{x \in I}$ is an algebra iff A and I are classes and for each x in I, F_x is a partial map from $A^{\underset{\smile}{\omega}}$ to A. The class A is the *universe* of the algebra.

Definition 16.4. An algebra $\langle A, F_x\rangle_{x \in I}$ is a subalgebra of the algebra $\langle B, G_x\rangle_{x \in I}$ iff

$$A \subseteqq B \wedge (\forall x \in I)[\mathscr{D}(F_x) = A^{\underset{\smile}{\omega}} \cap \mathscr{D}(G_x) \wedge (\forall y \in \mathscr{D}(F_x)[F_x`y = G_x`y]].$$

Proposition 16.5. *Let* $\{\langle A_b, F_{bx}\rangle_{x \in I} | b \in B\}$ *be a collection of subalgebras of an algebra* $\mathfrak{A}$. *Then*

$$(1) \quad \left\langle \bigcap_{b \in B} A_b, \bigcap_{b \in B} F_{bx} \right\rangle_{x \in I}$$

is a subalgebra of $\mathfrak{A}$.

The proof is obvious.

Definition 16.6. The algebra (1) of Proposition 16.5 is the *intersection* algebra of the given collection of algebras.

Definition 16.7. (1) A class X is contained in an algebra $\langle A, F_x\rangle_{x \in I}$ iff $X \subseteqq A$.

(2) If X is contained in an algebra $\mathfrak{A}$, then the subalgebra of $\mathfrak{A}$ generated by X, is the intersection of all of the subalgebras of $\mathfrak{A}$ that contain X. The universe of this algebra we denote by $\mathfrak{A}(X)$.

Remark. Following a custom established in group theory, and other places, we will speak of $\mathfrak{A}(X)$ as the subalgebra generated by X.

A mapping π from A to B induces a natural mapping from $A^{\underset{\smile}{\omega}}$ to $B^{\underset{\smile}{\omega}}$:

Definition 16.8. If $\pi : A \to B$, then $\tilde{\pi}$ is the mapping from $A^{\underset{\smile}{\omega}}$ to $B^{\underset{\smile}{\omega}}$ induced by π and defined by

$$\tilde{\pi}`\langle a_1, \ldots, a_n\rangle \triangleq \langle \pi`a_1, \ldots, \pi`a_n\rangle.$$

Remark. In dealing with partial functions on a class A it is convenient to introduce an extended equality relation that enables us to use quantification over all of A and without reference to the domains of the function involved.

Definition 16.9. If F and G are partial functions on A, then $\forall x \in A$

$$F\text{`}x \simeq G\text{`}x \quad \text{iff } x \in \mathscr{D}(F) \cap \mathscr{D}(G) \wedge F\text{`}x = G\text{`}x \quad \text{or} \quad x \notin \mathscr{D}(F) \wedge x \notin \mathscr{D}(G).$$

Definition 16.10. By a monomorphism from an algebra $\langle A, F_x \rangle_{x \in I}$ to an algebra $\langle B, G_x \rangle_{x \in I}$ we mean a mapping π such that

(1) $\pi \colon A \xrightarrow{1-1} B$,

and

(2) $(\forall x \in I)[\pi \circ F_x = G_x \circ \tilde{\pi}]$,

that is, $(\forall \mathbf{a} \in A^{(\omega)})(\forall x \in I)[\pi\text{`}F_x\text{`}\mathbf{a} \simeq G_x\text{`}\tilde{\pi}\text{`}\mathbf{a}]$.

Definition 16.11. An algebra $\langle A, F_i \rangle_{i \in \omega}$ is a Silver machine, or simply a machine, iff

(1) $A = \mathrm{On}$ or $A \in \mathrm{On}$,

and

(2) $F_0\text{`}\langle \alpha_1, \alpha_2 \rangle = 0$ iff $\alpha_1, \alpha_2 \in A \wedge \alpha_1 < \alpha_2$,

otherwise F_0 is undefined.

Remark. As a notational convenience we will, throughout the remainder of this section, think of a fixed machine $\mathfrak{A}$ that we denote by $\langle A, F_i \rangle_{i \in \omega}$. We will also use the notation $F(x)$ and $F\text{`}x$ interchangeably.

Definition 16.12. For each ordinal ζ in A

$$\mathfrak{A}^{\zeta} \triangleq \langle \zeta, F_i^{\zeta} \rangle_{i \in \omega}$$

is the machine defined by

$$F_i^{\zeta}(\boldsymbol{\alpha}) \triangleq F_i(\boldsymbol{\alpha}) \quad \text{if } \boldsymbol{\alpha} \in \zeta^{(\omega)} \wedge F_i(\boldsymbol{\alpha}) < \zeta,$$

otherwise F_i^{ζ} is undefined.

Definition 16.13. A function $\pi \colon \zeta \to \eta$ is a strong $\mathfrak{A}$-map iff π is a monomorphism from $\mathfrak{A}^{\zeta}$ to $\mathfrak{A}^{\eta}$. Such a function is a medium $\mathfrak{A}$-map iff there exists a $\delta \leqq \eta$ such that π is a strong $\mathfrak{A}$-map from ζ into δ.

Remark. (1) Obviously, a strong $\mathfrak{A}$-map is medium, but the converse is not true. Find a counter-example.

(2) If $\pi \colon \xi \to \eta$ is a medium $\mathfrak{A}$-map, then π is order preserving, that is, $(\forall \alpha, \beta < \xi)[\alpha < \beta \to \pi(\alpha) < \pi(\beta)]$.

(3) Let $\pi: \xi \to \eta$ be a medium $\mathfrak{A}$-map and let $\delta = \sup\{\pi(\alpha) + 1 \mid \alpha < \xi\}$. Then $\pi: \xi \to \delta$ is a strong $\mathfrak{A}$-map.

Proposition 16.14. *If $\pi: \xi \to \eta$ is a strong $\mathfrak{A}$-map and $\delta \leqq \eta$ is an ordinal such that $\mathscr{W}(\pi) \subseteq \delta$, then $\pi: \xi \to \delta$ is a strong $\mathfrak{A}$-map.*

PROOF. For any $\boldsymbol{\alpha} \in \xi^{\omega}$,

$$\begin{aligned} F_i^{\xi}(\boldsymbol{\alpha}) \simeq \beta &\to F_i^{\eta}(\pi(\boldsymbol{\alpha})) \simeq \pi(\boldsymbol{\beta}) \wedge \pi(\boldsymbol{\beta}) < \delta \\ &\to F_i^{\delta}(\pi(\boldsymbol{\alpha})) \simeq \pi(\boldsymbol{\beta}) \\ F_i^{\xi}(\boldsymbol{\alpha}) \quad \text{undefined} &\to F_i^{\eta}(\pi(\boldsymbol{\alpha})) \quad \text{undefined} \\ &\to F_i^{\delta}(\pi(\boldsymbol{\alpha})) \quad \text{undefined.} \end{aligned}$$

$\square$

Proposition 16.15. *If $\pi: \xi \to \eta$ is a medium $\mathfrak{A}$-map and $\gamma \leqq \xi$, then $\pi \upharpoonright \gamma: \gamma \to \eta$ is a medium $\mathfrak{A}$-map.*

PROOF. We may assume that π is strong. Let $\delta = \sup\{\pi(\alpha) + 1 \mid \alpha < \gamma\}$. We will show that $\pi \upharpoonright \gamma: \gamma \to \delta$ is a strong $\mathfrak{A}$-map. For any $\boldsymbol{\alpha} \in \gamma^{\omega}$,

$$\begin{aligned} F_i^{\gamma}(\boldsymbol{\alpha}) \simeq \beta &\to F_i^{\xi}(\boldsymbol{\alpha}) \simeq \beta \wedge \beta < \gamma \\ &\to F_i^{\eta}(\pi(\boldsymbol{\alpha})) \simeq \pi(\beta) \wedge \pi(\beta) < \delta \\ &\to F_i^{\delta}(\pi(\boldsymbol{\alpha})) \simeq \pi(\beta). \\ F_i^{\gamma}(\boldsymbol{\alpha}) \quad \text{undefined} &\to F_i^{\xi}(\boldsymbol{\alpha}) \quad \text{undefined or} \quad F_i^{\xi}(\boldsymbol{\alpha}) \geqq \gamma \\ &\to F_i^{\eta}(\pi(\boldsymbol{\alpha})) \quad \text{undefined or} \quad F_i^{\eta}(\pi(\boldsymbol{\alpha})) \geqq \delta \\ &\to F_i^{\delta}(\pi(\boldsymbol{\alpha})) \quad \text{undefined.} \end{aligned}$$

$\square$

Proposition 16.16. *If $\pi_1: \xi_1 \to \xi_2$ and $\pi_2: \xi_2 \to \xi_3$ are strong $\mathfrak{A}$-maps, then $\pi_2 \circ \pi_1: \xi_1 \to \xi_3$ is a strong $\mathfrak{A}$-map.*

The proof is left to the reader.

Proposition 16.17. *If $\pi_1: \xi_1 \to \xi_2$ and $\pi_2: \xi_2 \to \xi_3$ are medium $\mathfrak{A}$-maps, then $\pi_2 \circ \pi_1: \xi_1 \to \xi_3$ is a medium $\mathfrak{A}$-map.*

PROOF. Let $\delta_2 = \sup\{\pi_1(\alpha) + 1 \mid \alpha < \xi_1\}$ and let $\delta_3 = \sup\{\pi_2(\beta) + 1 \mid \beta < \delta_2\}$. Then $\pi_1: \xi_1 \to \delta_2$ and $\pi_2 \upharpoonright \delta_2: \delta_2 \to \delta_3$ are strong $\mathfrak{A}$-maps. Thus $\pi_2 \circ \pi_1 = (\pi_2 \upharpoonright \delta_2) \circ \pi_1$ is a strong $\mathfrak{A}$-map from ξ_1 to δ_3. $\square$

Proposition 16.18. *If $\pi_1: \xi_1 \to \eta$ and $\pi_2: \xi_2 \to \eta$ are medium $\mathfrak{A}$-maps such that $\mathscr{W}(\pi_1) \subseteq \mathscr{W}(\pi_2)$, then $\pi_2^{-1} \circ \pi_1: \xi_1 \to \xi_2$ is a medium $\mathfrak{A}$-map*

PROOF. Let $\delta_1 = \sup\{\pi(\alpha) + 1 \mid \alpha < \xi_1\}$ and let $\delta_2 = \sup\{\pi_2^{-1}(\pi_1(\alpha)) + 1 \mid \alpha < \xi_1\}$. Then $\pi_1: \xi_1 \to \delta_1$ and $\pi_2 \upharpoonright \delta_2: \delta_2 \to \delta_1$ are strong $\mathfrak{A}$-maps. Thus $\pi_2^{-1} \circ \pi_1 = (\pi_2 \upharpoonright \delta_2)^{-1} \circ \pi_1$ is a strong $\mathfrak{A}$-map from ξ_1 into δ_2. $\square$

Theorem 16.19. *If $\pi: \xi \to \eta$ is a strong $\mathfrak{A}$-map and $X \subseteq \xi$, then $\pi``\mathfrak{A}^\xi(X) = \mathfrak{A}^\eta(\pi``X)$.*

PROOF. It is easy to see that $\pi``\mathfrak{A}^\xi(X)$ is a subalgebra of $\mathfrak{A}^\eta$ that contains $\pi``X$, and thus $\mathfrak{A}^\eta(\pi``X) \subseteq \pi``\mathfrak{A}^\xi(X)$. We shall show that $Y = (\pi^{-1})``\mathfrak{A}^\eta(\pi``X)$ is a subalgebra of $\mathfrak{A}^\xi$ that contains X. From this we see that $\pi``\mathfrak{A}^\xi(X) \subseteq \mathfrak{A}^\eta(\pi``X)$. Since $\pi``X \subseteq \mathfrak{A}^\eta(\pi``X)$, it is clear that $X \subseteq Y$. Let $\boldsymbol{\alpha} \in Y^{\omega}$ and assume that $F_i^\xi(\boldsymbol{\alpha})$ is defined. Then $\pi(\boldsymbol{\alpha}) \in \mathfrak{A}^\eta(\pi``X)$ and $F_i^\eta(\pi(\boldsymbol{\alpha}))$ is defined. $\mathfrak{A}^\eta(\pi``X)$ is a subalgebra of $\mathfrak{A}^\eta$, and hence $F_i^\eta(\pi(\boldsymbol{\alpha})) \in \mathfrak{A}^\eta(\pi``X)$. Since $(\pi^{-1})`F_i^\eta(\pi(\boldsymbol{\alpha})) = F_i^\xi(\boldsymbol{\alpha})$, we have $F_i^\xi(\boldsymbol{\alpha}) \in Y$. Thus Y is a subalgebra of $\mathfrak{A}^\xi$. □

Remark. Note that if X is a set of ordinals then by Corollary 7.52 there is a unique ordinal ξ and a unique order isomorphism from ξ onto X. We call this order isomorphism the *collapsing map* of X, and we call ξ the *order type* of X.

Definition 16.20. A machine $\mathfrak{A} = \langle A, F_i \rangle_{i \in \omega}$ has the collapsing property iff for every $\eta \in A$ and every subalgebra X of $\mathfrak{A}^\eta$ of order type ξ, the collapsing map of X is a strong $\mathfrak{A}$-map from ξ into η.

Remark. If $\mathfrak{A}$ has the collapsing property and X is a subalgebra of $\mathfrak{A}^\eta$ with order type ξ, then the collapsing map of X gives the isomorphism with $\mathfrak{A}^\xi$ and X as algebras.

Definition 16.21. $\mathfrak{A}$ has the finiteness property iff for every $\eta \in A$, there exists a finite set $H_\eta \subseteq \eta$ such that

$$\mathfrak{A}^{\eta+1}(X \cup \{\eta\}) \subseteq \mathfrak{A}^\eta(X \cup H_\eta) \cup \{\eta\} \quad \text{for all } X \subseteq \eta.$$

Lemma 16.22. *Let κ be an arbitrary limit ordinal and let η be the least ordinal greater than or equal to κ such that for some $\alpha < \kappa$ and some finite $P \subseteq \eta$, the order type of $\mathfrak{A}^\eta(\alpha \cup P)$ is not less than κ. If $\mathfrak{A}$ has the finiteness property, then η is a limit ordinal.*

PROOF. Take $\alpha < \kappa$ and let $P \subseteq \eta$ be a finite set so chosen that the order type of $\mathfrak{A}^\eta(\alpha \cup P)$ is greater than or equal to κ. Assume $\eta = \nu + 1$ for some ν. Since $\mathfrak{A}$ has the finiteness property, there exists a finite set $H_\nu \subseteq \nu$ which satisfies

$$\mathfrak{A}^\eta(\alpha \cup P) \subseteq \mathfrak{A}^\nu(\alpha \cup (P \cap \nu) \cup H_\nu) \cup \{\nu\}.$$

Since κ is a limit ordinal, we see that the order type of $\mathfrak{A}^\nu(\alpha \cup (P \cap \nu) \cup H_\nu)$ is greater than or equal to κ. This contradicts the minimality of η. Thus η is a limit ordinal. □

Definition 16.23. A partially ordered set $\langle I, \leqq \rangle$ is called a directed set if $(\forall i, j \in I)(\exists k \in I)[i \leqq k \wedge j \leqq k]$.

Definition 16.24. Let $\langle I, \leqq\rangle$ be a directed set. Then $\Pi = \langle \eta_i, \pi_{ij}\rangle_{i \leqq j}$ is a direct system of order preserving maps iff

(1) $\pi_{ij}: \eta_i \to \eta_j$ is an order preserving map for $i \leqq j$,

(2) π_{ii} is the identity map, and

(3) $\pi_{jk} \circ \pi_{ij} = \pi_{ik}$ if $i \leqq j \leqq k$.

If each $\pi_{ij}: \eta_i \to \eta_j$ is a strong $\mathfrak{A}$-map (medium $\mathfrak{A}$-map), then Π is a direct system of strong $\mathfrak{A}$-maps (medium $\mathfrak{A}$-maps).

Remark. Let $\Pi = \langle \eta_i, \pi_{ij}\rangle_{i \leqq j}$ be a direct system of order preserving maps with an index set $\langle I, \leqq\rangle$. We next define the limit of Π.

Definition 16.25. Let $\Pi = \langle \eta_i, \pi_{ij}\rangle_{i \leqq j}$ be a direct system of order preserving maps with an index set $\langle I, \leqq\rangle$. Then $\varinjlim \Pi \triangleq \langle\langle M/\equiv, <\rangle, \pi_{i\infty}\rangle_{i \in I}$ where $M/\equiv$, $<$, and $\pi_{i\infty}$ are defined as follows.

$$M \triangleq \bigcup_{i \in I} \{i\} \times \eta_i.$$

On M we define an equivalence relation $\equiv$:

$$\langle i, \alpha\rangle \triangleq \langle j, \beta\rangle \quad \text{iff } (\exists k \in I)[i \leqq k \wedge j \leqq k \wedge \pi_{ik}(\alpha) = \pi_{jk}(\beta)].$$

We define equivalence classes for each $\langle i, \alpha\rangle \in M$

$$[i, \alpha] \triangleq \{\langle j, \beta\rangle \in M \mid \langle i, \alpha\rangle \equiv \langle j, \beta\rangle\}.$$

Then we define the class of equivalence classes

$$M/\equiv \triangleq \{[i, \alpha] \mid \langle i, \alpha\rangle \in M\}.$$

On $M/\equiv$ we define a linear ordering:

$$[1, \alpha] < [j, \beta] \overset{\triangle}{\leftrightarrow} (\exists k \in I)[i \leqq k \wedge j \leqq k \wedge \pi_{ik}(\alpha) < \pi_{jk}(\beta)].$$

Finally we define $\pi_{i\infty}$, the canonical order preserving map from η_i to $M/\equiv$:

$$\pi_{i\infty}(\alpha) \triangleq [i, \alpha], \qquad \alpha \in \eta_i.$$

Remark. The characteristics of $\varinjlim \Pi$ are given by the following theorem.

Theorem 16.26. (1) *$\langle M/\equiv, <\rangle$ is a linearly ordered set.*

(2) *$\pi_{i\infty}: \eta_i \to M/\equiv$ is an order preserving map.*

(3) *$\pi_{i\infty} = \pi_{j\infty} \circ \pi_{ij}$ if $i \leqq j$.*

(4) *$M/\equiv = \bigcup_{i \in I} \mathscr{W}(\pi_{i\infty})$.*

If $\langle N, <'\rangle$ and $p_i: \eta_i \to N$ $(i \in I)$ satisfy (1)–(3), *then there exists an order preserving map $f: M/\equiv \to N$ such that $f \circ \pi_{i\infty} = p_i$ for all $i \in I$.*

Definition 16.27. Π is well founded $\overset{\triangle}{\leftrightarrow} \langle M/\equiv, <\rangle$ is well ordered.

Remark. If Π is well founded, then we identify $M/\equiv$ with its order type and denote it by η_∞.

Lemma 16.28. *Π is well founded iff there are no sequences $\langle i_n \mid n < \omega\rangle$ and $\langle \sigma_n \mid n < \omega\rangle$ such that $i_0 \leqq i_1 \leqq \cdots \leqq i_n \leqq \cdots$ and $\pi_{i_n i_{n+1}}(\sigma_n) > \sigma_{n+1}$.*

Proof. If such sequences exist, then $\langle \pi_{i_n\infty}(\sigma_n) \mid n < \omega\rangle$ is an infinite descending sequence in $M/\equiv$. Thus Π is not well founded. Suppose Π is not well founded and $\langle [j_n, \tau_n] \mid n < \omega\rangle$ is an infinite descending sequence in $\varinjlim \Pi$. Then we can find a sequence $\langle i_n \mid n < \omega\rangle$ such that $i_0 \leqq i_1 \leqq \cdots \leqq i_n \leqq \cdots$ and $j_n \leqq i_n$ for all $n < \omega$. Let $\sigma_n = \pi_{j_n i_n}(\tau_n)$. Then for all $n < \omega$,

$$\begin{aligned}\pi_{i_n i_{n+1}}(\sigma_n) &= \pi_{i_n i_{n+1}}(\pi_{j_n i_n}(\tau_n)) = \pi_{j_n i_{n+1}}(\tau_n)\\ &> \pi_{j_{n+1} i_{n+1}}(\tau_{n+1}) = \sigma_{n+1}.\end{aligned}$$ ☐

Definition 16.29. $\mathfrak{A}$ has the direct limit property iff for every well-founded direct system $\Pi = \langle \eta_i, \pi_{ij}\rangle_{i \leqq j}$ of strong $\mathfrak{A}$-maps (medium $\mathfrak{A}$-maps), each $\pi_{i\infty}: \eta_i \to \eta_\infty$ is a strong $\mathfrak{A}$-map (medium $\mathfrak{A}$-map).

Definition 16.30. If $\boldsymbol{\alpha} = \langle \alpha_1, \ldots, \alpha_m\rangle$, then

$$\max(\boldsymbol{\alpha}) \triangleq \max\{\alpha_1, \ldots, \alpha_m\}.$$

Definition 16.31. For finite sequences of ordinals $\boldsymbol{\alpha} = \langle \alpha_1, \ldots, \alpha_n\rangle$ and $\boldsymbol{\beta} = \langle \beta_1, \ldots, \beta_m\rangle$, $\boldsymbol{\alpha} < \boldsymbol{\beta}$ iff

(1) $\max(\boldsymbol{\alpha}) < \max(\boldsymbol{\beta})$, or

(2) $\boldsymbol{\beta}$ is not a permutation of $\boldsymbol{\alpha}$ and $\alpha_i = \max(\boldsymbol{\alpha}) = \max(\boldsymbol{\beta}) = \beta_j \wedge \langle \alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_m\rangle < \langle \beta_1, \ldots, \beta_{j-1}, \beta_{j+1}, \ldots, \beta_m\rangle$, or

(3) $\boldsymbol{\beta}$ is a permutation of $\boldsymbol{\alpha}$, and $\boldsymbol{\alpha}$ is less than $\boldsymbol{\beta}$ in the lexicographic ordering, that is, $\alpha_{i_0} < \beta_{i_0}$ where $i_0 = \min\{i \mid \alpha_i \neq \beta_i\}$.

Remark. It is easy to verify that $<$ is a well ordering on $\mathrm{On}^{\underline{\omega}}$.

Definition 16.32. The pairing machine $\mathscr{P}_< = \langle \mathrm{On}, F_0, J, C_i\rangle$ is the machine defined by:

J: $\langle \mathrm{On}^{\underline{\omega}}, <\rangle \to \langle \mathrm{On}, <\rangle$ is the order isomorphism $C_i(\langle\alpha\rangle) = \beta_i$, if $J(\langle \beta_0, \ldots, \beta_{n-1}\rangle) = \alpha$ and $i < n$, $C_i(\boldsymbol{\alpha})$ is undefined, otherwise.

Exercises

(1) Compute $J(\langle\ \rangle)$, $J(\langle 0\rangle)$, $J(\langle 1\rangle)$, $J(\langle 2\rangle), \ldots$. Find the least ordinal closed under J, that is, the first ordinal α such that $J``\alpha^{\underline{\omega}} = \alpha$.

(2) Prove that $\mathscr{P}_<$ and $\mathscr{P}^\eta_<$ are absolute with respect to L, the class of constructible sets.

Proposition 16.33.

(1) $\max(\boldsymbol{\alpha}) \leqq J(\boldsymbol{\alpha})$.

(2) $C_i(\langle\alpha\rangle) \leqq \alpha$ if $C_i(\langle\alpha\rangle)$ *is defined.*

PROOF. (1) Since the function $f(\alpha) = J``\alpha^{\underline{\omega}}$ is increasing, it follows that $\alpha \leqq J``\alpha^{\underline{\omega}}$ for all α. Thus if $\alpha_i = \max(\boldsymbol{\alpha})$, then $\alpha_i \leqq J``\alpha_i^{\underline{\omega}} \leqq J(\boldsymbol{\alpha})$.

(2) Let $\alpha = J(\langle\alpha_0, \ldots, \alpha_{n-1}\rangle)$. Then by (1), $\max(\alpha_0, \ldots, \alpha_{n-1}) \leqq \alpha$ and so $C_i(\langle\alpha\rangle) = \alpha_i \leqq \alpha$ for each $i < n$. □

Theorem 16.34. *The pairing machine $\mathscr{P}_<$ has the collapsing property.*

PROOF. Let η be an ordinal, let $X \subseteqq \eta$ be a subalgebra of $\mathscr{P}^\eta_<$ and let $\pi: \xi \to X$ be the collapsing map of X. We want to show that $\pi: \xi \to \eta$ is a strong $\mathscr{P}_<$-map.

Let $Z = \{\boldsymbol{\alpha} \in \mathrm{On}^{\underline{\omega}} \mid J(\boldsymbol{\alpha}) \in X\}$. Then, for any $\boldsymbol{\alpha} = \langle\alpha_0, \ldots, \alpha_{n-1}\rangle$,

$$\begin{aligned}\boldsymbol{\alpha} \in Z &\to J(\boldsymbol{\alpha}) \in X \subseteqq \eta\\ &\to \alpha_i = C_i^\eta(\langle J(\boldsymbol{\alpha})\rangle) \in X \quad \text{for all } i < n\\ &\to \boldsymbol{\alpha} \in X^{\underline{\omega}}.\end{aligned}$$

Hence $Z \subseteqq X^{\underline{\omega}}$.

Z is an initial segment of $X^{\underline{\omega}}$:

$$\begin{aligned}\boldsymbol{\alpha} \in Z \wedge \boldsymbol{\beta} \in X^{\underline{\omega}} \wedge \boldsymbol{\beta} < \boldsymbol{\alpha} &\to J(\boldsymbol{\beta}) < J(\boldsymbol{\alpha}) \in X \subseteqq \eta\\ &\to J(\boldsymbol{\beta}) = J^\eta(\boldsymbol{\beta}) \in X\\ &\to \boldsymbol{\beta} \in Z.\end{aligned}$$

It is obvious, from the definition of Z, that J^η maps Z onto X. Since π induces an order isomorphism between $\xi^{\underline{\omega}}$ and $X^{\underline{\omega}}$, $(\pi^{-1})``Z$ is an initial segment of $\xi^{\underline{\omega}}$ of order type ξ. Therefore,

$$\pi(J^\xi(\boldsymbol{\alpha})) \simeq J^\eta(\tilde{\pi}(\boldsymbol{\alpha})) \quad \text{for all } \boldsymbol{\alpha} \in \xi^{\underline{\omega}}.$$

From this, we also see that

$$\pi(C_i^\xi(\boldsymbol{\alpha})) \simeq C_i^\eta(\tilde{\pi}(\boldsymbol{\alpha})) \quad \text{for all } \boldsymbol{\alpha} \in \xi^{\underline{\omega}}.$$ □

Theorem 16.35. *$\mathscr{P}_<$ has the finiteness property.*

PROOF. Let η be an arbitrary ordinal and $\beta_0, \ldots, \beta_{n-1}$ be such that $J(\langle\beta_0, \ldots, \beta_{n-1}\rangle) = \eta$, then $\beta_i \leqq \eta$ for all $i < n$. Let $H_\eta = \{\beta_i \mid \beta_i \neq \eta \wedge i < n\}$. We shall prove that for all $X \subseteqq \eta$

$$\mathscr{P}^{\eta+1}_<(X \cup \{\eta\}) \subseteqq \mathscr{P}^\eta_<(X \cup H_\eta) \cup \{\eta\}.$$

Let $Y = \mathscr{P}^\eta_<(X \cup H_\eta) \cup \{\eta\}$. We want to show that Y is a subalgebra of $\mathscr{P}^{\eta+1}_<$. If $\alpha \in \mathscr{P}^\eta_<(X \cup H_\eta)$, then $C_i^{\eta+1}(\langle\alpha\rangle) = C_i^\eta(\langle\alpha\rangle)$ and hence $C_i^{\eta+1}(\langle\alpha\rangle) \in$

$\mathscr{P}^{\eta}_{<}(X \cup H_{\eta}) \subseteqq Y$. Clearly, $C_i^{\eta+1}(\langle\eta\rangle) \in H_{\eta} \cup \{\eta\} \subseteqq Y$. Thus Y is closed under $C_i^{\eta+1}$. It is easy to verify that Y is closed under F_0 and J, so we leave it to the reader. $\square$

Exercise

Complete the discussion about C_i in the above proof.

Theorem 16.36. *$\mathscr{P}_{<}$ has the direct limit property.*

Proof. Let $\Pi = \langle\eta_i, \pi_{ij}\rangle_{i \leqq j}$ be a well-founded direct system of strong $\mathscr{P}_{<}$ maps with an index set $\langle I, \leqq\rangle$. First we shall show by induction on $J(\tilde{\pi}_{i\infty}(\boldsymbol{\alpha}))$ that if $J^{\eta_i}(\boldsymbol{\alpha}) \simeq \beta$, then $J(\tilde{\pi}_{i\infty}(\boldsymbol{\alpha})) = \pi_{i\infty}(\beta)$. Let $\boldsymbol{\delta} \in \eta_{\infty}^{\omega}$ be such that $\boldsymbol{\delta} < \tilde{\pi}_{i\infty}(\boldsymbol{\alpha})$. Then there exist $j \geqq i$ and $\boldsymbol{\delta}' \in \eta_j^{\omega}$ such that $\tilde{\pi}_{j\infty}(\boldsymbol{\delta}') = \boldsymbol{\delta}$. If we put $\boldsymbol{\alpha}' = \tilde{\pi}_{ij}(\boldsymbol{\alpha})$, then we have $\tilde{\pi}_{j\infty}(\boldsymbol{\delta}') < \tilde{\pi}_{i\infty}(\boldsymbol{\alpha}) = \tilde{\pi}_{j\infty}(\boldsymbol{\alpha}')$, and so $J(\tilde{\pi}_{j\infty}(\boldsymbol{\delta}')) < J(\tilde{\pi}_{i\infty}(\boldsymbol{\alpha}))$. By the induction hypothesis, we see that $J(\boldsymbol{\delta}) = \pi_{j\infty}(J(\boldsymbol{\delta}'))$. Since $\pi_{j\infty}$ is order preserving and $\tilde{\pi}_{j\infty}(\boldsymbol{\delta}') < \tilde{\pi}_{j\infty}(\boldsymbol{\alpha}')$, we have $\boldsymbol{\delta}' < \boldsymbol{\alpha}'$. Noting that $J(\boldsymbol{\alpha}') = J^{\eta_j}(\tilde{\pi}_{ij}(\boldsymbol{\alpha})) = \pi_{ij}(J^{\eta_i}(\boldsymbol{\alpha})) = \pi_{ij}(\beta)$, we have:

$$J(\boldsymbol{\delta}) = \pi_{j\infty}(J(\boldsymbol{\delta}')) < \pi_{j\infty}(J(\boldsymbol{\alpha}')) = \pi_{j\infty}(\pi_{ij}(\beta)) = \pi_{i\infty}(\beta).$$

Then $J(\tilde{\pi}_{i\infty}(\boldsymbol{\alpha})) \leqq \pi_{i\infty}(\beta)$. Assume $J(\tilde{\pi}_{i\infty}(\boldsymbol{\alpha})) < \pi_{i\infty}(\beta)$. Let $j \geqq i$ and $\sigma < \eta_j$ be such that $\pi_{j\infty}(\sigma) = J(\tilde{\pi}_{i\infty}(\boldsymbol{\alpha}))$. If $\boldsymbol{\alpha}' = \tilde{\pi}_{ij}(\boldsymbol{\alpha})$ and $\beta' = \pi_{ij}(\beta)$, then $\pi_{j\infty}(\sigma) < \pi_{j\infty}(\beta') = \pi_{j\infty}(J(\boldsymbol{\alpha}'))$ because $J(\boldsymbol{\alpha}') = J^{\eta_j}(\tilde{\pi}_{ij}(\boldsymbol{\alpha})) = \pi_{ij}(J^{\eta_i}(\boldsymbol{\alpha})) = \pi_{ij}(\beta) = \beta'$. Hence there exists a $\boldsymbol{\gamma} \in \eta_j^{\omega}$ such that $\boldsymbol{\gamma} < \boldsymbol{\alpha}'$ and $J(\boldsymbol{\gamma}) = \sigma$. Since $\boldsymbol{\gamma} < \boldsymbol{\alpha}'$, we see that $J(\tilde{\pi}_{j\infty}(\boldsymbol{\gamma})) < J(\tilde{\pi}_{j\infty}(\boldsymbol{\alpha}')) = J(\tilde{\pi}_{i\infty}(\boldsymbol{\alpha}))$. By the induction hypothesis, $J(\tilde{\pi}_{j\infty}(\boldsymbol{\gamma})) = \pi_{j\infty}(J(\boldsymbol{\gamma})) = \pi_{j\infty}(\sigma)$. Therefore $J(\tilde{\pi}_{j\infty}(\boldsymbol{\gamma})) = J(\tilde{\pi}_{i\infty}(\boldsymbol{\alpha}))$. This is a contradiction, and we conclude that $J(\tilde{\pi}_{i\infty}(\boldsymbol{\alpha})) = \pi_{i\infty}(\beta)$.

Next we shall show that if $J^{\eta_\infty}(\tilde{\pi}_{i\infty}(\boldsymbol{\alpha}))$ is defined, then $J^{\eta_i}(\boldsymbol{\alpha})$ is also defined. From this we have

$$\pi_{i\infty}(J^{\eta_i}(\boldsymbol{\alpha})) \simeq J^{\eta_\infty}(\tilde{\pi}_{i\infty}(\boldsymbol{\alpha})) \quad \text{for all } \boldsymbol{\alpha} \in \eta_i^{\omega}.$$

Let $\beta = J^{\eta_\infty}(\tilde{\pi}_{i\infty}(\boldsymbol{\alpha}))$, then there exist $j \geqq i$ and $\beta' < \eta_j$ such that $\pi_{j\infty}(\beta') = \beta$. If $\boldsymbol{\alpha}' = \tilde{\pi}_{ij}(\boldsymbol{\alpha})$, then

$$J^{\eta_\infty}(\tilde{\pi}_{j\infty}(\boldsymbol{\alpha}')) = \pi_{j\infty}(\beta').$$

If we show that $J(\boldsymbol{\alpha}') \leqq \beta'$, then $J^{\eta_j}(\boldsymbol{\alpha}') = J^{\eta_j}(\tilde{\pi}_{ij}(\boldsymbol{\alpha}))$ is defined, and hence $J^{\eta_i}(\boldsymbol{\alpha})$ is also defined because $\pi_{ij}\colon \eta_i \to \eta_j$ is a strong $\mathscr{P}_{<}$-map. Suppose $J(\boldsymbol{\alpha}') > \beta'$, then there exists a $\boldsymbol{\gamma} \in \eta_j^{\omega}$ such that $J(\boldsymbol{\gamma}) = \beta'$, and hence

$$J(\tilde{\pi}_{j\infty}(\boldsymbol{\gamma})) = \pi_{j\infty}(\beta') = J(\tilde{\pi}_{j\infty}(\boldsymbol{\alpha}')).$$

This is a contradiction since $\boldsymbol{\gamma} < \boldsymbol{\alpha}'$. $\square$

Remark. We next introduce a ramified language $\mathscr{L}$, to provide a notation for each member of L, the class of constructible sets. The symbols of $\mathscr{L}$ are the following:

Variables: $x_0, x_1, \ldots, x_n, \ldots \quad (n \in \omega)$.
Relation symbols: $\in$, $=$.
Propositional connectives: $\neg$, $\vee$.
Quantifiers: $\exists^\alpha$ $(\alpha \in \mathrm{On})$.
Abstraction operators: $^{\wedge\alpha}$ $(\alpha \in \mathrm{On})$.
Parenthesis: (,).

Definition 16.37.

(1) $g(\exists^\alpha) \triangleq 2\alpha + 1$,

(2) $g(^{\wedge\alpha}) \triangleq 2\alpha + 2$

and for any finite sequence s of symbols of $\mathscr{L}$

(3) $g(s)$ is the maximum of $g(\exists^\alpha)$ and $g(^{\wedge\alpha})$ for all $\exists^\alpha$ and $^{\wedge\alpha}$ which occur in s.

Definition 16.38. Formulas and constant terms of $\mathscr{L}$ are defined as follows.

(1) If each of t_1 and t_2 is a constant term or a variable, then $(t_1 \in t_2)$ and $(t_1 = t_2)$ are formulas of $\mathscr{L}$.

(2) If φ and ψ are formulas, then $(\neg\varphi)$ and $(\varphi \vee \psi)$ are formulas.

(3) If φ is a formula, then $(\exists^\alpha x_i \varphi)$ is a formula.

(4) If $\varphi(x_i)$ is a formula without free variables other than x_i such that $g(\varphi(x_i)) < g(^{\wedge\alpha})$, then $(\hat{x}_i^\alpha \varphi(x_i))$ is a constant term.

(5) Formulas and terms are only those obtained by a finite number of applications of (1)–(4).

Definition 16.39. $T_\alpha \triangleq \{t \mid t$ is a constant term of the form $(\hat{x}_i^\beta \varphi)$ with $\beta < \alpha\}$ and $T \triangleq \bigcup_{\alpha \in \mathrm{On}} T_\alpha$.

Definition 16.40. For each atomic sentence φ of $\mathscr{L}$, a sentence $E(\varphi)$ of $\mathscr{L}$ is defined as follows.

(1) $E(\hat{x}_i^\alpha \varphi(x_i) = \hat{x}_j^\beta \psi(x_j))$

$$\overset{\triangle}{\leftrightarrow} \forall^\gamma x_k[\exists^\alpha x_i(x_i = x_k \wedge \varphi(x_i)) \leftrightarrow \exists^\beta x_j(x_j = x_k \wedge \psi(x_j))]$$

where $\gamma = \max(\alpha, \beta)$ and x_k is the first variable occurring in neither $\hat{x}_i^\alpha \varphi(x_i)$ nor $\hat{x}_j^\beta \psi(x_j)$.

(2) If $\alpha < \beta$, then

$$E(\hat{x}_i^\alpha \varphi(x_i) \in \hat{x}_j^\beta \psi(x_j)) \overset{\triangle}{\leftrightarrow} \psi(\hat{x}_i^\alpha \varphi(x_i)).$$

(3) If $\alpha \geqq \beta$, then

$$E(\hat{x}_i^\alpha \varphi(x_i) \in \hat{x}_j^\beta \psi(x_j)) \overset{\triangle}{\leftrightarrow} \exists^\alpha x_k[\forall^\alpha x_i(x_i \in x_k \leftrightarrow \varphi(x_i)) \\ \wedge \exists^\beta x_j(x_j = x_k \wedge \psi(x_j))],$$

where x_k is the first variable occurring in neither $\hat{x}_i^\alpha \varphi(x_i)$ nor $\hat{x}_j^\beta \psi(x_j)$.

Remark. Let $\mathscr{P}_<$ be the pairing machine of Definition 16.32. Using J, we code the symbols of $\mathscr{L}$ by ordinals.

Definition 16.41.

(1) $\ulcorner \in \urcorner = J(\langle 0, 0\rangle)$,

(2) $\ulcorner = \urcorner = J(\langle 0, 1\rangle)$,

(3) $\ulcorner \neg \urcorner = J(\langle 0, 2\rangle)$,

(4) $\ulcorner \vee \urcorner = J(\langle 0, 3\rangle)$,

(5) $\ulcorner (\urcorner = J(\langle 0, 4\rangle)$,

(6) $\ulcorner) \urcorner = J(\langle 0, 5\rangle)$,

(7) $\ulcorner x_i \urcorner = J(\langle 0, 6 + i\rangle)$,

(8) $\ulcorner \exists^\alpha \urcorner = J(\langle 0, \omega + \alpha\rangle)$,

and

(9) $\ulcorner \wedge_\alpha \urcorner = J(\langle 0, \omega + \alpha, \omega + \alpha\rangle)$.

For any finite sequence $s_1, \ldots, s_n$ of symbols of $\mathscr{L}$,

(10) $\ulcorner s_1, \ldots, s_n \urcorner = J(\langle 1, \ulcorner s_1 \urcorner, \ldots, \ulcorner s_n \urcorner\rangle)$.

Remark. From now on, we identify formulas and terms of $\mathscr{L}$ with their codes. We need the following properties of codes.

(1) The codes of different formulas (terms) are different.

(2) If $\pi\colon \xi \to \eta$ is a medium $\mathscr{P}_<$-map, then for each formula (term) φ, $\pi(\varphi)$ is the formula (term) obtained from φ by replacing each occurrence of $\exists^\alpha$ and $\wedge_\alpha$ by $\exists^{\pi(\alpha)}$ and $\wedge_{\pi(\alpha)}$ respectively.

(3) If $(\exists^\alpha x_i\, \varphi(x_i))$ is a sentence and $t \in T_\alpha$, then $\varphi(t) < (\exists^\alpha x_i\, \varphi(x_i))$ and $t < (\exists^\alpha x_i\, \varphi(x_i))$.

(4) If $(\hat{x}_i^\alpha \varphi(x_i))$ is a constant term and $t \in T_\alpha$, then $\varphi(t) < (\hat{x}_i^\alpha \varphi(x_i))$ and $t < (\hat{x}_i^\alpha \varphi(x_i))$.

(5) If φ is an atomic sentence, then $E(\varphi) < \varphi$.

(6) $\varphi < (\neg \varphi)$, $\varphi < (\varphi \vee \psi)$ and $\psi < (\varphi \vee \psi)$.

(1), and (3)–(6) are easily verified. We shall only prove (2). Let $\pi\colon \xi \to \eta$ be a medium $\mathscr{P}_<$-map, and let $\delta = \sup\{\pi(\alpha) + 1 \mid \alpha < \xi\}$. Then $\pi\colon \xi \to \delta$ is a strong $\mathscr{P}_<$-map. Note that $\pi(0) = \pi(J^\xi(\langle\ \rangle)) = J^\delta(\pi(\langle\ \rangle)) = J^\delta(\langle\ \rangle) = 0$ and $\pi(1) = \pi(J^\xi(\langle 0\rangle)) = J^\delta(\pi(\langle 0\rangle)) = J^\delta(\langle \pi(0)\rangle) = J^\delta(\langle 0\rangle) = 1$. Furthermore each ordinal less than ω^2 can be expressed in the form $J(\langle i_0, \ldots, i_{n-1}\rangle)$ with each of $i_0, \ldots, i_{n-1}$ less than or equal to 1. Therefore, if $\alpha < \min(\xi, \omega^2)$, then $\pi(\alpha) = \alpha$. Thus we have $\pi(\ulcorner s \urcorner) = \ulcorner s \urcorner$ for all symbols of $\mathscr{L}$ except $\exists^\alpha$ and $\wedge_\alpha$. If $\omega + \alpha < \xi$, then

$$\pi(\omega + \alpha) = \begin{cases} \omega + \alpha & \text{if } \alpha < \omega^2 \\ \omega + \pi(\alpha) & \text{if } \alpha \geqq \omega^2. \end{cases}$$

From this, we have $\pi(\exists^\alpha) = \exists^{\pi(\alpha)}$ and $\pi(\wedge_\alpha) = \wedge_{\pi(\alpha)}$.

EXERCISE

Find the first ordinal $\lambda > 0$ such that for every medium $\mathscr{P}_<$-map, $\pi: \xi \to \eta$, $\pi \upharpoonright (\xi \cap \lambda)$ is the identity map.

Definition 16.42.

(1) $D(t_1 \in t_2) \overset{\Delta}{\leftrightarrow} D(E(t_1 \in t_2))$.

(2) $D(t_1 = t_2) \overset{\Delta}{\leftrightarrow} D(E(t_1 = t_2))$.

(3) $D(\neg \varphi) \overset{\Delta}{\leftrightarrow} \neg D(\varphi)$.

(4) $D(\varphi \vee \psi) \overset{\Delta}{\leftrightarrow} D(\varphi) \vee D(\psi)$.

(5) $D(\exists^\alpha x_i \varphi(x_i)) \overset{\Delta}{\leftrightarrow} (\exists t \in T_\alpha) D(\varphi(t))$.

(6) $D(\hat{x}_i^\alpha \varphi(x_i)) \triangleq \{D(t) \mid t \in T_\alpha \wedge D(\varphi(t))\}$.

$L_\alpha \triangleq \{D(t) \mid t \in T_\alpha\}$ and $L \triangleq \bigcup_{\alpha \in \mathrm{On}} L_\alpha$.

For each sentence φ of $\mathscr{L}$, $L \models \varphi \overset{\Delta}{\leftrightarrow} D(\varphi)$.

EXERCISE

Prove that the class L defined here is the class of constructible sets defined in Chapter 15. Prove also that all of the notions involving L are absolute with respect to L.

Remark. (1) Recall that each of the sentences and terms of $\mathscr{L}$ is an ordinal. Consequently the operator D of Definition 16.42 is well defined.

(2) For all constant terms t_1 and t_2, it is easily seen that

$$D(t_1 = t_2) \leftrightarrow D(t_1) = D(t_2).$$
$$D(t_1 \in t_2) \leftrightarrow D(t_1) \in D(t_2).$$

(3) $L_0 = 0$

$L_{\alpha+1} = \{x \subseteq L_\alpha \mid x \text{ is first-order definable over } \langle L_\alpha, \in\rangle \text{ with parameters from } L_\alpha\}$

$L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha,\ \lambda \in K_{\mathrm{II}}$

Definition 16.43. The L-machine $M = \langle \mathrm{On}, F_0, J, C_i, T, K\rangle_{i<\omega}$ is defined as follows:

(1) $\mathscr{P}_< = \langle \mathrm{On}, F_0, J, C_i\rangle_{i<\omega}$ is the pairing machine of Definition 16.32.

(2) If φ is a sentence of $\mathscr{L}$, then

$$T(\langle\varphi\rangle) = \begin{cases} 1 & \text{if } L \models \varphi, \\ 0 & \text{otherwise.} \end{cases}$$

(3) If $\exists^\alpha x_i \varphi(x_i)$ is a sentence of $\mathscr{L}$ such that $L \models \exists^\alpha x_i \varphi(x_i)$, then $K(\langle \exists^\alpha x_i \varphi(x_i)\rangle)$ is the least term $t \in T_\alpha$ such that $L \models \varphi(t)$.

(4) $T(\boldsymbol{\alpha})$ and $K(\boldsymbol{\alpha})$ are undefined except for the cases specified in (2) and (3).

Theorem 16.44. *The L-machine M has the collapsing property.*

Proof. Let η be an ordinal, let X be a subalgebra of M^η; and let $\pi\colon \bar\eta \to X$ be the collapsing map of X. We shall prove that $\pi\colon \bar\eta \to \eta$ is a strong M-map. Since $\mathscr{P}_<$ has the collapsing property, $\pi\colon \bar\eta \to \eta$ is a strong $\mathscr{P}_<$-map.

First we show, by induction on $\bar\varphi$, that for any sentence $\bar\varphi < \bar\eta$ of $\mathscr{L}$, $L \models \bar\varphi$ iff $L \models \pi(\bar\varphi)$. We consider only the case where $\bar\varphi$ is $\exists^{\bar\alpha} x_i \bar\theta(x_i)$. Other cases can be treated similarly. Let $\alpha = \pi(\bar\alpha)$ and $\theta = \pi(\bar\theta)$. Then $\pi(\bar\varphi) = \exists^\alpha x_i \theta(x_i)$ and

$$\begin{aligned} L \models \bar\varphi &\to (\exists\, \bar t \in T_{\bar\alpha})[L \models \bar\theta(\bar t)] \\ &\to (\exists\, t \in T_\alpha)[L \models \theta(t)] \quad \text{(by the induction hypothesis)} \\ &\to L \models \exists^\alpha x_i \theta(x_i) \\ &\to L \models \varphi. \end{aligned}$$

Conversely, suppose $L \models \varphi$ and let $t = K(\langle\varphi\rangle) \in T_\alpha$. Since $t < \varphi$, $t = K^\eta(\langle\varphi\rangle)$ and hence $t \in X$. Let $\bar t < \bar\eta$ be such that $\pi(\bar t) = t$. Then $\bar t \in T_{\bar\alpha}$, and by the induction hypothesis, $L \models \bar\theta(\bar t)$. Therefore $L \models \bar\varphi$.

It is easily seen that $\pi(T^{\bar\eta}(\boldsymbol{\alpha})) \simeq T^\eta(\tilde\pi(\boldsymbol{\alpha}))$ for all $\boldsymbol{\alpha} \in \bar\eta^{\underline{\omega}}$.

Next we shall prove that $\pi(K^{\bar\eta}(\boldsymbol{\alpha})) \simeq K^\eta(\tilde\pi(\boldsymbol{\alpha}))$ for all $\boldsymbol{\alpha} \in \bar\eta^{\underline{\omega}}$. Let $\exists^\alpha x_i \bar\theta(x_i) < \bar\eta$ be a sentence of $\mathscr{L}$ such that $L \models \exists^{\bar\alpha} x_i \bar\theta(x_i)$. Let $\alpha = \pi(\bar\alpha)$ and $\theta = \pi(\bar\theta)$, then we have seen that $L \models \exists^\alpha x_i \theta(x_i)$. Let $\bar t_1, \bar t_2 \in T_{\bar\alpha}$ and $t_1, t_2 \in T_\alpha$ be such that $\pi(\bar t_1) = t_1$, $\pi(\bar t_2) = t_2$, $\bar t_1 = K(\langle\exists^{\bar\alpha} x_i \bar\theta(x_i)\rangle)$ and $t_2 = K(\langle\exists^\alpha x_i \theta(x_i)\rangle)$. Then we have $L \models \bar\theta(\bar t_1)$, $L \models \bar\theta(\bar t_2)$, $L \models \theta(t_1)$ and $L \models \theta(t_2)$. Therefore $\bar t_1 \leqq \bar t_2$, $t_2 \leqq t_1$, and hence $\bar t_1 = \bar t_2$. Thus we have proved that if $K^{\bar\eta}(\langle\exists^{\bar\alpha} x_i \bar\theta(x_i)\rangle) \simeq \bar t_1$, then $K^\eta(\langle\exists^\alpha x_i \theta(x_i)\rangle) \simeq t_1$. If $K^\eta(\langle\exists^\alpha x_i \theta(x_i)\rangle)$ is defined, then $L \models \exists^\alpha x_i \theta(x_i)$, and so $L \models \exists^{\bar\alpha} x_i \bar\theta(x_i)$. Hence $K^{\bar\eta}(\langle\exists^{\bar\alpha} x_i \bar\theta(x_i)\rangle)$ is defined. □

Theorem 16.45. *M has the finiteness property.*

Proof. Let η be an arbitrary ordinal, and let H_η be the finite subset of η defined in the proof of Theorem 16.35. If $H'_\eta = H_\eta \cup \{K(\langle\eta\rangle)\}$, then, as in the proof of Theorem 16.35, $M^\eta(X \cup H'_\eta) \cup \{\eta\}$ is a subalgebra of $M^{\eta+1}$. Thus for all $X \subseteq \eta$,

$$M^{\eta+1}(X \cup \{\eta\}) \subseteq M^\eta(X \cup H'_\eta) \cup \{\eta\}. \qquad \square$$

Theorem 16.46. *The L-machine M has the direct limit property.*

Proof. Let $\Pi = \langle \eta_i, \pi_{ij}\rangle_{i \leqq j}$ be a well-founded direct system of strong M-maps with an index set $\langle I, \leqq\rangle$. We have already seen that each $\pi_{i\infty}\colon \eta_i \to \eta_\infty$ is a strong $\mathscr{P}_<$-map.

Let φ be a sentence of $\mathscr{L}$. We shall prove by induction on φ that if $\varphi < \eta_i$, then

(1) $L \models \varphi$ iff $L \models \pi_{i\infty}(\varphi)$.

If φ is $\exists^\alpha x\theta(x)$, then $\pi_{i\infty}(\varphi) = \exists^{\pi_{i\infty}(\alpha)}x\pi_{i\infty}(\theta)(x)$. Suppose $L \models \varphi$. Then there exists a $t \in T_\alpha$ such that $L \models \theta(t)$, and hence, by the induction hypothesis, $L \models \pi_{i\infty}(\theta)(\pi_{i\infty}(t))$ with $\pi_{i\infty}(t) \in T_{\pi_{i\infty}(\alpha)}$. Thus we have $L \models \pi_{i\infty}(\varphi)$. Now suppose $L \models \pi_{i\infty}(\varphi)$. Then there exists a $t' \in T_{\pi_{i\infty}(\alpha)}$ such that $L \models \pi_{i\infty}(\theta)(t')$. Let $j \geqq i$ and $t \in T_{\pi_{ij}(\alpha)}$ be such that $\pi_{j\infty}(t) = t'$. By the induction hypothesis, $L \models \pi_{ij}(\theta)(t)$, and hence $L \models \pi_{ij}(\varphi)$. Since $\pi_{ij}: \eta_i \to \eta_j$ is a strong M-map, we see that $L \models \varphi$.

From (1) it is easy to see that

$$(2) \quad \pi_{i\infty}(T^{\eta_i}(\boldsymbol{\alpha})) \simeq T^{\eta_\infty}(\bar{\pi}_{i\infty}(\boldsymbol{\alpha})) \quad \text{for all } \boldsymbol{\alpha} \in \eta_i^\omega.$$

It remains to verify that

$$(3) \quad \pi_{i\infty}(K^{\eta_i}(\boldsymbol{\alpha})) \simeq K^{\eta_\infty}(\bar{\pi}_{i\infty}(\boldsymbol{\alpha})) \quad \text{for all } \boldsymbol{\alpha} \in \eta_i^\omega.$$

Let $\varphi = \exists^\alpha x\theta(x)$ be a sentence that is less than η_i. Then by (1), $K^{\eta_i}(\langle\varphi\rangle)$ is defined iff $K^{\eta_\infty}(\langle\pi_{i\infty}(\varphi)\rangle)$ is defined. Suppose $L \models \varphi$ and $t = K^{\eta_i}(\langle\varphi\rangle)$. We want to show that $\pi_{i\infty}(t) = K^{\eta_\infty}(\langle\pi_{i\infty}(\varphi)\rangle)$. Let $s' = K^{\eta_\infty}(\langle\pi_{i\infty}(\varphi)\rangle)$, then by (1), it is obvious that $s' \leqq \pi_{i\infty}(t)$. Let $j \geqq i$ and $s \in T_{\pi_{ij}(\alpha)}$ be such that $\pi_{j\infty}(s) = s'$ and $L \models \pi_{ij}(\theta)(s)$. Since $\pi_{ij}: \eta_i \to \eta_j$ is a strong M-map, we see that $\pi_{ij}(t) = K^{\eta_j}(\langle\pi_{ij}(\varphi)\rangle)$. Hence $\pi_{ij}(t) \leqq s$, and $\pi_{i\infty}(t) = \pi_{j\infty}(\pi_{ij}(t)) \leqq \pi_{j\infty}(s) = s'$. Consequently, $\pi_{i\infty}(t) = s'$. □

Exercises

(1) Prove that if $\Pi = \langle\eta_i, \pi_{ij}\rangle_{i \leqq j}$ is a well-founded direct system of medium M-maps, then each $\pi_{i\infty}: \eta_i \to \eta_\infty$ is also a medium M-map.

(2) M and M^η are absolute with respect to L.

CHAPTER 17

Applications of Silver Machines

In this chapter, we give two applications of the L-machine M.

Definition 17.1. For each ordinal α,

$$\check{\alpha} = \hat{x}_0^{\alpha} \operatorname{Ord}(x_0).$$

Remark. By induction on α it is easy to see that

$$D(\check{\alpha}) = \alpha.$$

Theorem 17.2. $V = L \rightarrow \mathrm{GCH}$.

Proof. We shall show that

$$\mathscr{P}(\aleph_\alpha) \subseteq L_{\aleph_{\alpha+1}}.$$

Suppose that $a \subseteq \aleph_\alpha$ and let t be a term such that $D(t) = a$. Let $\eta = \max(\aleph_{\alpha+1}, \bar{\bar{t}}^{+})$ and let $X = M^{\eta}(\aleph_\alpha \cup \{t\})$, where $\bar{\bar{t}}$ is the cardinality of t and $\bar{\bar{t}}^{+}$ is the next cardinal after $\bar{\bar{t}}$. Obviously $\bar{\bar{X}} = \aleph_\alpha$. Since M has the collapsing property, it follows that if $\pi: \eta' \rightarrow X$ is the collapsing map of X, then $\pi: \eta' \rightarrow \eta$ is a strong M-map. Let t' be a term such that $t' < \eta'$ and $\pi(t') = t$. Noting that $\pi(\beta) = \beta$ for all $\beta < \aleph_\alpha$, we have that for any $\beta < \aleph_\alpha$,

$$\begin{aligned} \beta \in a &\leftrightarrow L \models \check{\beta} \in \pi(t') \\ &\leftrightarrow L \models \check{\beta} \in t' \\ &\leftrightarrow \beta \in D(t'). \end{aligned}$$

Hence $a = D(\hat{x}^{\gamma+1}(x \in \check{\aleph}_\alpha \wedge x \in t'))$, where $\gamma = \max(\aleph_\alpha, t') < \aleph_{\alpha+1}$. Thus

$$a \in L_{\aleph_{\alpha+1}}. \qquad \square$$

Definition 17.3. Let κ be a limit ordinal.

(1) C is closed in $\kappa \overset{\Delta}{\leftrightarrow} C \subseteq \kappa \wedge (\forall X \subseteq C)[\cup X < \kappa \rightarrow \cup X \in C]$.

(2) C is unbounded in $\kappa \leftrightarrow C \subseteq \kappa \wedge \cup C = \kappa$.

(3) S is stationary in κ iff $S \subseteq \kappa$ and $S \cap C \neq 0$ for all closed unbounded subsets C in κ.

Definition 17.4. The sequence $\langle S_\alpha : \alpha < \kappa \rangle$ is a $\diamond(\kappa)$-sequence iff

(1) $(\forall \alpha \in \kappa)[S_\alpha \subseteq \alpha]$, and

(2) $(\forall X \subseteq \kappa)[\{\alpha < \kappa \mid X \cap \alpha = S_\alpha\}$ is stationary in $\kappa]$.

Remark. We read $\diamond(\kappa)$ simply as "diamond kappa."

Definition 17.5. $\diamond(\kappa) \overset{\Delta}{\leftrightarrow}$ there exists a $\diamond(\kappa)$-sequence.

Lemma 17.6. *Let κ be a regular uncountable cardinal and let λ be an ordinal such that $\kappa < \lambda$. For each set X with $X \subseteq \lambda$ and $\overline{\overline{X}} < \kappa$, there exists a subalgebra Y of M^λ such that X is a subset of Y, $\overline{\overline{Y}} < \kappa$ and $Y \cap \kappa \in \kappa$.*

Proof. By induction on n, we define α_n and Y_n as follows:

$$Y_0 = X$$
$$\alpha_n = \sup\{\alpha + 1 \mid \alpha \in Y_n \cap \kappa\}$$
$$Y_{n+1} = M^\lambda(Y_n \cup \alpha_n).$$

Finally if $Y = \bigcup_{n \in \omega} Y_n$, then it is easy to verify that Y has the desired properties. $\square$

Theorem 17.7. *Assume that $V = L$. Then*

$$(\forall \kappa)[\kappa \textit{ is a regular cardinal} \wedge \kappa > \omega \rightarrow \diamond(\kappa)].$$

Proof. Let κ be an arbitrary regular uncountable cardinal. We define a sequence $\langle\langle S_\alpha, C_\alpha\rangle : \alpha < \kappa\rangle$ as follows:

(a) $S_0 = C_0 = 0$,

(b) $S_{\alpha+1} = C_{\alpha+1} = 0$,

(c) if α is a limit ordinal and there exists a pair $\langle S, C\rangle$ of subsets of α such that C is closed and unbounded in α and

$$(\forall \gamma \in C)[S \cap \gamma \neq S_\gamma],$$

then $\langle S_\alpha, C_\alpha\rangle$ is the $<_L$-least such pair $\langle S, C\rangle$, where $<_L$ is the canonical well ordering on L,

(d) otherwise, $S_\alpha = C_\alpha = 0$.

We claim $\langle S_\alpha : \alpha < \kappa \rangle$ is a $\diamondsuit(\kappa)$-sequence. If not, there exists a pair $\langle S, C \rangle$ of subsets of κ such that C is closed and unbounded in κ and $(\forall \gamma \in C)[S \cap \gamma \neq S_\gamma]$. We take the $<_L$-least such pair $\langle S, C \rangle$.

Let $X = \{\kappa, \kappa^+\}$ and let Y be a subalgebra of $M^{\kappa^{++}}$ such that $X \subseteq Y$, $\overline{\overline{Y}} < \kappa$ and $Y \cap \kappa \in \kappa$. If $\alpha_0 = Y \cap \kappa$, then α_0 is closed under J, and so α_0 is a limit ordinal.

Let $\pi : \eta' \to Y$ be the collapsing map of Y, then $\eta' < \kappa$ and $\pi : \eta' \to \kappa^{++}$ is a strong M-map.

By a similar proof to that of Theorem 17.2, it can be seen that $\langle \langle S_\alpha, C_\alpha \rangle : \alpha < \kappa \rangle \in L_{\kappa^+}$. There exists a formula $\varphi(x_0)$ of $\mathscr{L}$ with only the quantifier $\exists^{\kappa^+}$, such that $\varphi(x_0)$ defines $\langle \langle S_\alpha, C_\alpha \rangle : \alpha < \kappa \rangle$ and the only constant term occurring in $\varphi(x_0)$ is $\check{\kappa}$. Let $t = K(\langle \exists^{\kappa^+} x_0 \varphi(x_0) \rangle)$, then $D(t) = \langle \langle S_\alpha, C_\alpha \rangle : \alpha < \kappa \rangle$. Since Y is a subalgebra of $M^{\kappa^{++}}$ that contains κ and κ^+, it follows that $t \in Y$. Let t' be a term such that $t' < \eta'$ and $\pi(t') = t$.

We let

$$s_x = \hat{x}_1^{\kappa^+}[\exists^{\kappa^+} x_2 \, \exists^{\kappa^+} x_3 (\langle x, \langle x_2, x_3 \rangle \rangle \in t \wedge x_1 \in x_2)],$$

$$c_x = \hat{x}_1^{\kappa^+}[\exists^{\kappa^+} x_2 \, \exists^{\kappa^+} x_3 (\langle x, \langle x_2, x_3 \rangle \rangle \in t \wedge x_1 \in x_3)].$$

Then for each $\beta < \alpha_0$, we have that $s_{\check{\beta}}, c_{\check{\beta}} \in Y$, $D(s_{\check{\beta}}) = S_\beta$ and $D(c_{\check{\beta}}) = C_\beta$. Similarly, we define s'_x and c'_x by replacing each occurrence of κ^+ and t by $\pi^{-1}(\kappa^+)$ and t', respectively. We then see that

$$(\forall \beta < \alpha_0)[\pi(s'_{\check{\beta}}) = s_{\check{\beta}} \wedge \pi(c'_{\check{\beta}}) = c_{\check{\beta}}].$$

We shall show by induction on β that

(i) $D(s'_{\check{\beta}}) = S_\beta$ and $D(c'_{\check{\beta}}) = C_\beta$.

The only case we have to consider is that $\langle S_\beta, C_\beta \rangle$ is defined by (c). Let $\psi(x_1, x_2)$ be the following sentence of $\mathscr{L}$: $\langle x_1, x_2 \rangle$ is the $<_L$-least pair of subsets of $\check{\beta}$ such that x_1 is closed and unbounded in $\check{\beta}$ and $\forall^\beta x (x \in \check{\beta} \to x_1 \cap x \neq s'_x)$. Since $L \models \pi(\psi)(s_{\check{\beta}}, c_{\check{\beta}})$, it follows that $L \models \psi(s'_{\check{\beta}}, c'_{\check{\beta}})$. By the induction hypothesis, $D(s'_{\check{\gamma}}) = S_\gamma$ and $D(c'_{\check{\gamma}}) = C_\gamma$ for all $\gamma < \beta$. Hence $\langle D(s'_{\check{\beta}}), D(c'_{\check{\beta}}) \rangle$ satisfies (c), and thus $\langle D(s'_{\check{\beta}}), D(c'_{\check{\beta}}) \rangle = \langle S_\beta, C_\beta \rangle$.

Let s and c be terms of $\mathscr{L}$ such that $s, c \in Y$, $D(s) = S$ and $D(c) = C$. Their existence can be proved in the same way as above. Let s' and c' be such that $s', c' < \eta'$, $\pi(s') = s$ and $\pi(c') = c$. Since $\pi \restriction \alpha_0$ is the identity map and $\pi(\alpha_0) = \kappa$, we can easily show that

(ii) $D(s') = S \cap \alpha_0$ and $D(c') = C \cap \alpha_0$.

By the same proof as (i), we can show that

(iii) $D(s') = S_{\alpha_0}$, $D(c') = C_{\alpha_0}$ and $\langle S_{\alpha_0}, C_{\alpha_0} \rangle$ satisfies (c).

From (ii) and (iii), $S \cap \alpha_0 = S_{\alpha_0}$ and $C \cap \alpha_0 = C_{\alpha_0}$. But since C_{α_0} is unbounded in α_0, we have $\alpha_0 = \sup(C \cap \alpha_0) \in C$. This contradicts the assumption that $(\forall \gamma \in C)[S \cap \gamma \neq S_\gamma]$. $\square$

Definition 17.8. The ordered triple $\langle \delta, \alpha, P \rangle$ is an acceptable triple iff

(1) $\delta, \alpha \in \mathrm{On}$ and $\alpha \leqq \delta$,

(2) P is a finite subset of δ, and

(3) $\delta = M^\delta(\alpha \cup P)$.

Definition 17.9. The mapping $\pi: \langle \delta, \alpha, P \rangle \to \langle \delta', \alpha', P' \rangle$ is an acceptable map iff

(1) $\langle \delta, \alpha, P \rangle$ and $\langle \delta', \alpha', P' \rangle$ are acceptable,

(2) $\pi: \delta \to \delta'$ is a medium M-map.

(3) $\alpha \leqq \alpha'$, and

(4) $\pi \restriction \alpha$ is the identity map on α.

Definition 17.10. A transitive set a is elementarily equivalent to a transitive set b iff for every sentence φ of the language

$$\langle a, \varepsilon \rangle \models \varphi \leftrightarrow \langle b, \varepsilon \rangle \models \varphi.$$

Lemma 17.11. *Let M be a transitive set which is elementarily equivalent to L_κ for some uncountable cardinal in L. If $\pi: \langle \delta, \alpha, P \rangle \to \langle \delta', \alpha', P' \rangle$ is an acceptable map such that $\langle \delta, \alpha, P \rangle, \langle \delta', \alpha', P' \rangle \in M$, then $\pi \in M$.*

Proof. Note that each of F_0, J, C_i, T and K is absolute with respect to M. Let $F_1 = J$, $F_2 = T$, $F_3 = K$ and $F_{i+4} = C_i$ $(i < \omega)$. We define a sequence $\langle X_n: n < \omega \rangle \in M$ as follows:

$$X_0 = \alpha \cup P,$$
$$X_{n+1} = \bigcup_{i<\omega} \{F_i^\delta(\mathbf{v}): \mathbf{v} \in X_n^\omega\}.$$

Then $\delta = M^\delta(\alpha \cup P) = \bigcup_{n<\omega} X_n$. Let $P = \{\sigma_1, \ldots, \sigma_k\}$ and $\sigma_i' = \pi(\sigma_i)$. Then π is inductively defined in M as follows:

$$\pi(v) = v \quad \text{if } v < \alpha$$
$$\pi(\sigma_i) = \sigma_i' \qquad (i = 1, \ldots, k)$$
$$\pi(F_i^\delta(\mathbf{v})) = F_i(\bar{\pi}(\mathbf{v})) \quad \text{if } \mathbf{v} \in X_n^\omega,\ F_i^\delta(\mathbf{v}) \text{ is defined, and } F_i^\delta(\mathbf{v}) \notin \bigcup_{m \leqq n} X_m.$$

Thus we have $\pi \in M$. □

Definition 17.12. Let κ be a limit ordinal. Then $\Pi = \langle \langle \delta_i, \alpha_i, P_i \rangle, \pi_{ij} \rangle_{i, j \in I}$ is a κ-direct limit system iff

(1) $\langle I, \leqq \rangle$ is a partially ordered set such that $(\forall i, j \in I)(\exists k \in I)[i < k \wedge j < k]$,

(2) $\langle \delta_i, \pi_{ij} \rangle_{i \leqq j}$ is a direct system of medium M-maps,

(3) if $i < j$, then $\pi_{ij}: \langle \delta_i, \alpha_i, P_i \rangle \to \langle \delta_j, \alpha_j, P_j \rangle$ is an acceptable map such that $\sup\{\pi_{ij}(\nu) + 1 \mid \nu < \delta_i\} < \delta_j$,

(4) $\delta_i < \kappa$, and

(5) $\{\alpha_i \mid i \in I\}$ is cofinal in κ.

Remark. If Π is well founded, then it is easily seen that $\delta_\infty \geqq \kappa$ and $\sup\{\pi_{i\infty}(\nu) + 1 \mid \nu < \delta_i\} < \delta_\infty$ for all $i \in I$.

For every set X of ordinals, we use $o(X)$ to denote the order type of X.

Lemma 17.13. *Let μ and κ be limit ordinals such that $\mu \leqq \kappa$. If $(\forall \eta < \mu)$ $(\forall \alpha < \kappa)(\forall Q \subseteq \eta)$ $[Q$ is finite $\to o(M^\eta(\alpha \cup Q)) < \kappa]$, then there exists a well-founded κ-direct limit system whose limit is μ.*

Proof. Let $I = \{\langle \eta, \alpha, Q \rangle \mid \eta < \mu \wedge \alpha < \kappa \wedge \alpha \leqq \eta \wedge Q$ is a finite subset of $\eta\}$. For each $i = \langle \eta, \alpha, Q \rangle \in I$, we set $\eta_i = \eta$, $\alpha_i = \alpha$ and $Q_i = Q$. For any $i, j \in I$, we define $i < j$ by

$$i < j \overset{\triangle}{\leftrightarrow} \eta_i \leqq \eta_j \wedge \alpha_i \leqq \alpha_j \wedge Q_i \subseteq Q_j \wedge \eta_i \in Q_j,$$

Furthermore $i \leqq j \overset{\triangle}{\leftrightarrow} i < j \vee i = j$. It is easy to see that $(\forall i, j \in I)(\exists k \in I)$ $[i < k \wedge j < k]$. Let $X_i = M^{\eta_i}(\alpha_i \cup Q_i)$, then $o(X_i) < \kappa$ by our assumption. For each $i \in I$, let $\rho_i: \delta_i \to X_i$ be the collapsing map of X_i. Then $\delta_i < \kappa$ and $\rho_i: \delta_i \to \eta_i$ is a strong M-map. If $i, j \in I$ and $i < j$, then by Proposition 16.18, $\rho_j^{-1} \circ \rho_i$ is a medium M-map since $\mathscr{W}(\rho_i) = X_i \subseteq X_j = \mathscr{W}(\rho_j)$, and also we have that $\sup\{\rho_j^{-1} \circ \rho_i(\nu) + 1 \mid \nu < \delta_i\} < \delta_j$ since $\eta_i \in Q_j \subseteq X_j$. Let $\pi_{ij} = \rho_j^{-1} \circ \rho_i$ and $P_i = (\rho_i^{-1})``Q_i$. We want to show that $\Pi = \langle\langle \delta_i, \alpha_i, P_i \rangle, \pi_{ij} \rangle_{i, j \in I}$ is a well-founded $\kappa$-direct limit system with $\mu$ as its limit. By Theorem 16.19, $\rho_i``M^{\delta_i}(\alpha_i \cup P_i) = X_i$ and hence $\delta_i = M^{\delta_i}(\alpha_i \cup P_i)$. Consequently, each $\langle \delta_i, \alpha_i, P_i \rangle$ is an acceptable triple and $\pi_{ij}: \langle \delta_i, \alpha_i, P_i \rangle \to \langle \delta_j, \alpha_j, P_j \rangle$ is an acceptable map if $i < j$. It is clear that $\{\alpha_i \mid i \in I\}$ is cofinal with κ.

We need to prove that μ is the limit of Π. By the definition of π_{ij}, we have

$$\rho_j \circ \pi_{ij} = \rho_i \qquad (i \leqq j).$$

In view of Theorem 16.26, it suffices to show that $\mu = \bigcup_{i \in I} \mathscr{W}(\rho_i)$. Let $\eta < \mu$ and $i = \langle \eta + 1, 0, \{\eta\} \rangle$, then $\eta \in X_i = \mathscr{W}(\rho_i)$. This complete the proof. $\square$

Lemma 17.14. *Let κ be a limit ordinal, and let $\langle I_1, \leqq_1 \rangle$ and $\langle I_2, \leqq_2 \rangle$ be directed sets such that $I_1 \cap I_2 = 0$. If $\Pi_1 = \langle\langle \xi_i, \alpha_i, P_i \rangle, \rho_{ij} \rangle_{i, j \in I_1}$ and $\Pi_2 = \langle\langle \eta_i, \beta_i, Q_i \rangle, \theta_{ij} \rangle_{i, j \in I_2}$ are well-founded κ-direct limit systems with limits μ_1 and μ_2, and if $\mu_1 \leqq \mu_2$, then there exists a κ-direct limit system $\Pi = \langle\langle \zeta_i, \gamma_i, R_i \rangle, \pi_{ij} \rangle_{i, j \in I}$ such that*

(1) $I = I_1 \cup I_2$,

(2) $i, j \in I \wedge i \leqq j \rightarrow (i, j \in I_1 \wedge i \leqq_1 j) \vee (i, j \in I_2 \wedge i \leqq_2 j)$
$\vee (i \in I_1 \wedge j \in I_2)$,

(3) $\langle \zeta_i, \gamma_i, R_i \rangle = \begin{cases} \langle \xi_i, \alpha_i, P_i \rangle & \text{if } i \in I_1, \\ \langle \eta_i, \beta_i, Q_i \rangle & \text{if } i \in I_2, \end{cases}$

(4) $\pi_{ij} = \begin{cases} \rho_{ij} & \text{if } i, j \in I_1, \\ \theta_{ij} & \text{if } i, j \in I_2. \end{cases}$

PROOF. We need to define $i < j$ and π_{ij} when $i \in I_1$ and $j \in I_2$: If $i \in I_1$ and $j \in I_2$, then

$$i < j \overset{\Delta}{\leftrightarrow} \xi_i \leqq \eta_j \wedge \alpha_i \leqq \beta_j \wedge \rho_{i\infty}``P_i \subseteq \mathscr{W}(\theta_{j\infty}) \wedge \sup\{\rho_{i\infty}(\nu) + 1 \mid \nu < \xi_i\} \in \mathscr{W}(\theta_{j\infty}).$$

It is easily seen that

$$\mathscr{W}(\rho_{i\infty}) = \rho_{i\infty}``\xi_i = \rho_{i\infty}``M^{\xi_i}(\alpha_i \cup P_i) \subseteq \mathscr{W}(\varphi_{j\infty}).$$

Hence $\theta_{j\infty}^{-1} \circ \rho_{i\infty}$ is a medium M-map, by Proposition 16.18, and so we set $\pi_{ij} = \theta_{j\infty}^{-1} \circ \rho_{i\infty}$. It is necessary to show that $\langle I, \leqq \rangle$ is a directed set. To see this, it suffices to show that

$$(\forall i \in I_1)(\exists j \in I_2)(i < j).$$

Let $i \in I_1$. Since P_i is finite and $\sup\{\rho_{i\infty}(\nu) + 1 \mid \nu < \xi_i\} < \mu_1 \leqq \mu_2$, there exists a $j \in I_2$ such that $\xi_i \leqq \beta_j$, $\rho_{i\infty}``P_i \subseteq \mathscr{W}(\theta_{j\infty})$ and $\sup\{\rho_{i\infty}(\nu) + 1 \mid \nu < \xi_i\} \in \mathscr{W}(\theta_{j\infty})$. Then $i < j$. □

Definition 17.15. Let A and B be sets. A mapping $h: A \rightarrow B$ is an elementary embedding, iff for every formula $\varphi(x_1, \ldots, x_n)$ where $x_1, \ldots, x_n$ are the only free variables

$$\langle A, \varepsilon \rangle \models \varphi(a_1, \ldots, a_n) \rightarrow \langle B, \varepsilon \rangle \models \varphi(h(a_1), \ldots, h(a_n)).$$

Lemma 17.16. *Let κ be an infinite cardinal in L and $h: L_{\bar{\kappa}} \rightarrow L_\kappa$ be an elementary embedding. Let $\bar{\Pi} = \langle \langle \bar{\delta}_i, \bar{\alpha}_i, \bar{P}_i \rangle, \bar{\pi}_{ij} \rangle_{i,j \in I}$ be a well-founded $\bar{\kappa}$-direct limit system with limit $\bar{\mu}$. Also suppose that $\Pi = h(\bar{\Pi}) = \langle \langle h(\bar{\delta}_i), h(\bar{\alpha}_i), h(\bar{P}_i) \rangle, h(\bar{\pi}_{ij}) \rangle_{i,j \in I}$ is a well-founded κ-direct limit system with limit μ. Then there exists a medium M-map $h^*: \bar{\mu} \rightarrow \mu$ such that $h^* \upharpoonright \bar{\kappa} = h \upharpoonright \bar{\kappa}$. (By Lemma 17.11, $\bar{\pi}_{ij} \in L_{\bar{\kappa}}$ and hence $h(\bar{\pi}_{ij})$ is defined.)*

PROOF. Let $\delta_i = h(\bar{\delta}_i)$, $\alpha_i = h(\bar{\alpha}_i)$, $P_i = h(\bar{P}_i)$ and $\pi_{ij} = h(\bar{\pi}_{ij})$. For each $\bar{\sigma} < \bar{\delta}_i$, we let $h^*(\bar{\pi}_{i\infty}(\bar{\sigma})) = \pi_{i\infty}(h(\bar{\sigma}))$. First, we must show that h^* is well defined. Let $\bar{\sigma}$ and $\bar{\tau}$ be such that $\bar{\sigma} < \bar{\delta}$, $\bar{\tau} < \bar{\delta}$ and $\bar{\pi}_{i\infty}(\bar{\delta}) = \bar{\pi}_{j\infty}(\bar{\tau})$, then there is a $k \in I$ with $k \geqq i, j$ such that $\bar{\pi}_{ik}(\bar{\sigma}) = \bar{\pi}_{jk}(\bar{\tau})$, and hence $\pi_{ik}(h(\bar{\sigma})) = \pi_{jk}(h(\bar{\tau}))$. Thus we have $\pi_{i\infty}(h(\bar{\sigma})) = \pi_{j\infty}(h(\bar{\tau}))$ if $\bar{\pi}_{i\infty}(\bar{\sigma}) = \bar{\pi}_{j\infty}(\bar{\tau})$.

Since each $\bar{\pi}_{i\infty}(\pi_{i\infty})$ is the identity on $\bar{\alpha}_i(\alpha_i)$ and $\{\bar{\alpha}_i \mid i \in I\}$ is cofinal with $\bar{\kappa}$, we have that $h^*(\bar{\sigma}) = h(\bar{\sigma})$ for all $\bar{\sigma} < \bar{\kappa}$. Consequently, h^* is an extension of $h \upharpoonright \bar{\kappa}$.

Let $\mu' = \sup\{h^*(\bar{\sigma}) + 1 \mid \bar{\sigma} < \bar{\mu}\}$. We want to show that $h^*: \bar{\mu} \to \mu'$ is a strong M-map. Let F be one of F_0, J, C_i, T and K, and $\bar{\sigma}, \bar{\tau} < \bar{\mu}$ be such that $F(\bar{\sigma}) = \bar{\tau}$. Then

$$(\exists\, i \in I)(\exists\, \bar{\sigma}', \bar{\tau}' < \bar{\delta}_i)[\bar{\pi}_{i\infty}(\bar{\sigma}') = \bar{\sigma} \wedge \bar{\pi}_{i\infty}(\bar{\tau}') = \bar{\tau}].$$

Since $\bar{\pi}_{i\infty}: \bar{\delta}_i \to \bar{\mu}$ is a medium M-map,

$$F(\bar{\sigma}') = \bar{\tau}'.$$

But since h is elementary,

$$F(h(\bar{\sigma}')) = h(\bar{\tau}').$$

Hence

$$\pi_{i\infty}(F(h(\bar{\sigma}'))) = \pi_{i\infty}(h(\bar{\tau}')).$$

Since $\pi_{i\infty}: \delta_i \to \mu$ is a medium M-map,

$$F(\pi_{i\infty}(h(\bar{\sigma}'))) = \pi_{i\infty}(h(\bar{\tau}')).$$

And hence,

$$F(h^*(\bar{\sigma})) = h^*(\bar{\tau}).$$

Thus we have

$$F^{\bar{\mu}}(\bar{\sigma}) \simeq \bar{\tau} \to F^{\mu'}(h^*(\bar{\sigma})) \simeq h^*(\bar{\tau}).$$

Let $\bar{\sigma}$ and τ be such that $\bar{\sigma} < \bar{\mu}$, $\tau < \mu'$ and $F(h^*(\bar{\sigma})) = \tau$. Then

$$(\exists\, i \in I)(\exists\, \bar{\sigma}', \bar{\tau}' < \bar{\delta}_i)[\bar{\pi}_{i\infty}(\bar{\sigma}') = \bar{\sigma} \wedge \tau \leqq h^*(\bar{\pi}_{i\infty}(\bar{\tau}'))].$$

Then $\tau < \sup\{\pi_{i\infty}(v) + 1 \mid v < \delta_i\} = \delta_i'$, and hence $F^{\delta_i'}(\pi_{i\infty}(h(\bar{\sigma}')))$ is defined. Since $\pi_{i\infty}: \delta_i \to \delta_i'$ is a strong M-map, $F^{\delta_i}(h(\bar{\sigma}'))$ is defined. It is easy to see that $h \restriction \bar{\delta}_i : \bar{\delta}_i \to \delta_i$ is a strong M-map because h is elementary. Hence, $F^{\bar{\delta}_i}(\bar{\sigma}')$ is defined, and therefore $F^{\bar{\mu}}(\bar{\sigma})$ is defined. Thus

$$h^*(F^{\bar{\mu}}(\bar{\sigma})) \simeq F^{\mu'}(h^*(\bar{\sigma})) \quad \text{for all } \bar{\sigma} < \bar{\mu}. \qquad \square$$

Definition 17.17. $S \overset{\triangle}{\leftrightarrow} (\forall x \subseteqq \mathrm{On})[\bar{\bar{x}} > \omega \to (\exists\, y \in L)\,(x \subseteqq y \wedge \bar{\bar{x}} = \bar{\bar{y}})]$.

Definition 17.18. $2^{\underline{\lambda}} \triangleq \sup_{\mu<\lambda} \overline{\overline{2^{\mu}}}$.

Theorem 17.19. *Assume S.*

(1) *If λ is a regular cardinal in L and $\lambda \geqq \aleph_2$, then* $\mathrm{cf}(\lambda) = \bar{\bar{\lambda}}$.

(2) *If λ is a singular cardinal, then λ is also a singular cardinal in L.*

(3) *For every singular cardinal λ, $\lambda^+ = (\lambda^+)^L$.*

(4) *If λ is a singular cardinal and if τ is a cardinal such that* $\mathrm{cf}(\lambda) \leqq \tau < \lambda$, *then* $\overline{\overline{\lambda^{\tau}}} = \max(\overline{\overline{2^{\tau}}}, \lambda^+)$.

(5) *For each singular cardinal* λ,

$$\overline{\overline{2^\lambda}} = \begin{cases} 2^{\underset{=}{\lambda}} & \textit{if } (\exists\, \tau < \lambda)(\overline{\overline{2^\tau}} = 2^{\underset{=}{\lambda}}), \\ (2^{\underset{=}{\lambda}})^+ & \textit{otherwise.} \end{cases}$$

PROOF. (1) Let $x \subseteqq \lambda$ be such that $\bar{\bar{x}} = \mathrm{cf}(\lambda)$ and $\cup x = \lambda$. Then by S there exists a $y \in L$ such that $x \subseteqq y \subseteqq \lambda$ and $\bar{\bar{y}} = \max(\bar{\bar{x}}, \aleph_1)$. Since $\cup y = \lambda$ and λ is regular in L, we see that $\bar{\bar{y}}^L = \lambda$. Hence we have $\bar{\bar{y}} = \bar{\bar{\lambda}} \geqq \aleph_2$. Thus, $\mathrm{cf}(\lambda) = \bar{\bar{x}} = \bar{\bar{y}} = \bar{\bar{\lambda}}$.

(2) Note that $\lambda \geqq \aleph_\omega > \aleph_2$. By (1), λ is singular in L.

(3) Obviously $(\lambda^+)^L \leqq \lambda^+$. Suppose $(\lambda^+)^L < \lambda^+$. From (1) $\mathrm{cf}((\lambda^+)^L) = \overline{\overline{(\lambda^+)^L}} = \lambda$. Hence λ is a regular cardinal.

(4) By the König lemma,

$$(\forall \tau)[\mathrm{cf}(\lambda) \leqq \tau < \lambda \to \lambda < \overline{\overline{\lambda^{\mathrm{cf}(\lambda)}}} \leqq \overline{\overline{\lambda^\tau}}].$$

Therefore,

$$(\forall \tau)[\mathrm{cf}(\lambda) \leqq \tau < \lambda \to \overline{\overline{\lambda^\tau}} \geqq \max(\overline{\overline{2^\tau}}, \lambda^+)].$$

Let τ be an arbitrary cardinal such that $\mathrm{cf}(\lambda) \leqq \tau < \lambda$. Since

$$\lambda^\tau = \cup \{a^\tau \mid a \subseteqq \lambda \wedge \bar{\bar{a}} = \tau\},$$

we have

$$\overline{\overline{\lambda^\tau}} = \max(\overline{\overline{\{a \subseteqq \lambda \mid \bar{\bar{a}} = \tau\}}}, \overline{\overline{\tau^\tau}}) = \max(\overline{\overline{\{a \subseteqq \lambda \mid \bar{\bar{a}} = \tau\}}}, \overline{\overline{2^\tau}}).$$

Case 1. $\tau > \omega$: By S,

$$a \subseteqq \lambda \wedge \bar{\bar{a}} = \tau \to (\exists\, x \in L)[a \subseteqq x \subseteqq \lambda \wedge \bar{\bar{x}} = \tau].$$

Let $X = \{x \in L \mid x \subseteqq \lambda \wedge \bar{\bar{x}} = \tau\}$. Then

$$\bar{\bar{X}} \leqq \overline{\overline{\mathscr{P}^L(\lambda)}} \leqq (\overline{\overline{2^\lambda}})^L = (\lambda^+)^L = \lambda^+ \text{ by (3).}$$

Therefore,

$$\overline{\overline{\lambda^\tau}} \leqq \max\left(\overline{\overline{\bigcup_{x \in X} \{a \subseteqq x \mid \bar{\bar{a}} = \tau\}}}, \overline{\overline{2^\tau}}\right) \leqq \max(\overline{\overline{2^\tau}}, \lambda^+).$$

Case 2. $\tau = \omega$: By S, we see that

$$a \subseteqq \lambda \wedge \bar{\bar{a}} = \aleph_0 \to (\exists\, x \in L)[a \subseteqq x \subseteqq \lambda \wedge \bar{\bar{x}} = \aleph_1].$$

Let $Y = \{x \in L \mid x \subseteqq \lambda \wedge \bar{\bar{x}} = \aleph_1\}$. Then,

$$\begin{aligned} \overline{\overline{\lambda^\tau}} &\leqq \max\left(\overline{\overline{\bigcup_{x \in Y} \{a \subseteqq x \mid \bar{\bar{a}} = \aleph_0\}}}, \overline{\overline{2^\tau}}\right) \\ &\leqq \max(\lambda^+, \overline{\overline{\aleph_1^{\aleph_0}}}, \overline{\overline{2^{\aleph_0}}}) = \max(\overline{\overline{2^{\aleph_0}}}, \lambda^+). \end{aligned}$$

(5) Let $\kappa = \mathrm{cf}(\lambda)$ and let λ_ν for $\nu < \kappa$ be such that $\lambda_\nu < \lambda$ and $\lambda = \bigcup_{\nu<\kappa} \lambda_\nu$. Then

$$\overline{\overline{2^\lambda}} = \overline{\overline{2^{\bigcup_{\nu<\kappa}\lambda_\nu}}} = \overline{\overline{\prod_{\nu<\kappa} 2^{\lambda_\nu}}} \leqq \overline{\overline{\prod_{\nu<\kappa} 2^{\underset{\smile}{\lambda}}}} = \overline{\overline{(2^{\underset{\smile}{\lambda}})^\kappa}}.$$

If $\overline{\overline{2^\tau}} = 2^{\underset{\smile}{\lambda}}$ for some $\tau < \lambda$, such that $\kappa \leqq \tau$, then

$$\overline{\overline{2^\lambda}} \leqq \overline{\overline{(2^\lambda)^\kappa}} = \overline{\overline{2^{\tau\times\kappa}}} \leqq 2^{\underset{\smile}{\lambda}}.$$

and hence $\overline{\overline{2^\lambda}} = 2^{\underset{\smile}{\lambda}}$. Suppose $(\forall\, \tau < \lambda)(\overline{\overline{2^\tau}} < 2^{\underset{\smile}{\lambda}})$. Then by (4) we have

$$\overline{\overline{(2^{\underset{\smile}{\lambda}})^\kappa}} = \max(\overline{\overline{2^\kappa}}, (2^{\underset{\smile}{\lambda}})^+) = (2^{\underset{\smile}{\lambda}})^+.$$

Therefore, $\overline{\overline{2^\lambda}} \leqq (2^{\underset{\smile}{\lambda}})^+$. On the other hand, $\overline{\overline{(2^{\underset{\smile}{\lambda}})^\kappa}} \leqq \overline{\overline{(2^\lambda)^\kappa}} \leqq \overline{\overline{2^\lambda}}$. ☐

Definition 17.20. $\mathrm{I} \overset{\Delta}{\leftrightarrow} (\forall\, \bar{\mu})(\forall\, \pi)$ [If $\bar{\mu}$ is a regular cardinal $\wedge$ $\pi\colon \bar{\mu} \to \mathrm{On}$ is a medium M-map, then π is the identity map on $\bar{\mu}$].

Remark. Our next objective is to prove the following theorem.

Theorem 17.21. $\mathrm{I} \to S$.

Remark. To prove Theorem 17.21 we will prove a sequence of lemmas that will enable us to prove the contrapositive $\neg\, S \to \neg\, \mathrm{I}$. So from this point until we complete the proof of the contrapositive of Theorem 17.21 we assume $\neg\, S$. That is, we assume

$$(\exists\, X \subseteq \mathrm{On})[\overline{\overline{X}} > \omega \wedge (\forall\, Y \in L)(\overline{\overline{X}} = \overline{\overline{Y}} \to X \nsubseteq Y)].$$

Let $\kappa = \min\{\lambda \in \mathrm{On} \mid (\exists\, X \subseteq \lambda)[\overline{\overline{X}} > \omega \wedge (\forall\, Y \in L)(\overline{\overline{X}} = \overline{\overline{Y}} \to X \nsubseteq Y)]\}$, and let X be a subset of κ such that

$$(*) \quad \overline{\overline{X}} > \omega \wedge (\forall\, Y \in L)(\overline{\overline{X}} = \overline{\overline{Y}} \to X \nsubseteq Y).$$

Lemma 17.22.

(1) $(\forall\, Y \in L)(X \subseteq Y \to \overline{\overline{Y}}^L \geqq \kappa)$.

(2) $L \models [\kappa$ *is a cardinal*$]$.

(3) $\cup X = \kappa$.

(4) $\overline{\overline{X}} < \bar{\kappa}$.

Proof. (1) Assume

$$(\exists\, Y \in L)(X \subseteq Y \wedge \overline{\overline{Y}}^L < \kappa),$$

and let $\lambda = \overline{\overline{Y}}^L$. Then

$$(\exists\, f \in L)(f\colon \lambda \xrightarrow[\text{onto}]{1-1} Y)$$

Let $X' = (f^{-1})``X$, then it is easily seen that $X' \subseteq \lambda$ and X' satisfies (*). This contradicts the minimality of κ.

(2) As in (1) there is no one-to-one-onto mapping in L from some $\lambda < \kappa$ to κ.

(3) and (4) are trivial. ☐

Lemma 17.23. *There exists an elementary embedding* $h: L_{\bar\kappa} \to L_\kappa$ *such that*

(1) $X \subseteq \mathscr{W}(h)$ *and* $\overline{\overline{X}} = \overline{\overline{\mathscr{W}(h)}}$;

(2) *if* $\bar\Pi = \langle\langle\bar\delta_i, \bar\alpha_i, \bar P_i\rangle, \bar\pi_{ij}\rangle_{i,j\in I}$ *is a well-founded* $\bar\kappa$*-direct limit system, then* $h(\bar\Pi) = \langle\langle h(\bar\delta_i), h(\bar\alpha_i), h(\bar P_i)\rangle, h(\bar\pi_{ij})\rangle_{i,j\in I}$ *is a well-founded* κ*-direct limit system.*

Remark. We shall prove this lemma later.

Proof of Theorem 17.21. Let $h: L_{\bar\kappa} \to L_\kappa$ be as in Lemma 17.23. In view of (3) and (4) of Lemma 17.22, $h \upharpoonright \bar\kappa$ is not the identity. Let $\bar\mu$ be an arbitrary limit ordinal with $\bar\mu \geq \bar\kappa$.

Claim: $(\forall \bar\eta < \bar\mu)(\forall \bar\alpha < \bar\kappa)(\forall \bar Q \subseteq \bar\eta)$ $[\bar Q$ is finite $\to o(M^{\bar\eta}(\bar\alpha \cup \bar Q)) < \bar\kappa]$. From this claim and Lemma 17.13, there exists a well-founded $\bar\kappa$-direct limit system $\bar\Pi$ with $\bar\mu$ as its limit. By Lemma 17.23 $h(\bar\Pi)$ is also well founded and hence by Lemma 17.16, there exists a medium M-map, $h^*: \bar\mu \to \mathrm{On}$, such that $h^* \upharpoonright \bar\kappa = h \upharpoonright \bar\kappa$. Then h^* is not the identity.

If our claim were false, then there would be an $\bar\eta < \bar\mu$ such that

$$(\exists\, \bar\alpha < \bar\kappa)(\exists\, \bar Q \subseteq \bar\eta)[\bar Q \text{ is finite} \wedge o(M^{\bar\eta}(\bar\alpha \cup \bar Q)) \geq \bar\kappa].$$

Take the least such $\bar\eta$ and let $\bar\alpha < \bar\kappa$, $\bar Q \subseteq \bar\eta$ be such that

$$\bar Q \text{ is finite} \wedge o(M^{\bar\eta}(\bar\alpha \cup \bar Q)) \geq \bar\kappa.$$

Then by Lemma 16.22, $\bar\eta$ is a limit ordinal that is greater than or equal to $\bar\kappa$.

Let $\pi: \tilde\eta \to M^{\bar\eta}(\bar\alpha \cup \bar Q)$ be the collapsing map of $M^{\bar\eta}(\bar\alpha \cup \bar Q)$. Then $\bar\kappa \leq \tilde\eta \leq \bar\eta$ and $\pi: \tilde\eta \to \bar\eta$ is a strong M-map. If $\bar P = (\pi^{-1})``Q$, then by Theorem 16.19, $\pi``M^{\tilde\eta}(\bar\alpha \cup \bar P) = M^{\bar\eta}(\bar\alpha \cup \bar Q)$. Hence $\tilde\eta = M^{\tilde\eta}(\bar\alpha \cup \bar P)$. But by the minimality of $\bar\eta$, we have $\tilde\eta = \bar\eta$. Thus we have shown that

$$(\exists\, \bar P \subseteq \bar\eta)[\bar P \text{ is finite} \wedge \bar\eta = M^{\bar\eta}(\bar\alpha \cup \bar P)].$$

Since $\bar\eta$ satisfies the hypothesis of Lemma 17.13, there exists a medium M-map, $h^*: \bar\eta \to \mathrm{On}$ such that $h^* \upharpoonright \bar\kappa = h \upharpoonright \bar\kappa$. Let $\eta = \sup\{h^*(\bar\sigma) + 1 \mid \bar\alpha < \bar\eta\}$. Then $h^*: \bar\eta \to \eta$ is a strong M-map. If $Y = M^{\bar\eta}(h(\bar\alpha) \cup h^*``\bar P)$, then $Y \in L$ and from Lemma 17.22.2

$$\overline{\overline{Y}}^L = \overline{\overline{h(\bar\alpha)}}^L < \kappa,$$

because $h(\bar\alpha) < \kappa$. We shall show that $X \subseteq Y$. Since $h^*: \bar\eta \to \eta$ is a strong M-map, we have by Theorem 16.19 that

$$\mathscr{W}(h^*) = h^*``M^{\bar\eta}(\bar\alpha \cup \bar P) = M^\eta((h^*``\bar\alpha) \cup (h^*``\bar P)) \subseteq Y.$$

But $X \subseteq \mathscr{W}(h^*)$, since $X \subseteq \mathscr{W}(h)$. Thus $X \subseteq Y$. This contradicts Lemma 17.22.1, and the proof is completed. □

Remark. To prove Lemma 17.23 we need a result that requires the following definition.

Definition 17.24. Let $\Pi = \langle\langle\delta_i, \alpha_i, P_i\rangle, \pi_{ij}\rangle_{i,j\in I}$ be a κ-direct limit system, and $Z \subseteqq L_\kappa$.

(1) $\Pi \subseteqq Z \overset{\Delta}{\leftrightarrow} (\forall i \in I)[\delta_i, \alpha_i, P_i \in Z] \wedge (\forall i, j \in I)[i \leqq j \to \pi_{ij} \in Z]$.

(2) Π is Z-well-founded iff there are no sequences $\langle i_n : n < \omega\rangle$ and $\langle \sigma_n : n < \omega\rangle$ such that $i_n \in I$, $\sigma_n \in Z \cap \delta_{i_n}$ and $(\forall n < \omega)[i_n \leqq i_{n+1} \wedge \pi_{i_n i_{n+1}}(\sigma_n) > \sigma_{n+1}]$.

Remark. Let $h: L_{\bar\kappa} \to L_\kappa$ be an elementary embedding and $\bar\Pi$ be a $\bar\kappa$-direct limit system. Also let $Z = \mathscr{W}(h)$ and $\Pi = h(\bar\Pi)$. Then $\bar\Pi$ is well founded iff Π is Z-well-founded.

Definition 17.25. Let A be a subset of a set B. Then A is an elementary submodel of B iff the embedding map of A into B is an elementary embedding. We write $A \prec B$ which is read "A is an elementary submodel of B."

Exercise

Let Z be an elementary submodel of L_κ. Then there exists a $\bar\kappa$ and an elementary embedding $h: L_{\bar\kappa} \to L_\kappa$ such that $\mathscr{W}(h) = Z$.

Lemma 17.26. *Let Z be such that $Z \prec L_\kappa$, $X \subseteqq Z$ and $\overline{\overline{X}} = \overline{\overline{Z}}$. Then there exists a Z' an elementary submodel of L_κ such that*

(1) $Z \subseteqq Z'$, $\overline{\overline{Z'}} = \overline{\overline{X}}$, *and*

(2) *if $\Pi \subseteqq Z$ is a κ-direct limit system which is not well founded, then Π is not Z'-well-founded.*

Proof. Let $h: L_{\bar\kappa} \to L_\kappa$ be an elementary embedding such that $\mathscr{W}(h) = Z$. Suppose there is a well-founded κ-direct limit system $\bar\Pi$ such that $h(\bar\Pi)$ is not well founded. Let $\bar\mu$ be the least ordinal that is the limit of a well-founded $\bar\kappa$-direct limit system $\bar\Pi_1 = \langle\langle\bar\xi_i, \bar\alpha_i, \bar P_i\rangle, \bar P_{ij}\rangle_{i,j\in I_1}$ such that $h(\bar\Pi_1) = \langle\langle\xi_i, \alpha_i, P_i\rangle, \rho_{ij}\rangle_{i,j\in I_1}$ is not well founded. Then there are sequences $\langle i_n : n \in \omega\rangle$ and $\langle\sigma_n : n \in \omega\rangle$ such that

$$(\forall n \in \omega)[i_n \in I_1 \wedge \sigma_n < \delta_{i_n}]$$

and

$$(\forall n \in \omega)[i_n \leqq i_{n+1} \wedge \rho_{i_n i_{n+1}}(\sigma_n) > \sigma_{n+1}].$$

Take $Z' \prec L_\kappa$ such that $Z \cup \{\sigma_n : n \in \omega\} \subseteqq Z'$ and $\overline{\overline{Z'}} = \overline{\overline{X}}$. Let

$$\Pi_2 = \langle\langle\eta_i, \beta_i, Q_i\rangle, \theta_{ij}\rangle_{i,j\in I_2} \subseteqq Z$$

be an arbitrary κ-direct limit system which is not well founded. We want to show that Π_2 is not Z'-well-founded. We may assume that Π_2 is Z-well-founded. Then there exists a well-founded $\bar\kappa$-direct limit system $\bar\Pi_2 = \langle\langle\bar\eta_i, \bar\beta_i, \bar Q_i\rangle, \bar\theta_{ij}\rangle_{i,j\in I_2}$ such that $h(\bar\Pi_2) = \Pi_2$. By the minimality of $\bar\mu$ the limit of $\bar{\Pi}_2$ is greater than or equal to $\bar{\mu}$. Let $\bar{\Pi} = \langle\langle\bar{\xi}_i, \bar{\gamma}_i, \bar{R}_i\rangle, \bar{\pi}_{ij}\rangle_{i,j\in I}$ be a $\bar{\kappa}$-direct limit system which satisfies (1)–(3) of Lemma 17.14 (clearly we may assume $I_1 \cap I_2 = 0$). Then there exists a sequence $\langle j_n : n \in \omega\rangle$ of elements of I_2 such that $i_n \leqq j_n$ for all $n \in \omega$. Let $\sigma_n' = \pi_{i_n j_n}(\sigma_n)$, where $\pi_{ij} = h(\bar{\pi}_{ij})$. Since $\pi_{i_n j_n} \in Z$ and $\sigma_n \in Z'$, we have $\{\sigma_n' : n \in \omega\} \subseteqq Z'$. Also

$$\begin{aligned}\theta_{j_n j_{n+1}}(\sigma_n') &= \pi_{j_n j_{n+1}}(\pi_{i_n j_n}(\sigma_n))\\ &= \pi_{i_n j_{n+1}}(\sigma_n)\\ &= \pi_{i_{n+1} j_{n+1}}(\pi_{i_n i_{n+1}}(\sigma_n))\\ &> \pi_{i_{n+1} j_{n+1}}(\sigma_{n+1}) = \sigma_{n+1}'.\end{aligned}$$

Thus Π_2 is not Z'-well-founded. □

Proof of Lemma 17.23 ($\mathrm{cf}(\kappa) = \omega$ case). We want to show that there exists Z, an elementary submodel of L_κ, such that

(1) $X \subseteqq Z, \bar{\bar{Z}} = \bar{\bar{X}}$; and

(2) If $\Pi \subseteqq Z$ is a Z-well-founded κ-direct limit system, then Π is well founded.

First we shall prove that (2) can be replaced by the following (2′):

(2′) If $\Pi \subseteqq Z$ is a countable Z-well-founded κ-direct limit system, then Π is well founded, where Π is countable iff its index set is countable.

Let Z be an elementary submodel of L_κ that satisfies (1) and (2′). Let $\Pi = \langle\langle\delta_i, \alpha_i, P_i\rangle, \pi_{ij}\rangle_{i,j\in I} \subseteqq Z$ be a Z-well-founded κ-direct limit system. Suppose that Π is not well founded. Then there are sequences $\langle i_n : n \in \omega\rangle$ and $\langle\sigma_n : n \in \omega\rangle$ such that $(\forall n \in \omega)[i_n \in I \wedge \sigma_n < \delta_{i_n}]$ and

$$(\forall n \in \omega)[i_n \leqq i_{n+1} \wedge \pi_{i_n i_{n+1}}(\sigma_n) > \sigma_{n+1}].$$

Since $\{\alpha_i \mid i \in I\}$ is cofinal with κ and $\mathrm{cf}(\kappa) = \omega$, there is a sequence $\langle j_n : n \in \omega\rangle$ such that $\{\alpha_{j_n} \mid n \in \omega\}$ is cofinal with κ. Let J be a subset of I such that

(i) $\{i_n : n \in \omega\} \cup \{j_n : n \in \omega\} \subseteqq J$;

(ii) $\bar{\bar{J}} = \omega$;

and

(iii) $\langle J, \leqq \cap J^2\rangle$ is directed.

Then $\Pi \restriction J = \langle\langle\delta_i, \alpha_i, P_i\rangle, \pi_{ij}\rangle_{i,j\in J}$ is a countable κ-direct limit system that is a subset of Z and which is not well founded. Clearly $\Pi \restriction J$ is Z-well-founded. This contradicts (2′). Thus Π is well founded.

Using Lemma 17.26, we can construct a sequence $\langle Z_\alpha : \alpha < \aleph_1\rangle$ such that

(a) $Z_\alpha \prec L_\kappa$, $X \subseteqq Z_\alpha$ and $\bar{\bar{Z}}_\alpha = \bar{\bar{X}}$;

(b) $\alpha < \beta \rightarrow Z_\alpha \subseteqq Z_\beta$; and

(c) if $\Pi \subseteqq Z_\alpha$ is a κ-direct limit system which is not well founded, then Π is not $Z_{\alpha+1}$-well-founded.

Let $Z = \bigcup_{\alpha<\aleph_1} Z_\alpha$. Then $Z \prec L_\kappa$, $X \subseteqq Z$ and $\overline{\overline{Z}} = \overline{\overline{X}}$. We want to show that (2′) holds for this Z. Let $\Pi \subseteqq Z$ be a countable Z-well-founded κ-direct limit system. Then $\Pi \subseteqq Z_\alpha$ for some $\alpha < \aleph_1$. If Π were not well founded, then Π would not be $Z_{\alpha+1}$-well-founded, and hence not Z-well-founded. This is a contradiction. Thus the lemma has been proved in the case where $\mathrm{cf}(\kappa) = \omega$. □

From now on, we assume that $\mathrm{cf}(\kappa) > \omega$. From (3) of Lemma 17.22, we see that X is unbounded in κ. We may assume that X is closed in κ, (if necessary, consider the closure of X in place of X).

Exercise

We denote the closure of X by $C(X)$. Prove $\overline{\overline{X}} = \overline{\overline{C(X)}}$.

Let $S = \{\lambda \in X \mid \mathrm{cf}(\lambda) = \omega\}$.

Lemma 17.27. *Let Z be an elementary submodel of L_κ such that $X \subseteqq Z$ and $\overline{\overline{Z}} = \overline{\overline{X}}$. Then there exists a Z', an elementary submodel of L_κ such that*

(1) *$Z \subseteqq Z'$, $\overline{\overline{Z'}} = \overline{\overline{X}}$; and*

(2) *for any $\lambda \in S$, if $\Pi \subseteqq Z$ is a λ-direct limit system which is not well founded, then Π is not Z'-well-founded.*

Proof. Let $h: L_{\bar{\kappa}} \to L_{\bar{\kappa}}$ be an elementary embedding such that $\mathscr{W}(h) = Z$. For any $\lambda \in S$, by the same proof as that of Lemma 17.26, we can find a countable set A_λ such that $A_\lambda \subseteqq \lambda$ and if $Z \cup A_\lambda \subseteqq Z' \prec L_\kappa$, then Z' satisfies (2) for λ. Let Z' be an elementary submodel of L_κ such that $Z \cup \bigcup_{\lambda\in S} A_\lambda \subseteqq Z'$ and $\overline{\overline{Z'}} = \overline{\overline{X}}$. Then Z' satisfies (1) and (2). □

Corollary 17.28. *There exists a Z, an elementary submodel of L_κ such that*

(1) *$X \subseteqq Z$, $\overline{\overline{Z}} = \overline{\overline{X}}$; and*

(2) *for any $\lambda \in S$, if $\Pi \subseteqq Z$ is a countable λ-direct limit system which is not well founded, then Π is not Z-well-founded.*

Proof. We can construct a sequence $\langle Z_\alpha : \alpha < \aleph_1 \rangle$ such that

(a) $Z_\alpha \prec L_\kappa$, $X \subseteqq Z_\alpha$ and $\overline{\overline{Z_\alpha}} = \overline{\overline{X}}$;

(b) $\alpha < \beta \to Z_\alpha \subseteqq Z_\beta$;

(c) for any $\lambda \in S$, if $\Pi \subseteqq Z_\alpha$ is a λ-direct limit system which is not well founded, then Π is not $Z_{\alpha+1}$-well-founded.

Then $Z = \bigcup_{\alpha<\aleph_1} Z_\alpha$ has the desired property. □

Proof of Lemma 17.23. ($\mathrm{cf}(\kappa) > \omega$ case).

Let Z be as in Corollary 17.28. It is sufficient to prove that if $\Pi \subseteqq Z$ is a κ-direct limit system which is not well founded, then Π is not Z-well-founded.

Let $\Pi = \langle\langle\delta_i, \alpha_i, P_i\rangle, \pi_{ij}\rangle_{i,j\in I} \subseteqq Z$ be a κ-direct limit system which is not well founded. Then there exist sequences $\langle i_n : n \in \omega\rangle$ and $\langle \sigma_n : n \in \omega\rangle$ such that $(\forall n \in \omega)\,[i_n \in I \wedge \sigma_n < \delta_{i_n}]$ and

$$(\forall n \in \omega)[i_n \leqq i_{n+1} \wedge \pi_{i_n i_{n+1}}(\sigma_n) > \sigma_{n+1}].$$

Claim: There exists a sequence $\langle j_n : n \in \omega\rangle$ such that

(i) $(\forall n \in \omega)[j_n \in I \wedge i_n \leqq j_n \leqq j_{n+1}]$;

(ii) $\lambda \triangleq \sup_{n<\omega} \delta_{j_n} \in S$;

(iii) $\Pi' = \Pi \restriction \{j_n : n \in \omega\}$ is a countable λ-direct limit system.

We let $\sigma'_n = \pi_{i_n j_n}(\sigma_n)$. Then

$$\begin{aligned}\pi_{j_n j_{n+1}}(\sigma'_n) &= \pi_{j_n j_{n+1}}(\pi_{i_n j_n}(\sigma_n))\\ &= \pi_{i_n j_{n+1}}(\sigma_n)\\ &= \pi_{i_{n+1} j_{n+1}}(\pi_{i_n i_{n+1}}(\sigma_n))\\ &> \pi_{i_{n+1} j_{n+1}}(\sigma_{n+1}) = \sigma'_{n+1}.\end{aligned}$$

Hence Π' is not well founded, and hence not Z-well-founded. Thus Π is not Z-well-founded.

We shall now prove the claim. By recursion on n, we define $\gamma_n \in X$ and $j_n \in I$. We choose $\gamma_0 \in X$ arbitrarily. Since $\{\alpha_j : j \in I\}$ is cofinal with κ, there exists a $j_0 \in I$ such that $i_0 \leqq j_0$ and $\gamma_0 < \alpha_{j_0}$. Suppose that γ_n and j_n are already defined. Since X is unbounded in κ, there exists a $\gamma_{n+1} \in X$ such that $\delta_{j_n} < \gamma_{n+1}$. Let $j_{n+1} \in I$ be such that $i_{n+1} \leqq j_{n+1}, j_n \leqq j_{n+1}$ and $\gamma_{n+1} < \alpha_{j_{n+1}}$. Then (i) is clear.

(ii) Let $\lambda = \sup_{n\in\omega} \delta_{j_n}$. By definition, we have

$$\lambda = \sup_{n\in\omega} \gamma_n = \sup_{n\in\omega} \alpha_{j_n}.$$

Since $\mathrm{cf}(\kappa) > \omega$ and X is closed in κ, we see that $\lambda \in S$.

(iii) It suffices to show that $\{\alpha_{j_n} | n \in \omega\}$ is cofinal with λ. But this is clear from the above. □

Definition 17.29. The mapping π: On $\to$ On is a strong M-map iff π is a monomorphism from M to itself.

Definition 17.30. (1) A formula φ is bounded iff all the quantifiers occurring in φ are of the form $\exists\, x \in y$.

(2) A formula φ is Σ_1 iff it is of the form $\exists\, x\psi$, where ψ is a bounded formula.

(3) A mapping $h: L \to L$ is a Σ_1 elementary embedding iff for any Σ_1 formula $\varphi(x_1, \ldots, x_n)$ and any $a_1, \ldots, a_n \in L$,

$$L \models \varphi(a_1, \ldots, a_n) \qquad \text{iff } L \models \varphi(h(a_1), \ldots, h(a_n)).$$

Lemma 17.31. *If $\pi\colon \mathrm{On} \to \mathrm{On}$ is a strong M-map, then there exists a Σ_1 elementary embedding $h\colon L \to L$ such that $\pi = h \upharpoonright \mathrm{On}$.*

Proof. For any term t of $\mathscr{L}$, we let $h(D(t)) = D(\pi(t))$. Then h is well defined and for all $\alpha \in \mathrm{On}$, $h(\alpha) = h(D(\check{\alpha})) = D(\pi(\check{\alpha})) = D(\widecheck{\pi(\alpha)}) = \pi(\alpha)$.

Let $\psi(x_0, \ldots, x_n)$ be a bounded formula and let $t_0, \ldots, t_n \in T_\alpha$. Let $\psi^\alpha(x_0, \ldots, x_n)$ be the formula of $\mathscr{L}$ obtained from ψ by replacing each quantifier $\exists\, x \in y(\ldots)$ in ψ by $\exists^\alpha\, x(x \in y \wedge \ldots)$. Then

$$\begin{aligned} L \models \psi(D(t_0), \ldots, D(t_n)) &\leftrightarrow T(\langle \psi^\alpha(t_0, \ldots, t_n)\rangle) = 1 \\ &\leftrightarrow T(\langle \psi^{\pi(\alpha)}(\pi(t_0), \ldots, \pi(t_n))\rangle) = 1 \\ &\leftrightarrow L \models \psi(D(\pi(t_0)), \ldots, D(\pi(t_n))). \end{aligned}$$

Let $\varphi(x_1, \ldots, x_n)$ be $(\exists\, x_0)\psi(x_0, x_1, \ldots, x_n)$ and let $t_1, \ldots, t_n$ be terms of $\mathscr{L}$. Then,

$$\begin{aligned} &L \models \varphi(D(t_1), \ldots, D(t_n)) \\ &\quad\to (\exists\, \alpha)(\exists\, t_0)(t_0, t_1, \ldots, t_n \in T_\alpha \wedge L \models \psi(D(t_0), \ldots, D(t_n))) \\ &\quad\to L \models (\exists\, \alpha)(\exists\, t_0)(t_0, t_1, \ldots, t_n \in T_\alpha \wedge \psi(D(\pi(t_0)), \ldots, D(\pi(t_n))) \\ &\quad\to L \models \varphi(D(\pi(t_1)), \ldots, D(\pi(t_n))). \end{aligned}$$

We have to show the converse. Suppose that

$$L \models \varphi(D(\pi(t_1)), \ldots, D(\pi(t_n))).$$

Then

$$(\exists\, \beta)(\exists\, a \in L_\beta)[\pi(t_1), \ldots, \pi(t_n) \in T_\beta \wedge L \models \psi(a, D(\pi(t_1)), \ldots, D(\pi(t_n)))].$$

Take $\alpha \in \mathrm{On}$ such that $\pi(\alpha) \geqq \beta$. Then

$$L \models \exists^{\pi(\alpha)}\, x_0\, \psi^{\pi(\alpha)}(x_0, \pi(t_1), \ldots, \pi(t_n)).$$

Since $\pi\colon \mathrm{On} \to \mathrm{On}$ is a strong M-map, we have

$$L \models \exists^\alpha x_0\, \psi^\alpha(x_0, t_1, \ldots, t_n).$$

Thus

$$L \models \varphi(D(t_1), \ldots, D(t_n)). \qquad \square$$

Theorem 17.32. *Assume $\neg S$. Then there exists a Σ_1 elementary embedding from L into itself which is not the identity.*

Proof. Let $h\colon L_{\bar{\kappa}} \to L_\kappa$ be as in Lemma 17.23. We shall show that there exists a strong M-map $h^*\colon \mathrm{On} \to \mathrm{On}$ such that $h^* \upharpoonright \bar{\kappa} = h \upharpoonright \bar{\kappa}$. Then, by Lemma 17.31, there is a Σ_1 elementary embedding from L into L which extends h^* and hence is not the identity.

Let $I = \{\langle \bar{\eta}, \bar{\alpha}, \bar{Q}\rangle \mid \bar{\eta} \in \mathrm{On} \wedge \bar{\alpha} < \bar{\kappa} \wedge \bar{\alpha} \leqq \bar{\eta} \wedge \bar{Q}$ is a finite subset of $\bar{\eta}\}$. In the proof of Theorem 17.21, we showed that if $\langle \bar{\eta}, \bar{\alpha}, \bar{Q}\rangle \in I$, then

$o(M^{\bar\eta}(\bar\alpha \cup \bar Q)) < \bar\kappa$. If $i = \langle \bar\eta, \bar\alpha, \bar Q\rangle$, then we set $\bar\eta_i = \bar\eta$, $\bar\alpha_i = \bar\alpha$ and $\bar Q_i = \bar Q$. For any, $i, j \in I$, we define $i < j$ and $i \leqq j$ by

$$i < j \overset{\triangle}{\leftrightarrow} \bar\eta_i \leqq \bar\eta_j \wedge \bar\alpha_i \leqq \bar\alpha_j \wedge \bar Q_i \subseteqq \bar Q_j \wedge \eta_i \in Q_j,$$

$$i \leqq j \overset{\triangle}{\leftrightarrow} i < j \vee i = j.$$

Then $\langle I, \leqq\rangle$ is directed, but note that I is a proper class in the present situation. For each $i \in I$, let $X_i = M^{\bar\eta_i}(\bar\alpha_i \cup \bar Q_i)$ and $\bar\rho_i: \bar\delta_i \to X_i$ be the collapsing map of X_i. Also let $\bar P_i = (\bar\rho_i^{-1})``\bar Q_i$. When $i \leqq j$, we set $\bar\pi_{ij} = \bar\rho_j^{-1} \circ \bar\rho_i$. Then $\bar\pi = \langle\langle\bar\delta_i, \bar\alpha_i, \bar P_i\rangle, \bar\pi_{ij}\rangle_{i,j\in I}$ is a $\bar\kappa$-direct limit system whose limit is On.

If we put $\Pi = h(\bar\Pi)$, then Π is a κ-direct limit system such that $\varinjlim \Pi$ is linearly ordered and has no infinite descending sequence. We define $h^*: \mathrm{On} \to \varinjlim \Pi$ by

$$h^*(\bar\pi_{i\infty}(\bar\sigma)) = \pi_{i\infty}(h(\bar\sigma))$$

for each $\bar\sigma < \bar\delta_i$. In Lemma 17.16, we saw that h^* is well defined and is an extension of $h \restriction \bar\kappa$. Obviously, h^* is order preserving. We claim that

(1) if $x \in \varinjlim \Pi$, then $\{y \in \varinjlim \Pi \mid y < x\}$ is a set.

By (1) $\varinjlim \Pi$ is well ordered and hence is order isomorphic to On. Thus we may assume that $\varinjlim \Pi = \mathrm{On}$. As is known from Lemma 17.16, h^* is a strong M-map from On into itself.

Now we shall prove (1). Let $\bar\lambda$ be an arbitrary regular cardinal such that $\bar\lambda \geqq \bar\kappa$ and $A = \{x \in \varinjlim \Pi \mid x < h^*(\bar\lambda)\}$. We must show that A is a set. To see this, it suffices to show that for any $i \in I$ there is a $j \in I$ such that $\bar\eta_j < \bar\lambda$ and $\mathscr{W}(\pi_{i\infty}) \cap A \subseteqq \mathscr{W}(\pi_{j\infty})$, because this implies that

$$A = \bigcup_{i\in I} \mathscr{W}(\pi_{i\infty}) \cap A \subseteqq \cup\{\mathscr{W}(\pi_{j\infty}) \mid j \in I \wedge \bar\eta_j < \bar\lambda\}.$$

Let $i \in I$. We may assume that $\bar\eta_i \geqq \bar\lambda$. Since $\bar\lambda$ is regular and $\bar\kappa \leqq \bar\lambda$, there exists an $\bar\eta < \bar\lambda$ such that $\mathscr{W}(\bar\pi_{i\infty}) \cap \bar\lambda \subseteqq \bar\eta$. Let $X = M^{\bar\eta_i}(\bar\eta \cup \bar\alpha_i \cup \bar Q_i)$ and let $\rho: \bar\xi \to X$ be the collapsing map of X. If $j = \langle \bar\xi, \bar\alpha_i, (\rho^{-1})``\bar Q_i\rangle$, then obviously, $j \in I$ and $\bar\eta_j = \xi < \bar\lambda$. We want to show that $\mathscr{W}(\pi_{i\infty}) \cap A \subseteqq \mathscr{W}(\pi_{j\infty})$. Since $\mathscr{W}(\bar\pi_{i\infty}) \cap \bar\lambda = \mathscr{W}(\bar\pi_{i\infty}) \cap \bar\eta = \mathscr{W}(\pi_{j\infty}) \cap \bar\eta$, it follows that $\mathscr{W}(\bar\pi_{i\infty}) \cap \bar\lambda \subseteqq \mathscr{W}(\bar\pi_{j\infty})$. We choose $k \in I$ and $\bar\mu < \bar\delta_k$ so that $\bar\pi_{k\infty}(\bar\mu) = \bar\lambda$. For any $l \in I$ with $i, j, k \leqq l$, it is easy to see that $\mathscr{W}(\bar\pi_{il}) \cap \bar\pi_{kl}(\bar\mu) \subseteqq \mathscr{W}(\bar\pi_{jl})$. Therefore, in $L_{\bar\kappa}$

$$(\forall x)[x \in \mathscr{W}(\bar\pi_{il}) \wedge x < \bar\pi_{kl}(\bar\mu) \to x \in \mathscr{W}(\bar\pi_{jl})].$$

And so in L_κ,

$$(\forall x)[x \in \mathscr{W}(\pi_{il}) \wedge x < \pi_{kl}(h(\bar\mu)) \to x \in \mathscr{W}(\pi_{jl})].$$

Thus

$$(\forall x)[x \in \mathscr{W}(\pi_{i\infty}) \wedge x < \pi_{k\infty}(h(\bar\mu)) \to x \in \mathscr{W}(\pi_{j\infty})].$$

This means that $\mathscr{W}(\pi_{i\infty}) \cap A \subseteqq \mathscr{W}(\pi_{j\infty})$, and we have completed the proof of the theorem. $\square$

CHAPTER 18

Introduction to Forcing

In proving that AC and GCH are consistent with ZF, Gödel used the so called method of internal models. From the assumption that the universe V is a model of ZF Gödel prescribed a method for producing a submodel L that is also a model of $V = L$, AC and GCH. This submodel is defined as the class of all sets having a certain property, i.e.,

$$L = \{x \mid (\exists\, \alpha)[x = F`\alpha]\}.$$

Indeed since $x = F`\alpha$ is absolute w.r.t. every standard transitive model M it follows that if

$$L^M = \{x \mid (\exists\, \alpha \in M)[x = F`\alpha]\}$$

then L^M is a submodel of M that is also a model of $V = L$.

If $V = L$ is valid in every model then $V = L$ must be provable in ZF and conversely if $V = L$ is not provable in ZF then $V = L$ is not valid in some model. Can we hope to find such a model by the method of internal models? That is, can we hope to produce a property $\varphi(a)$ such that

$$\{x \mid \varphi(x)\}$$

is a model of ZF + $V \neq L$? There are compelling reasons for believing that this method cannot succeed. The arguments turn upon the assumption that there is a set that is a standard model of ZF.

Theorem 18.1. *If there exists a set that is a standard model of ZF then there exists one and only one set M_0 such that*

(1) *M_0 is a countable standard transitive model of* ZF + $V = L$ *and*

(2) *M_0 is a submodel of every standard transitive model of* ZF.

Proof. From Mostowski's theorem (Theorem 12.8) every standard model is $\in$-isomorphic to a standard transitive model. Therefore the existence of a set that is a standard model of ZF implies the existence of a set that is a standard transitive model. For transitive models the property of being an ordinal is absolute. That is, those sets in a transitive model that play the role of ordinals are ordinals. Furthermore, from transitivity, if α is in the model, then all smaller ordinals are in the model. But a standard transitive model that is a set cannot contain all ordinals.

If α is the smallest ordinal not contained in such a model then α is the class of ordinals for that model. But the existence of such an ordinal implies the existence of a smallest such ordinal, α_0, that is the set of all ordinals in some standard transitive model N_0.

Since N_0 is a model of ZF it follows that if

$$M_0 \triangleq L^{N_0} \doteq \{x \mid (\exists\, \alpha \in N_0)[x = F`\alpha]\} = \{F`\alpha \mid \alpha < \alpha_0\}$$

then M_0 is a model of ZF + $V = L$.

If N is any standard transitive model of ZF then N is closed w.r.t. the fundamental operations. Therefore since $\alpha_0 \subseteq N$ it follows that $M_0 \subseteq N$. From this we see that M_0 is unique, for if M_0 and M' are each standard transitive models with the prescribed properties then

$$M_0 \subseteq M' \quad \text{and} \quad M' \subseteq M_0.$$

Finally, from the Lowenheim–Skolem theorem, M_0 contains a countable standard submodel. But this submodel must be $\in$-isomorphic to a countable standard transitive model that must contain M_0 as a submodel. Hence M_0 is countable. □

Remark. The unique model M_0 described in Theorem 18.1 is called the *minimal model.* Its existence follows from the existence of a set that is a standard model. It should be observed that the minimal model contains no proper transitive submodel. Thus the existence of a model of ZF does not imply the existence of a standard model. (See Appendix.) We clearly cannot prove the existence of a standard model, for that would prove the consistency of ZF. We therefore postulate the existence of such a model

Standard Model Hypothesis: $(\exists\, m)\, \mathrm{SM}\,(m, \mathrm{ZF})$.

From this assumption and Theorem 18.1 we are assured of the existence of the minimal model M_0 that is (1) countable, hence contains a countable collection of ordinals α_0, (2) a model of ZF + $V = L$, and (3) a submodel of every standard transitive model. Indeed from Mostowski's theorem every standard model contains a submodel $\in$-isomorphic to M_0.

From the existence of the minimal model it follows that an attempt to prove $V = L$ and ZF independent by the method of internal models is doomed to failure. Suppose that we could produce a wff $\varphi(a)$ for which it is provable in ZF that

$$\{x \mid \varphi(x)\}$$

is a model of ZF + $V \neq L$. It then follows that this theorem relativized to M_0 is also a theorem. That is

$$\{x \in M_0 \mid \varphi^{M_0}(x)\}$$

is a submodel of M_0 that is also a model of ZF + $V \neq L$. Since M_0 is a model of $V = L$ this submodel is a proper submodel of M_0. But such a submodel of M_0 must be isomorphic to a standard transitive proper submodel of M_0 that must in turn contain M_0 as a submodel. This is impossible.

The independence of $V = L$ must then be established by some method other than that of internal models. Cohen's approach is to extend the minimal model M_0 in the following way. Since M_0 is countable and $\mathscr{P}(\omega)$ is not countable, there exists a set a such that $a \subseteq \omega$ and $a \notin M_0$. We adjoin such a set a to M_0 to obtain $M_0 \cup \{a\}$ and we define $M_0[a]$ to be the result of closing $M_0 \cup \{a\}$ under the eight fundamental operations. Can we select a set a so that $M_0[a]$ is a model of ZF? We will show that we can. Moreover we will show that a can be selected so that the ordinals in $M_0[a]$ are precisely the ordinals in M_0. It then follows that

$$L^{M_0[a]} = \{x \mid (\exists\, \alpha \in M_0)[x = F\text{`}\alpha]\} = M_0.$$

This tells us that in the universe $M_0[a]$ the class of constructible sets is M_0. Since $a \in M_0[a]$ but $a \notin M_0$ it follows that a is not constructible relative to the universe $M_0[a]$. That is, $M_0[a]$ is a model of $V \neq L$.

To prove the existence of a set a that is not in M_0 and for which $M_0[a]$ is a model of ZF we will develop a general theory for adjoining a set G to any countable standard transitive model of ZF. If M is such a model of ZF then the result of adjoining G to M we denote by $M[G]$. However in order for $M[G]$ to be a model of ZF with new properties, the set G must be especially selected. Let us first describe the special properties of G that we require.

Definition 18.2. A structure $\mathscr{P} = \langle P, \leqq \rangle$ is said to be a partially-ordered structure if for every $a, b, c \in P$, the following conditions are satisfied.

(1) $a \leqq b \wedge b \leqq a \leftrightarrow a = b.$

(2) $a \leqq b \wedge b \leqq c \rightarrow a \leqq c.$

Remark. For the remainder of the section let M be a countable transitive model of ZF, and let $\mathscr{P} = \langle P, \leqq \rangle$ be a partially ordered structure in M, i.e., $\mathscr{P} \in M$.

Definition 18.3. A subset D of P (in M) is dense in P iff

$$(\forall\, p \in P)(\exists\, q \in D)(q \leqq p).$$

Remark. Let $[p] = \{q \in P \mid q \leqq p\}$. Let $\{[p] \mid p \in P\}$ be a base of open sets for a topology on P. Then D is dense in P iff D is dense in the sense of the topology, that is, $\bar{D} = P$, where $\bar{D}$ is the closure of D.

Definition 18.4. A subset G of P (outside M) is said to be $\mathscr{P}$-generic if

(1) $(\forall p, q \in G)(\exists r \in G)(r \leqq p \wedge r \leqq q)$,

(2) $(\forall p \in G)(\forall q \in P)[p \leqq q \rightarrow q \in G]$,

(3) $\forall D \in M$ with D dense in P, $D \cap G \neq 0$.

Lemma 1. *For every $p_0 \in P$, there exists a $\mathscr{P}$-generic set G such that $p_0 \in G$.*

Proof. Since M is countable, we can enumerate all dense subsets of P in M, say $D_1, D_2, \ldots$. We then define p_{n+1} by introduction on n such that

$$p_{n+1} \in D_{n+1} \wedge p_{n+1} \leqq p_n.$$

Let $G = \{q \in P \mid (\exists n)[p_n \leqq q]\}$. It is then obvious that G is $\mathscr{P}$-generic. □

Remark. We now introduce a ramified language, $\mathscr{L}(M, \mathbf{G})$, to give a notation for each member of the universe $M[G]$ which we are going to construct, that is, we first give the names of sets and later we will construct them. The symbols of $\mathscr{L}(M, \mathbf{G})$ are the following.

Variables: $x_0, x_1, \ldots, x_n, \ldots$ $(n \in \omega)$.
Symbols for special objects: $\mathbf{k}$ for every $k \in M$.
A symbol for a special set P: $\mathbf{P}$.
A symbol for a set G which will be defined: $\mathbf{G}$.
A relation symbol: $\in$.
Propositional connectives: $\neg, \wedge, \vee$.
Quantifiers: $\exists^\alpha$ $(\alpha \in M)$.
Abstraction operators: $\hat{}^\alpha$ $(\alpha \in M)$.
Parentheses: (,).

Definition 18.5.

(1) $g(\exists^\alpha) \triangleq 2\alpha + 1$,

(2) $g(\hat{}^\alpha) \triangleq 2\alpha + 2$,

(3) $g(\mathbf{k}) \triangleq 2\,\mathrm{rank}(k) + 3$

(4) $g(\mathbf{G}) \triangleq g(\mathbf{P}) \triangleq 2\,\mathrm{rank}(P) + 3$,

and for any finite sequence s of symbols of $\mathscr{L}(M, \mathbf{G})$

(5) $g(s)$ is the maximum of $g(\exists^\alpha)$, $g(\hat{}^\alpha)$, $g(\mathbf{k})$, $g(\mathbf{G})$, and $g(\mathbf{P})$, for all $\exists^\alpha$, $\hat{}^\alpha$, $\mathbf{k}$, $\mathbf{G}$, and $\mathbf{P}$ which occur in s.

Definition 18.6. Limited formulas and constant terms of $\mathscr{L}(M, \mathbf{G})$ are defined as follows.

(1) $\mathbf{k}$, $\mathbf{G}$ and $\mathbf{P}$ are constant terms.

(2) If each of t_1 and t_2 is a constant term or a variable, then $(t_1 \in t_2)$ is a limited formula of $\mathscr{L}(M, \mathbf{G})$.

(3) If φ and ψ are limited formulas, then $(\neg\varphi)$, $(\varphi \vee \psi)$, and $(\varphi \wedge \psi)$ are limited formulas.

(4) If φ is a limited formula, then $(\exists^\alpha x_i \varphi)$ is a limited formula.

(5) If $\varphi(x_i)$ is a limited formula without free variables other than x_i such that $g(\varphi(x_i)) < g(^{\wedge\alpha})$, then $(\hat{x}_i^\alpha \varphi(x_i))$ is a constant term. This constant term is called an abstraction term.

(6) Limited formulas and terms are only those obtained by a finite number of applications of (1)–(5).

Definition 18.7. $\rho(\hat{x}_i^\alpha \varphi(x_i)) \triangleq \alpha$, $\rho(\mathbf{k}) \triangleq \operatorname{rank}(k)$ and $\rho(\mathbf{G}) \triangleq \operatorname{rank}(P) \triangleq \rho(\mathbf{P})$.

Definition 18.8. $T_\alpha \triangleq \{t \mid t$ is a constant term, and $\rho(t) < \alpha\}$ and $T \triangleq \bigcup_{\alpha \in M} T_\alpha$.

Remark. We code the symbols of $\mathscr{L}(M, \mathbf{G})$ by members of M.

Definition 18.9.

(1) $\ulcorner \in \urcorner \triangleq \langle 0, 0\rangle$,

(2) $\ulcorner \neg \urcorner \triangleq \langle 0, 1\rangle$,

(3) $\ulcorner \wedge \urcorner \triangleq \langle 0, 2\rangle$,

(4) $\ulcorner \vee \urcorner \triangleq \langle 0, 3\rangle$,

(5) $\ulcorner (\urcorner \triangleq \langle 0, 4\rangle$,

(6) $\ulcorner) \urcorner \triangleq \langle 0, 5\rangle$,

(7) $\ulcorner \mathbf{G} \urcorner \triangleq \langle 0, 6\rangle$,

(8) $\ulcorner \mathbf{P} \urcorner \triangleq \langle 0, 7\rangle$,

(9) $\ulcorner x_i \urcorner \triangleq \langle 0, 9 + i\rangle$,

(10) $\ulcorner \exists^\alpha \urcorner \triangleq \langle 0, \omega + \alpha\rangle$,

(11) $\ulcorner {}^{\wedge\alpha} \urcorner \triangleq \langle 0, \omega + \alpha, \omega + \alpha\rangle$,

and

(12) $\ulcorner \mathbf{k} \urcorner = \langle 1, k\rangle$.

For any finite sequence $s_1, \ldots, s_n$ of symbols of $\mathscr{L}(M, \mathbf{G})$,

(13) $\ulcorner s_1, \ldots, s_n \urcorner \triangleq \langle 2, \ulcorner s_1 \urcorner, \ldots, \ulcorner s_n \urcorner\rangle$.

Remark. The codes of limited formulas or constant terms are members of M and the codes of different formulas and terms are different. From now on, we identify formulas and terms of $\mathscr{L}(M, \mathbf{G})$ with their codes. Then $T_\alpha \in M$ for every $\alpha \in M$ and T is a definable class of M, that is, there exists a formula $\varphi(x)$ such that $T = \{x \in M \mid M \models \varphi(x)\}$.

By transfinite induction on the ranks of their code, we assign the ordinals in M to limited formulas of $\mathscr{L}(M, \mathbf{G})$.

Definition 18.10. Let φ be a limited formula.

(1) $\mathrm{Ord}^1(\varphi)$ is the number of occurrences of the form $t_1 \in t_2$ in φ, where t_1 and t_2 are constant terms and $g(\varphi) = g(t_1)$.

(2) $\mathrm{Ord}^2(\varphi)$ is the number of logical symbols in φ which are not contained in any constant term in φ.

(3) $\mathrm{Ord}(\varphi) \triangleq \omega^2 g(\varphi) + \omega \cdot \mathrm{Ord}^1(\varphi) + \mathrm{Ord}^2(\varphi) + 1.$

Remark. $\mathrm{Ord}(\varphi)$ is called the ordinal of φ. It should be remarked that the ordinals of different formulas may be the same. Since $\mathrm{Ord}(\varphi)$ is defined by transfinite induction, $\mathrm{Ord}(\varphi)$ is definable in M, that is, there exists a formula $\psi(x, y)$ such that $\alpha = \mathrm{Ord}(\varphi)$ iff $M \models \psi(\ulcorner\varphi\urcorner, \alpha)$.

For a constant term t,

$$\mathrm{Ord}(t) = \omega^2 \cdot g(t).$$

Proposition 18.11.

(1) $t \in T_\alpha \to \mathrm{Ord}(\varphi(t)) < \mathrm{Ord}(\exists^\alpha x_i\, \varphi(x_i))$.

(2) $\mathrm{Ord}(\varphi) < \mathrm{Ord}(\neg\varphi)$.

(3) $\mathrm{Ord}(\varphi) < \mathrm{Ord}(\varphi \wedge \psi)$ *and* $\mathrm{Ord}(\psi) < \mathrm{Ord}(\varphi \wedge \psi)$.

(4) $t \in T_\alpha \to \mathrm{Ord}(\varphi(t)) < \mathrm{Ord}(\hat{x}_i^\alpha \varphi(x_i))$.

The proof is left to the reader.

Definition 18.12. Let G be a subset of P which is possibly outside of M. We define D^G as follows.

(1) $D^G(\hat{x}_i^\alpha \varphi(x_i)) \triangleq \{D^G(t) \mid t \in T_\alpha \wedge D^G(\varphi(t))\}$.

(2) $D^G(\neg\varphi) \overset{\triangle}{\leftrightarrow} \neg D^G(\varphi)$.

(3) $D^G(\varphi \wedge \psi) \overset{\triangle}{\leftrightarrow} D^G(\varphi) \wedge D^G(\psi)$.

(4) $D^G(\varphi \vee \psi) \overset{\triangle}{\leftrightarrow} D^G(\varphi) \vee D^G(\psi)$.

(5) $D^G(\exists^\alpha x_i\, \varphi(x_i)) \overset{\triangle}{\leftrightarrow} (\exists t \in T_\alpha) D^G(\varphi(t))$.

(6) $D^G(t_1 \in t_2) \overset{\triangle}{\leftrightarrow} D^G(t_1) \in D^G(t_2)$.

(7) $D^G(\mathbf{k}) \triangleq k$.

(8) $D^G(\mathbf{G}) \triangleq G$.

(9) $D^G(\mathbf{P}) \triangleq P$.

(10) $M[G] \triangleq \{D^G(t) \mid t \in T\}$.

Remark. It should be noted that D^G is well defined for all limited sentences and constant terms in $\mathscr{L}(M, \mathbf{G})$. The proof is by transfinite induction on their ordinals.

Proposition 18.13. *$M[G]$ is transitive and the ordinals in $M[G]$ are the ordinals in M.*

The proof is left to the reader.

Definition 18.14. For every limited sentence φ in $\mathscr{L}(M, \mathbf{G})$,

$$M[G] \models \varphi \overset{\Delta}{\leftrightarrow} D^G(\varphi).$$

Definition 18.15. For a certain type of atomic limited sentence φ of $\mathscr{L}(M, \mathbf{G})$, a limited sentence $E(\varphi)$ of $\mathscr{L}(M, \mathbf{G})$ is defined as follows.

(1) If φ is of the form $t \in \hat{x}_i^\alpha \varphi(x_i)$ and $g(t) < g(\hat{x}_i^\alpha \varphi(x_i))$, then

$$E(\varphi) \overset{\Delta}{\leftrightarrow} \varphi(t).$$

(2) If φ is of the form $t_1 \in t_2$ and $g(t_2) \leqq g(t_1)$, then

$$E(\varphi) \overset{\Delta}{\leftrightarrow} \exists^\alpha x_i(x_i \in t_2 \wedge \forall^\alpha x_j(x_i \in t_1 \leftrightarrow x_j \in x_i)),$$

where $\alpha = \rho(t_1)$ and x_i and x_j are the first and the second variables not occurring in t_1 or t_2 respectively.

Exercises

(1) If $E(\varphi)$ is defined, then

$$D^G(E(\varphi)) \leftrightarrow D^G(\varphi).$$

(2) $D^G(\mathbf{k}_1 \in \mathbf{k}_2) \leftrightarrow k_1 \in k_2$.

Proposition 18.16. *Let φ be an atomic sentence for which $E(\varphi)$ is defined. Then*

$$\operatorname{Ord}(E(\varphi)) < \operatorname{Ord}(\varphi).$$

Definition 18.17. Now we extend the language $\mathscr{L}(M, \mathbf{G})$ by introducing

a relation symbol: $\mathbf{M}$, and

quantifiers: $\exists$, $\forall$.

Definition 18.18. A formula in $\mathscr{L}(M, \mathbf{G})$ is defined as follows.

(1) Any limited formula of $\mathscr{L}(M, \mathbf{G})$ is a formula of $\mathscr{L}(M, \mathbf{G})$.

(2) If t is a constant term or a variable, then $\mathbf{M}(t)$ is a formula.

(3) If φ and ψ are formulas, then $\neg\varphi$, $[\varphi \vee \psi]$, and $[\varphi \wedge \psi]$ are formulas.

(4) If $\varphi(x_i)$ is a formula, then $(\exists x_i)\varphi(x_i)$ and $(\forall x_i)\psi(x_i)$ are formulas.

A formula which is not a limited formula is called an unlimited formula.

We defined $a = b$ to be an abbreviation of $\forall x(x \in a \leftrightarrow x \in b)$. Therefore $a = b$ is an unlimited formula.

We code unlimited formulas in the following way: $\ulcorner \mathbf{M} \urcorner = \langle 0, 8 \rangle$, $\ulcorner \exists \urcorner = \langle 3, 0 \rangle$, and $\ulcorner \forall \urcorner = \langle 3, 1 \rangle$. The quantifiers $\exists$ and $\forall$ are called unlimited quantifiers.

Remark. The intended interpretation of $\mathbf{M}(t)$ is "$D^G(t)$ is a member of M."

Lemma 18.19. *Let $\varphi(x_1, \ldots, x_n)$ be a first-order formula without any occurrence of $\exists^\alpha$, or $\hat{x}^\alpha$ and let $t_1, \ldots, t_n \in T$. Then*

$$D^G(\varphi(t_1, \ldots, t_n)) \quad \textit{iff } M[G] \models \varphi(D^G(t_1), \ldots, D^G(t_n)).$$

The proof is left to the reader.

CHAPTER 19
Forcing

Remark. We define $p \Vdash \varphi$, read p forces φ, by transfinite induction on Ord (φ).

Definition 19.1. Let $p \in P$ and let φ be a limited sentence in $\mathscr{L}(M, \mathbf{G})$. Then

(1) $p \Vdash \neg \varphi \overset{\triangle}{\leftrightarrow} (\forall q \leqq p) \neg (q \Vdash \varphi)$.

(2) $p \Vdash [\varphi_1 \wedge \varphi_2] \overset{\triangle}{\leftrightarrow} [p \Vdash \varphi_1] \wedge [p \Vdash \varphi_2]$.

$p \Vdash [\varphi_1 \vee \varphi_2] \overset{\triangle}{\leftrightarrow} [p \Vdash \varphi_1] \vee [p \Vdash \varphi_2]$.

(3) $p \Vdash \exists^\alpha x_i \varphi(x_i) \overset{\triangle}{\leftrightarrow} (\exists t \in T_\alpha)[p \Vdash \varphi(t)]$.

(4) If φ is not of the form $\mathbf{k}_1 \in \mathbf{k}_2$ and $E(\varphi)$ is defined for φ, then

$$p \Vdash \varphi \overset{\triangle}{\leftrightarrow} p \Vdash E(\varphi).$$

(5) If $E(t \in \mathbf{G})$ is not defined, then

$$p \Vdash [t \in \mathbf{G}] \overset{\triangle}{\leftrightarrow} (\exists q \geqq p)[p \Vdash \forall^\alpha x_i (x_i \in t \leftrightarrow x_i \in \mathbf{q})]$$

where $\alpha = \rho(\mathbf{G})$.

(6) $p \Vdash [\mathbf{k}_1 \in \mathbf{k}_2] \overset{\triangle}{\leftrightarrow} k_1 \in k_2$.

(7) If t is not of the form $\mathbf{k}'$ and $E(t \in \mathbf{k})$ is not defined, then

$$p \Vdash [t \in \mathbf{k}] \overset{\triangle}{\leftrightarrow} (\exists k_1 \in k)[p \Vdash \forall^\alpha x_i (x_i \in t \leftrightarrow x_i \in \mathbf{k}_1)],$$

where $\alpha = \rho(k)$ and x_i is the first variable not occurring in t.

Remark. Since $p \Vdash \varphi$ is defined by transfinite induction on Ord(φ), $p \Vdash \varphi$ is definable in M, that is, there exists a formula $\varphi(x, y)$ such that $p \Vdash \varphi$ iff $M \models \varphi(\mathbf{p}, \ulcorner\varphi\urcorner)$.

Now we extend forcing to unlimited formulas by adding the following.

Definition 19.2.

(1) For $t \in T_\alpha$, $p \Vdash \mathbf{M}(t) \overset{\triangle}{\leftrightarrow} p \Vdash [t \in R^M(\alpha + 1)]$ where $R^M(\alpha + 1) = M \cap R(\alpha + 1)$.

(2) $p \Vdash \exists x_i \varphi(x_i) \overset{\triangle}{\leftrightarrow} (\exists t \in T)[p \Vdash \varphi(t)]$.

(3) $p \Vdash \forall x_i \varphi(x_i) \overset{\triangle}{\leftrightarrow} (\forall t \in T)[p \Vdash \varphi(t)]$.

Remark. Let F_n be the set of sentences in which the number of unlimited quantifiers is less than or equal to n. Then from the M-definability of $p \Vdash \varphi$ for the limited sentence φ, it follows by induction on n that for the sentences φ in F_n, $p \Vdash \varphi$ is definable in M, that is, there exists a formula $\psi_n(x, y)$ such that $(p \Vdash \varphi) \wedge \ulcorner\varphi\urcorner \in F_n$ iff $M \models \psi_n(\mathbf{p}, \ulcorner\varphi\urcorner)$. However, if φ ranges over all the (limited and unlimited) sentences, then $p \Vdash \varphi$ is not definable in M. The reason is that the number of quantifiers in ψ_n increases if n increases. From now on, let G be a $\mathscr{P}$-generic set.

Lemma 19.3. $q \leqq p \wedge p \Vdash \varphi \rightarrow q \Vdash \varphi$.

PROOF. By transfinite induction on $\mathrm{Ord}(\varphi)$. □

The details are left to the reader.

EXERCISES

(1) We define $\varphi \rightarrow \psi$ to be $\neg\varphi \vee \psi$. Then $p \Vdash \varphi \vee p \Vdash [\varphi \rightarrow \psi] \rightarrow p \Vdash \psi$.

(2) $p \Vdash [\mathbf{k}_1 \notin \mathbf{k}_2] \leftrightarrow k_1 \notin k_2$.

(3) $p \Vdash [\mathbf{k}_1 = \mathbf{k}_2] \rightarrow k_1 = k_2$.

Lemma 19.4. $(\exists p \in G)(p \Vdash \varphi \vee p \Vdash \neg\varphi)$.

PROOF. Since $p \Vdash \varphi$ is M-definable, $D = \{p \in P \mid p \Vdash \varphi \vee p \Vdash \neg\varphi\}$ is a member of M. We claim that D is dense in P. Let $p \in P$. If $p \Vdash \neg\varphi$, then $p \in D$. If $\neg(p \Vdash \neg\varphi)$, then $(\exists q \leqq p)[q \Vdash \varphi]$. Therefore $(\exists q \in D)(q \leqq p)$. This shows that D is dense. Therefore there exists a $p \in G$ such that $p \in D$. □

Lemma 19.5. $\neg(p \Vdash \varphi \wedge p \Vdash \neg\varphi)$.

PROOF. Immediate from the definition of $p \Vdash \neg\varphi$. □

Lemma 19.6

$$(\forall p \in G)(\exists q \leqq p)(q \Vdash \varphi) \quad \leftrightarrow \quad (\exists p \in G)(p \Vdash \varphi).$$

PROOF. The "if" part is immediate from Lemma 19.3. For the "only if" part suppose that $(\forall p \in G)(\exists q \leqq p)(q \Vdash \varphi)$. If $(\exists p \in G)(p \Vdash \varphi)$ does not hold, then we have $(\exists p \in G)[p \Vdash \neg\varphi]$ by Lemma 19.4. This contradicts $(\forall p \in G)$ $(\exists q \leqq p)(q \Vdash \varphi)$. □

Theorem 19.7. *For a sentence φ, $(\exists p \in G)(p \Vdash \varphi)$ iff $M[G] \models \varphi$.*

PROOF. First we prove this for a limited sentence φ by transfinite induction on $\mathrm{Ord}(\varphi)$ then we prove it for an unlimited sentence φ by induction on the number of logical symbols in φ. Though there are many cases, almost all of the cases can be proved by a straightforward check of the definition and the induction hypothesis. Therefore we treat only the case where φ is $\neg\psi$ or $t \in \mathbf{G}$.

$$\begin{aligned}(\exists p \in G)(p \Vdash \varphi) &\leftrightarrow (\exists p \in G)(\forall q \leqq p)[\neg(q \Vdash \psi)]\\ &\leftrightarrow \neg(\forall p \in G)(\exists q \leqq p)[q \Vdash \psi].\\ &\leftrightarrow \neg(\exists q \in G)[q \Vdash \psi].\\ &\qquad \text{(Corollary of Lemma 19.6)}\\ &\leftrightarrow \neg M[G] \models \psi.\\ &\leftrightarrow M[G] \models \neg\psi.\end{aligned}$$

$$\begin{aligned}(\exists p \in G)[p \Vdash t \in \mathbf{G}] &\leftrightarrow (\exists p \in G)(\exists q \geqq p)[p \Vdash \forall^\alpha x_i(x_i \in t \leftrightarrow x_i \in \mathbf{q})]\\ &\to (\exists q \in G)[M[G] \models t = \mathbf{q}\\ &\to M[G] \models t \in \mathbf{G}\end{aligned}$$

$$\begin{aligned}M[G] \models t \in \mathbf{G} &\to (\exists q \in G)[M[G] \models t = \mathbf{q}]\\ &\to (\exists q \in G)[M[G] \models \forall^\alpha x_i(x_i \in t \leftrightarrow x_i \in \mathbf{q})]\\ &\to (\exists q \in G)(\exists p \in G)[p \Vdash \forall^\alpha x_i(x_i \in t \leftrightarrow x_i \in \mathbf{q})]\\ &\to (\exists p \in G)(\exists q \geqq p)[p \Vdash \forall^\alpha x_i(x_i \in t \leftrightarrow x_i \in \mathbf{q})]\\ &\to (\exists p \in G)[p \Vdash t \in \mathbf{G}].\end{aligned}$$ □

Corollary 19.8. *For a finite order sentence $\varphi(t_1, \ldots, t_n)$ without any occurrence of $\hat{x}^\alpha$ or $\exists^\alpha$, we have*

$$(\exists p \in G)(p \Vdash \varphi(t_1, \ldots, t_n)) \leftrightarrow M[G] \models \varphi(D^G(t_1), \ldots, D^G(t_n)).$$

PROOF. This is immediate from Theorem 19.7 and Lemma 18.19. □

Theorem 19.9. *$M[G]$ satisfies the axioms of* ZF. *If M satisfies* AC, *then $M[G]$ satisfies* AC *too.*

PROOF. (1) *Axiom of Pairing.* This is obvious since $\{D^G(t_1), D^G(t_2)\} = D^G(\hat{x}_i^\alpha(x_i = t_1 \vee x_i = t_2))$ where $\alpha = \max(\rho(t_1), \rho(t_2)) + 1$.

(2) *Axioms of Unions.* This is immediate since $(D^G(t)) = D^G(\hat{x}_i^{\alpha+1}\, \exists^\alpha x_k (x_i \in x_k \wedge x_k \in t))$ for some appropriate α, x_i and x_k.

(3) *Axiom Schema of Separation.* Let $t \in T_\alpha$ and $t_1, \ldots, t_n \in T$. We would like to show that $\{x \in D^G(t) \mid M[G] \models \varphi(x, D^G(t_1), \ldots, D^G(t_n))\}$ is a member

of $M[G]$. Let $k = \{\langle p, s\rangle \mid s \in T_\alpha \wedge p \Vdash s \in t \wedge \varphi(s, t_1, \ldots, t_n)\}$. Then $k \in M$. Let

$$t' = \hat{x}_1^\alpha(\exists^\beta p \in \mathbf{G})[\langle p, x_i\rangle \in \mathbf{k}].$$

Then

$$\begin{aligned} D^G(t') &= \{D^G(s) \mid (\exists\, p \in G)[s \in T_\alpha \wedge p \Vdash s \in t \wedge \varphi(s, t_1, \ldots, t_n)\} \\ &= \{D^G(s) \mid s \in T_\alpha \wedge D^G(s) \in D^G(t) \wedge \\ &\qquad\qquad M[G] \models \varphi(D^G(s), D^G(t_1), \ldots, D^G(t_n))\} \\ &= \{D^G(s) \in D^G(t) \mid M[G] \models \varphi(D^G(s), D^G(t_1), \ldots, D^G(t_n))\}. \end{aligned}$$

(4) *Axiom of Infinity.* This is obvious since ω is in T.

(5) *Axiom of Foundation.* This is obvious since $M[G]$ is a transitive model.

(6) *Axiom of Extensionality.* This is obvious since $M[G]$ is a transitive model.

(7) *Axiom Schema of Replacement.* Let $a \in M[G]$. We would like to show from the assumption $M[G] \models (\forall\, x)(\exists\, y)\varphi(x, y)$, that

$$(\exists\, b \in M[G])(M[G] \models (\forall\, x \in a)(\exists\, y \in b)\psi(x, y)).$$

Let $a = D^G(t)$ and $t \in T_\alpha$. Since the Axiom Schema of Replacement holds for M, there exists a $\beta \in M$ such that

$$(\forall\, t' \in T_\alpha)(\forall\, p \in P)((\exists\, s')[p \Vdash \varphi(t', s')] \rightarrow (\exists\, s' \in T_\beta)[p \Vdash \varphi(t', s')]).$$

Let $s = \hat{x}^\beta(x_0 \in x_0 \vee \neg x_0 \in x_0)$ and $b = D^G(s)$. It is easily seen that b satisfies the condition.

(8) *Axiom of Powers.* We would like to show that $M[G] \models (\forall\, y)(\exists\, z)(\forall\, x \subseteqq y)(x \in z)$. Let $t \in T_\alpha$. We will show that there exists an $s \in T$ such that $M[G] \models (\forall\, x \subseteqq t)(x \in s)$. For each $t_1 \in T$ let

$$B(t_1) = \{\langle p, s_0\rangle \mid p \in P \wedge s_0 \in T_\alpha \wedge p \Vdash [t_1 \subseteqq t] \wedge p \Vdash [s_0 \in t_1]\}.$$

Obviously $B(t_1) \in M$ for every $t_1 \in T$. For every $t_1, t_2 \in T$, we claim that

$$\text{(i)}\quad [M[G] \models t_1 \subseteqq t \wedge t_2 \subseteqq t \wedge t_1 \neq t_2] \rightarrow [B(t_1) \neq B(t_2)].$$

For this purpose, suppose $M[G] \models t_1 \subseteqq t \wedge t_2 \subseteqq t \wedge t_1 \neq t_2$. Then there exists an $s_0 \in T_\alpha$ such that $M[G] \models s_0 \in t_1 \wedge s_0 \notin t_2$ or $M[G] \models s_0 \notin t_1 \wedge s_0 \in t_2$. Therefore there exist $p \in G$ and $s_0 \in T_\alpha$ such that

$$p \Vdash t_1 \subseteqq t \wedge p \Vdash t_2 \subseteqq t \wedge p \Vdash s_0 \in t_1 \wedge \neg p \Vdash s_0 \in t_2$$

or

$$p \Vdash t_1 \subseteqq t \wedge p \Vdash t_2 \subseteqq t \wedge \neg p \Vdash s_0 \in t_1 \wedge p \Vdash s_0 \in t_2.$$

Therefore (i) is proved. From this, we have

$$[M[G] \models t_1 \subseteqq t \wedge t_2 \subseteqq t \wedge B(t_1) = B(t_2)] \rightarrow [M[G] \models t_1 = t_2].$$

Since $\{B(t_1) \mid t_1 \in T\} \subseteqq \mathscr{P}(P \times T_\alpha)$, there exists a $\beta \in M$ such that

$$(\forall t_1 \in T)(\exists t_2 \in T_\beta)[B(t_1) = B(t_2)].$$

It is now obvious that we can take s to be $\hat{x}_0^\beta(x_0 \in x_0 \vee x_0 \notin x_0)$.

(9) *Axiom of Choice.* Here we assume that M satisfies the Axiom of Choice.

Since $M[G]$ satisfies the axioms of ZF with the language containing the predicate $\mathbf{M}$, we can carry out every construction of T, $D^G(t)$, $p \Vdash t$ for M and G in $M[G]$.

Since M satisfies the Axiom of Choice, we can well order T_α for every $\alpha \in M$. Therefore we can well order $\{D^G(t) \mid t \in T_\alpha\}$ in $M[G]$. It is then obvious that the Axiom of Choice holds in $M[G]$. □

Remark. There are numerous applications of forcing to independence problems in set theory. Here we treat only the first application of Cohen's original paper, the independence of $V = L$ from ZFC + GCH.

Let M be a countable transitive model of ZF + $V = L$. In M, we define $\mathscr{P} = \langle P, \leqq \rangle$ as follows (Every notion should be relativized to M):

$$P \triangleq \{\langle p, q \rangle \mid p \subseteqq \omega \wedge q \subseteqq \omega \wedge \bar{\bar{p}} < \omega \wedge \bar{\bar{q}} < \omega \wedge p \cap q = 0\}.$$

$$\langle p_1, q_1 \rangle \leqq \langle p_2, q_2 \rangle \overset{\triangle}{\leftrightarrow} p_2 \subseteqq p_1 \wedge q_2 \subseteqq q_1.$$

Let G be $\mathscr{P}$-generic over M. Let $\langle p_1, q_1 \rangle$ and $\langle p_2, q_2 \rangle$ be members of G. Then there exists $\langle p_3, q_3 \rangle \in G$ such that $\langle p_3, q_3 \rangle \leqq \langle p_1, q_1 \rangle$ and $\langle p_3, q_3 \rangle \leqq \langle p_2, q_2 \rangle$. Therefore $p_1 \cap q_2 = 0$ and $p_2 \cap q_1 = 0$. Let $a = \cup\{p \mid (\exists q)[\langle p, q \rangle \in G]\}$ and $b = \cup\{q \mid (\exists p)[\langle p, q \rangle \in G]\}$. Then $a \subseteqq \omega$, $b \subseteqq \omega$ and $a \cap b = 0$. We claim that $b = \omega - a$. Suppose otherwise, that is, suppose there exists an $n \in \omega$ such that $n \notin a \cup b$. Then $D = \{\langle p, q \rangle \in P \mid n \in p \vee n \in q\}$ is obviously dense. Since G is $\mathscr{P}$-generic, $G \cap D \neq 0$ and $n \in a \cup b$.

G is obviously obtained from a by the following formula:

$$G = \{\langle p, q \rangle \mid p \subseteqq a \wedge q \subseteqq \omega - a \wedge \bar{\bar{p}} < \omega \wedge \bar{\bar{q}} < \omega\}.$$

Lemma. $G \notin M$.

Proof. Let $c \in M$ be $\mathscr{P}$-generic over M. Let $a' = \cup\{p \mid (\exists q)[\langle p, q \rangle \in c]\}$ and $b' = \cup\{q \mid (\exists p)[\langle p, q \rangle \in c]\}$. Then $a' \subseteqq \omega$ and $b' = \omega - a'$. Let $D = P - c$. It suffices to show that D is dense. Let $\langle p, q \rangle \in P$. Since p and q are finite, there exists an $n \in \omega$ such that $n \notin p \cup q$. Let $p' = p$ and $q' = q \cup \{n\}$ if $n \in a'$ and $p' = p \cup \{n\}$ and $q' = q$ otherwise. Obviously $\langle p', q' \rangle \leqq \langle p, q \rangle$ and $\langle p', q' \rangle \in D$. □

Theorem 19.10. *$M[G]$ is a model of* ZFC + GCH + $V \neq L$.

Proof. We have already proved that $M[G]$ is a model of ZFC. Since in $M[G]$ the class of constructible sets is M, it follows that $G \notin M$ implies that $M[G]$ does not satisfy $V = L$. So it suffices to show that $M[G]$ satisfies

GCH. If we use Theorem 15.44(2), $a \subseteq \omega \wedge V = L_a \to$ GCH, then we get the result immediately as follows: Obviously $M[G] \models a \subseteq \omega \wedge V = L_a$. Therefore $M[G] \models$ GCH. But we would like to prove this without using the Theorem 15.44(2).

First we will show that the notion of cardinality is the same in M and $M[G]$: Since $M \subseteq M[G]$, an ordinal α is a cardinal in M if α is a cardinal in $M[G]$. To show that α is a cardinal in $M[G]$ if α is so in M, let $\beta < \alpha$ and $M[G] \models f\colon \boldsymbol{\beta} \xrightarrow[\text{onto}]{} \boldsymbol{\alpha}$, where $f \in T$. Then there exists a $p_0 \in G$ such that

$$p_0 \Vdash (\forall x_0)(\forall x_1)(\forall x_2)(x_1 = f(x_0) \wedge x_2 = f(x_0) \to x_1 = x_2),$$

where $x_1 = f(x_0)$ is an abbreviation of some formula. From this we have,

$$p_0 \Vdash \gamma_1 = f(\gamma_0) \wedge p_0 \Vdash \gamma_2 = f(\gamma_0) \to \gamma_1 = \gamma_2.$$

Suppose $\alpha_1 < \alpha$. Then there exists a $\beta_1 < \beta$ such that $M[G] \models \boldsymbol{\alpha}_1 = f(\boldsymbol{\beta}_1)$. Therefore we have

$$(\exists p \in G)(p \leqq p_0 \wedge p \Vdash \boldsymbol{\alpha}_1 = f(\boldsymbol{\beta}_1)).$$

Therefore

$$\begin{aligned} \bar{\bar{\alpha}} &\leqq \overline{\overline{\{\langle p, \beta_1\rangle | p \leqq p_0 \wedge (\exists \alpha_1)(p \Vdash \boldsymbol{\alpha}_1 = f(\boldsymbol{\beta}_1))\}}} \\ &\leqq \overline{\overline{P \times \beta}} = \max(\omega, \bar{\bar{\beta}}), \end{aligned}$$

where the calculation of the cardinalities can be carried out in M because $\Vdash$ can be defined in M.

Next we show that $\overline{\overline{\mathscr{P}(\aleph_\alpha)}} = \aleph_{\alpha+1}$ holds in $M[G]$. First we note that $\aleph_\alpha \in T_{\aleph_{\alpha+1}}$ and $\gamma < \aleph_\alpha \to \gamma \in T_{\aleph_\alpha}$. Let $B(t_1)$ be as in the proof of the Axiom of Powers in Theorem 19.9 but with α replaced by $\aleph_\alpha$. Then the argument there shows that

$$\overline{\overline{\mathscr{P}^{M[G]}(\aleph_\alpha)}}^{M[G]} \leqq \overline{\overline{\mathscr{P}^M(\mathrm{P} \times \mathrm{T}_{\aleph_\alpha})}}^{M} = \overline{\overline{\mathscr{P}^M(\aleph_\alpha)}}^{M} = \aleph_{\alpha+1}$$

holds in $M[G]$. This completes our proof. □

Now we can show the independence of $V = L$ from ZFC + GCH. Suppose $V = L$ is provable from ZFC + GCH. Then $V = L$ is provable from a finite subsystem T of ZFC + GCH. Let the maximum number of quantifiers in the axioms in T be m. Let $(\text{ZFC} + \text{GCH})^m$ denote the axioms of ZFC + GCH in which the number of quantifiers is less than or equal to m. For any integer m we can define the truth definition T_m such that for every sentence φ, in which the number of quantifiers is less than m,

$$T_m(\ulcorner\varphi\urcorner) \leftrightarrow \varphi.$$

By the usual proof of the Skolem–Löwenheim theorem, we can construct a countable transitive model of $(\text{ZFC} + \text{GCH})^m$. Let M' be such a model. If m is sufficiently large relative to k, then we can develop forcing theory for those formulas whose number of logical symbol is less than or equal to k. Therefore we can show that $M'[G]$ is a model of $(\text{ZFC} + \text{GCH})^m + V \neq L$.

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Problem List

(1) Let A be an infinite set. Prove that the cardinality of the set of all automorphisms of A, i.e., one-to-one mappings of A onto A, is $\overline{\overline{2^A}}$. (*Hint*: Divide A into A_1, A_2, A_3 so that $\bar{\bar{A}}_1 = \bar{\bar{A}}_2 = \bar{\bar{A}}_3 = \bar{\bar{A}}$. For each $B \subseteq A_2$ find an automorphism π for which $\pi``(A_1 \cup B) = A_3 \cup (A_2 - B)$.

(2) Let A be a countable infinite set and $<_1$ be an order relation on A (Definition 6.19). Let R_0 be the set of rationals in the interval (0, 1). Find a one-to-one order-preserving map τ from A into R_0. (*Hint*: Let $A = \{a_0, a_1, \ldots\}$. Define $\tau(a_i)$ assuming that $\tau(a_0), \ldots, \tau(a_{i-1})$ have been defined.

(3) Let A_1 and A_2 be infinite countable sets. Let $<_1$ Or A_1, $<_2$ Or A_2, and both structures satisfy

(a) $(\forall x)(\exists y)[y < x]$
(b) $(\forall x)(\exists y)[x < y]$
(c) $(\forall x)(\forall y)(\exists z)[x < y \rightarrow x < z < y]$.

Prove: $(\exists f) f \operatorname{Isom}_{<_1, <_2}(A_1, A_2)$. (*Hint*: Let $A_1 = \{a_0, a_1, \ldots\}$ and $A_2 = \{b_0, b_1, \ldots\}$. Define $\tau_1 \operatorname{Isom}_{<_1, <_2}(A_1, A_2)$ and $\tau_2 \operatorname{Isom}_{<_2, <_1}(A_2, A_1)$ inductively in the order $\tau_1(a_0)$, $\tau_2(b_0)$, $\tau_1(a_1)$, $\tau_2(b_1), \ldots$, such that $\tau_1 \circ \tau_2$ and $\tau_2 \circ \tau_1$ are identity functions on A_2 and A_1 respectively.

(4) Let $W_\alpha = \{\langle \alpha_0, \ldots, \alpha_n \rangle \mid n < \omega \wedge (\forall i \leqq n)[\alpha_i < \alpha]\}$. Let $<_\alpha$ be the lexicographical ordering on W_α. Prove that if α is finite, $\langle W_\alpha, <_\alpha \rangle$ is isomorphic to $R_1 \times \omega$, where R_1 is the set of all rationals in the interval [0, 1), and $R_1 \times \omega$ is ordered lexicographically relative to the natural order on R_1 and on ω. What is the order type of $\langle W_\alpha, <_\alpha \rangle$ if $\alpha \geqq \omega$?

(5) Let $\mathscr{R}$ be the set of all real numbers and let f be a mapping from $\aleph_1$ into $\mathscr{R}$ that is monotone increasing. Prove: $(\exists\, \alpha < \aleph_1)(\forall\, \beta > \alpha)[f(\beta) = f(\alpha)]$. (*Hint*: $\mathscr{R}$ is separable and hence $\mathrm{cf}(f``\aleph_1) = \aleph_0$).

(6) Let $\mathscr{R}$ be the set of all real numbers and let f be a continuous mapping from $\aleph_1$ into $\mathscr{R}$. Prove: $(\exists\, \alpha < \aleph_1)(\forall\, \beta > \alpha)[f(\beta) = f(\alpha)]$.

(7) Let $\mathscr{R}$ be the set of all real numbers and let $f\colon \aleph_1 \xrightarrow{1-1} \mathscr{R}$. $\forall\, \alpha, \beta < \aleph_1$ define $\alpha \lll \beta$ iff $\alpha < \beta \wedge f(\alpha) < f(\beta)$. Prove the following:

- (a) The relation $\lll$ is a well-founded partial ordering on $\aleph_1$.
- (b) If $A \subseteq \aleph_1$, and $(\forall\, x, y \in A)[x \lll y \vee x = y \vee y \lll x]$ then A is countable.
- (c) If $A \subseteq \aleph_1$ and $(\forall\, x, y \in A)[x = y \vee \neg [x \lll y \vee y \lll x]]$ then A is countable.

(8) Let $A \subseteq [0, 1]$. Then $\forall\, x \in [0, 1]$, x is a κ-accumulation point of A iff

$$\forall\, N(x)[\overline{\overline{N(x) \cap A}} \geq \kappa].$$

(Here $N(x)$ is a neighborhood of x in the usual topology on $[0, 1]$.)

- (a) Prove that $\{x \in [0, 1] \mid x \text{ is an } \aleph_1\text{-accumulation point of } A\}$ is a closed set that is dense in itself.
- (b) Prove that $\{x \in [0, 1] \mid x \text{ is an } \aleph_2\text{-accumulation point of } A\}$ is a closed set that is dense in itself.

(9) If $\mathrm{cf}(\aleph_\alpha) < \aleph_\alpha$ and $(\forall\, \lambda < \alpha)(\exists\, \nu < \alpha)[\overline{\overline{2^\lambda}} < \overline{\overline{2^\nu}}]$ and if $\lambda = \sup_{\xi < \aleph_\alpha} \overline{\overline{\aleph_\xi^{\mathrm{cf}(\aleph_\xi)}}}$ prove that $\overline{\overline{2^{\aleph_\alpha}}} = \overline{\overline{\lambda^{\mathrm{cf}(\lambda)}}}$.

(10) If $\mathrm{cf}(\aleph_\alpha) < \aleph_\alpha$, if $\lambda < \alpha$, and if $(\forall\, \nu < \alpha)[\lambda \leq \nu \rightarrow \overline{\overline{2^{\aleph_\lambda}}} = \overline{\overline{2^{\aleph_\nu}}}]$ prove that $\overline{\overline{2^{\aleph_\alpha}}} = \overline{\overline{2^{\aleph_\lambda}}}$.

(11) Prove that if $\aleph_\alpha > \aleph_\beta \geq \mathrm{cf}(\aleph_\alpha)$ and $(\exists\, \gamma < \alpha)[\aleph_\alpha \leq \overline{\overline{\aleph_\gamma^{\aleph_\beta}}}]$ then

$$\overline{\overline{\aleph_\alpha^{\aleph_\beta}}} = \overline{\overline{\aleph_{\gamma_0}^{\aleph_\beta}}}$$

where $\gamma_0 = \mu_\gamma(\gamma < \alpha \wedge \aleph_\alpha \leq \overline{\overline{\aleph_\gamma^{\aleph_\beta}}})$.

(12) Let $\aleph_\alpha > \aleph_\beta \geq \mathrm{cf}(\aleph_\alpha)$ and $(\forall\, \gamma < \alpha)[\overline{\overline{\aleph_\gamma^{\aleph_\beta}}} < \aleph_\alpha]$. Prove that

$$\overline{\overline{\aleph_\alpha^{\mathrm{cf}(\aleph_\alpha)}}} = \overline{\overline{\aleph_\alpha^{\aleph_\beta}}}.$$

(13) Prove: If $\overline{\overline{2^{\mathrm{cf}(\aleph_\alpha)}}} < \aleph_\alpha$, if $(\exists\, \beta < \alpha)[\aleph_\alpha \leq \overline{\overline{\aleph_\beta^{\mathrm{cf}(\aleph_\beta)}}} \wedge \mathrm{cf}(\aleph_\beta) < \mathrm{cf}(\aleph_\alpha)]$ and if $\lambda = \mu_\gamma(\mathrm{cf}(\aleph_\gamma) \leq \mathrm{cf}(\aleph_\alpha) \wedge \aleph_\alpha \leq \overline{\overline{\aleph_\gamma^{\mathrm{cf}(\aleph_\gamma)}}})$, then $\overline{\overline{\aleph_\alpha^{\mathrm{cf}(\aleph_\alpha)}}} = \overline{\overline{\aleph_\lambda^{\mathrm{cf}(\aleph_\gamma)}}}$. (*Hint*: If $\nu = \mu_\gamma(\aleph_\alpha \leq \overline{\overline{\aleph_\gamma^{\mathrm{cf}(\aleph_\alpha)}}})$ then $\overline{\overline{\aleph_\alpha^{\mathrm{cf}(\aleph_\alpha)}}} = \overline{\overline{\aleph_\nu^{\mathrm{cf}(\aleph_\nu)}}}$ and $\lambda = \nu$).

(14) Let $F``\alpha_0$ be a model of ZF. Prove that $\{\ulcorner\varphi\urcorner \mid F``\alpha_0 \models \varphi\} \in L$.

(15) A set a is L-finite iff $(\forall\, x \in L)[x \subseteq a \rightarrow x \text{ is finite}]$. Assuming that $\overline{\overline{\mathscr{P}(\omega)^L}} = \omega$, prove that $(\exists\, x \subseteq \omega)$, x and $\omega - x$ are each L-finite.

(16) Find a model $M[G]$ of ZF + AC + $V \neq L$ such that M is a model of ZF + $V = L$ and G is L-finite.

(17) If $\vdash^1 \ulcorner\varphi\urcorner \stackrel{\Delta}{\leftrightarrow} (\exists \mathscr{P})[\mathscr{P} \Vdash \ulcorner\varphi\urcorner]$ and $\vdash^2 \ulcorner\varphi\urcorner \stackrel{\Delta}{\leftrightarrow} (\forall \mathscr{P})[\mathscr{P} \Vdash \ulcorner\varphi\urcorner)$ what logical rules do $\vdash^1$ and $\vdash^2$ satisfy.

(18) A sentence is called arithmetical if every quantifier in it is restricted to ω. Let φ be an arithmetic sentence and let $F``\alpha_0$ be a model of ZF. Prove

$$\varphi \leftrightarrow F``\alpha_0 \models \varphi.$$

(19) A sentence is called a $\mathscr{P}(\omega)$-sentence if every quantifier in it is restricted to $\mathscr{P}(\omega)$. Assuming the existence of the minimal model M_0 find a $\mathscr{P}(\omega)$-sentence φ such that

$$L \models \varphi \leftrightarrow M_0 \models \varphi$$

is false.

(20) In a forcing model M of $V \neq L$, find $a \subseteq \omega$ such that

(a) $L_a \neq M$,

(b) a and $\omega - a$ are L-finite.

Appendix

Let M, SM, and Consis(ZF) be statements that assert respectively:

(1) There exists a set that is a model of ZF.

(2) There exists a set that is a standard model of ZF.

(3) ZF is consistent.

Furthermore, let Consis(ZF) be so chosen that it is absolute w.r.t. every standard transitive model of ZF.

Theorem. $\neg \vdash_{ZF} M \to SM$.

Proof. Suppose

(1) $\vdash_{ZF} M \to SM$.

It is known that

(2) $\vdash_{ZF} M \leftrightarrow \text{Consis(ZF)}$.

Consequently, from (1) and (2)

(3) $\text{Consis(ZF)} \vdash_{ZF} SM$.

There exists a minimal standard transitive model of ZF, M_0. Clearly

(4) $M_0 \models \text{Consis(ZF)}$

Then relativizing (3) to M_0, using the fact that Consis(ZF) is absolute w.r.t. M_0, we have

$$M_0 \models SM.$$

This is a contradiction. □

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