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Jet Nestruev

Smooth Manifolds and Observables

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Editorial Board

S. Axler
Mathematics Department
San Francisco State University
San Francisco, CA 94132
USA
axler@sfsu.edu

F.W. Gehring
Mathematics Department
East Hall
University of Michigan
Ann Arbor, MI 48109
USA
fgehring@math.lsa.umich.edu

K.A. Ribet
Mathematics Department
University of California, Berekeley
Berkeley, CA 94720-3840
USA
ribet@math.berkeley.edu

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Preface to the English Edition

The author is very pleased that his book, first published in Russian in 2000 by MCCME Publishers, is now appearing in English under the auspices of such a truly classical publishing house as Springer-Verlag.

In this edition several pertinent remarks by the referees (to whom the author expresses his gratitude) were taken into account, and new exercises were added (mostly) to the first half of the book, thus achieving a better balance with the second half. Besides, some typos and minor errors, noticed in the Russian edition, were corrected. We are extremely grateful to all our readers who assisted us in this tiresome bug hunt. We are especially grateful to A. De Paris, I. S. Krasil'schik, and A. M. Verbovetski, who demonstrated their acute eyesight, truly of *degli Lincei* standards.

The English translation was carried out by A. B. Sossinsky (Chapters 1–8), I. S. Krasil'schik (Chapter 9), and S. V. Duzhin (Chapters 10–11) and reduced to a common denominator by the first of them; A. M. Astashov prepared new versions of the figures; all the TEX-nical work was done by M. M. Vinogradov.

In the process of preparing this edition, the author was supported by the Istituto Nazionale di Fisica Nucleare and the Istituto Italiano per gli Studi Filosofici. It is only thanks to these institutions, and to the efficient help of Springer-Verlag, that the process successfully came to its end in such a short period of time.

Jet Nestruev
Moscow–Salerno
April 2002

Preface

The limits of my language
are the limits of my world.
— L. Wittgenstein

This book is a self-contained introduction to smooth manifolds, fiber spaces, and differential operators on them, accessible to graduate students specializing in mathematics and physics, but also intended for readers who are already familiar with the subject. Since there are many excellent textbooks in manifold theory, the first question that should be answered is, Why another book on manifolds?

The main reason is that the good old differential calculus is actually a particular case of a much more general construction, which may be described as the *differential calculus over commutative algebras.* And this calculus, in its entirety, is just the consequence of properties of arithmetical operations. This fact, remarkable in itself, has numerous applications, ranging from delicate questions of algebraic geometry to the theory of elementary particles. Our book explains in detail why the differential calculus on manifolds is simply an aspect of commutative algebra.

In the standard approach to smooth manifold theory, the subject is developed along the following lines. First one defines the notion of smooth manifold, say M. Then one defines the algebra $\mathcal{F}_M$ of smooth functions on M, and so on. In this book this sequence is reversed: We begin with a certain commutative $\mathbb{R}$-algebra[1] $\mathcal{F}$, and then define the manifold $M = M_{\mathcal{F}}$

[1]Here and below $\mathbb{R}$ stands for the real number field. Nevertheless, and this is very important, nothing prevents us from replacing it by an arbitrary field (or even a ring) if this is appropriate for the problem under consideration.

as the $\mathbb{R}$-spectrum of this algebra. (Of course, in order that $M_{\mathcal{F}}$ deserve the title of a smooth manifold, the algebra $\mathcal{F}$ must satisfy certain conditions; these conditions appear in Chapter 3, where the main definitions mentioned here are presented in detail.)

This approach is by no means new: It is used, say, in algebraic geometry. One of its advantages is that from the outset it is not related to the choice of a specific coordinate system, so that (in contrast to the standard analytical approach) there is no need to constantly check that various notions or properties are independent of this choice. This explains the popularity of this viewpoint among mathematicians attracted by sophisticated algebra, but its level of abstraction discourages the more pragmatically inclined applied mathematicians and physicists.

But what is really new in this book is the motivation of the algebraic approach to smooth manifolds. It is based on the fundamental notion of *observable*, which comes from physics. It is this notion that creates an intuitively clear environment for the introduction of the main definitions and constructions. The concepts of *state of a physical system* and *measuring device* endow the very abstract notions of *point of the spectrum* and *element of the algebra* $\mathcal{F}_M$ with very tangible physical meanings.

One of the fundamental principles of contemporary physics asserts that *what exists is only that which can be observed.* In mathematics, which is not an experimental science, the notion of observability was never considered seriously. And so the discussion of any *existence problem* in the formalized framework of mathematics has nothing to do with reality. A present-day mathematician studies sets supplied with various structures without ever specifying (distinguishing) individual elements of those sets. Thus their observability, which requires some *means of observation*, lies beyond the limits of formal mathematics.

This state of affairs cannot satisfy the working mathematician, especially one who, like Archimedes or Newton, regards his science as natural philosophy. Now, physicists, for example, in their study of quantum phenomena, come to the conclusion that it is impossible in principle to completely distinguish the observer from the observed. Hence any adequate mathematical description of quantum physics must include, as an inherent part, an appropriate formalization of observability.

Scientific observation relies on measuring devices, and in order to introduce them into mathematics, it is natural to begin with its classical parts, i.e., those coming from classical physics. Thus we begin with a detailed explanation of why the classical measuring procedure can be translated into mathematics as follows:

Physics lab	$\longrightarrow$	Commutative unital $\mathbb{R}$-algebra A
Measuring device	$\longrightarrow$	Element of the algebra A
State of the observed physical system	$\longrightarrow$	Homomorphism of unital $\mathbb{R}$-algebras $h\colon A \to \mathbb{R}$
Output of the measuring device	$\longrightarrow$	Value of this function $h(a)$, $a \in A$

In the framework of this approach, smooth (i.e., differentiable) manifolds appear as $\mathbb{R}$-spectra of a certain class of $\mathbb{R}$-algebras (the latter are therefore called *smooth*), and their elements turn out to be the smooth functions defined on the corresponding spectra. Here the $\mathbb{R}$-spectrum of some $\mathbb{R}$-algebra A is the set of all its unital homomorphisms into the $\mathbb{R}$-algebra $\mathbb{R}$, i.e., the set that is "visible" by means of this algebra. Thus smooth manifolds are "worlds" whose observation can be carried out by means of smooth algebras. Because of the algebraic universality of the approach described above, "nonsmooth" algebras will allow us to observe "nonsmooth worlds" and study their singularities by using the differential calculus. But this differential calculus is not the naive calculus studied in introductory (or even "advanced") university courses; it is a much more sophisticated construction.

It is to the foundations of this calculus that the second part of this book is devoted. In Chapter 9 we "discover" the notion of differential operator *over a commutative algebra* and carefully analyze the main notion of the classical differential calculus, that of the derivative (or more precisely, that of the tangent vector). Moreover, in this chapter we deal with the other simplest constructions of the differential calculus from the new point of view, e.g., with tangent and cotangent bundles, as well as jet bundles. The latter are used to prove the equivalence of the algebraic and the standard analytic definitions of differential operators for the case in which the basic algebra is the algebra of smooth functions on a smooth manifold. As an illustration of the possibilities of this "algebraic differential calculus," at the end of Chapter 9 we present the construction of the Hamiltonian formalism over an arbitrary commutative algebra.

In Chapters 10 and 11 we study fiber bundles and vector bundles from the algebraic point of view. In particular, we establish the equivalence of the category of vector bundles over a manifold M and the category of finitely generated projective modules over the algebra $C^\infty(M)$. Chapter 11 is concluded by a study of jet modules of an arbitrary vector bundle and an explanation of the universal role played by these modules in the theory of differential operators.

Thus the last three chapters acquaint the reader with some of the simplest and most thoroughly elaborated parts of the new approach to the differential calculus, whose complete logical structure is yet to be determined. In fact, one of the main goals of this book is to show that the discovery of the differential calculus by Newton and Leibniz is quite similar to the discovery of the New World by Columbus. The reader is invited to continue the expedition into the internal areas of this beautiful new world, differential calculus.

Looking ahead beyond the (classical) framework of this book, let us note that the mechanism of *quantum observability* is in principle of cohomological nature and is an appropriate specification of those natural observation methods of solutions to (nonlinear) partial differential equations that have appeared in the *secondary differential calculus* and in the fairly new branch of mathematical physics known as *cohomological physics.*

The prerequisites for reading this book are not very extensive: a standard advanced calculus course and courses in linear algebra and algebraic structures. So as not to deviate from the main lines of our exposition, we use certain standard elementary facts without providing their proofs, namely, partition of unity, Whitney's immersion theorem, and the theorems on implicit and inverse functions.

* * *

In 1969 Alexandre Vinogradov, one of the authors of this book, started a seminar aimed at understanding the mathematics underlying quantum field theory. Its participants were his mathematics students, and several young physicists, the most assiduous of whom were Dmitry Popov, Vladimir Kholopov, and Vladimir Andreev. In a couple of years it became apparent that the difficulties of quantum field theory come from the fact that physicists express their ideas in an inadequate language, and that an adequate language simply does not exist (see the quotation preceding the Preface). If we analyze, for example, what physicists call the covariance principle, it becomes clear that its elaboration requires a correct definition of differential operators, differential equations, and, say, second-order differential forms.

For this reason in 1971 a mathematical seminar split out from the physical one, and began studying the structure of the differential calculus and searching for an analogue of algebraic geometry for systems of (nonlinear) partial differential equations. At the same time, the above-mentioned author began systematically lecturing on the subject.

At first, the participants of the seminar and the listeners of the lectures had to manage with some very schematic summaries of the lectures and their own lecture notes. But after ten years or so, it became obvious that all these materials should be systematically written down and edited. Thus Jet Nestruev was born, and he began writing an infinite series of books entitled *Elements of the Differential Calculus*. Detailed contents of the first installments of the series appeared, and the first one was written. It contained, basically, the first eight chapters of the present book.

Then, after an interruption of nearly fifteen years, due to a series of objective and subjective circumstances, work on the project was resumed, and the second installment was written. Amalgamated with the first one, it constitutes the present book. This book is a self-contained work, and we have consciously made it independent of the rest of the Nestruev project. In it the reader will find, in particular, the definition of differential operators on a manifold. However, Jet Nestruev has not lost the hope to explain, in the not too distant future, what a system of partial differential equations is, what a second-order form is, and some other things as well. The reader who wishes to have a look ahead without delay can consult the references appearing on page 217. A more complete bibliography can be found in [8].

Unlike a well-known French general, Jet Nestruev is a civilian and his personality is not veiled in military secrecy. So it is no secret that this book was written by A. M. Astashov, A. B. Bocharov, S. V. Duzhin, A. B. Sossinsky, A. M. Vinogradov, and M. M. Vinogradov. Its conception and its main original observations are due to A. M. Vinogradov. The figures were drawn by A. M. Astashov. It is a pleasure for Jet Nestruev to acknowledge the role of I. S. Krasil'schik, who carefully read the whole text of the book and made several very useful remarks, which were taken into account in the final version.

During the final stages, Jet Nestruev was considerably supported by *Istituto Italiano per gli Studi Filosofici* (Naples), *Istituto Nazionale di Fisica Nucleare* (Italy), and INTAS (grant 96-0793).

Jet Nestruev
Moscow–Pereslavl-Zalesski–Salerno
April 2000

Contents

1
Introduction

1.0. This chapter is a preliminary discussion of *finite-dimensional smooth* (*infinitely differentiable*) *real manifolds*, the main protagonists of this book.

Why are smooth manifolds important?

Well, we live *in* a manifold (a four-dimensional one, according to Einstein) and *on* a manifold (the Earth's surface, whose model is the sphere S^2). We are surrounded by manifolds: The surface of a coffee cup is a manifold (namely, the torus $S^1 \times S^1$, more often described as the surface of a doughnut or an anchor ring, or as the tube of an automobile tire); a shirt is a two-dimensional manifold with boundary.

Processes taking place in nature are often adequately modeled by points moving on a manifold, especially if they involve no discontinuities or catastrophes. (Incidentally, catastrophes — in nature or on the stock market — as studied in "catastrophe theory" may not be manifolds, but then they are smooth maps of manifolds.)

What is more important from the point of view of this book, is that *manifolds arise quite naturally in various branches of mathematics* (in *algebra* and *analysis* as well as in *geometry*) *and its applications* (especially *mechanics*). Before trying to explain what smooth manifolds are, we give some examples.

1.1. The configuration space Rot(3) **of a rotating solid in space.** Consider a solid body in space fixed by a hinge O that allows it to rotate in any direction (Figure 1.1). We want to describe the set of positions of the body, or, as it is called in classical mechanics, its *configuration space.* One way of going about it is to choose a coordinate system $Oxyz$ and

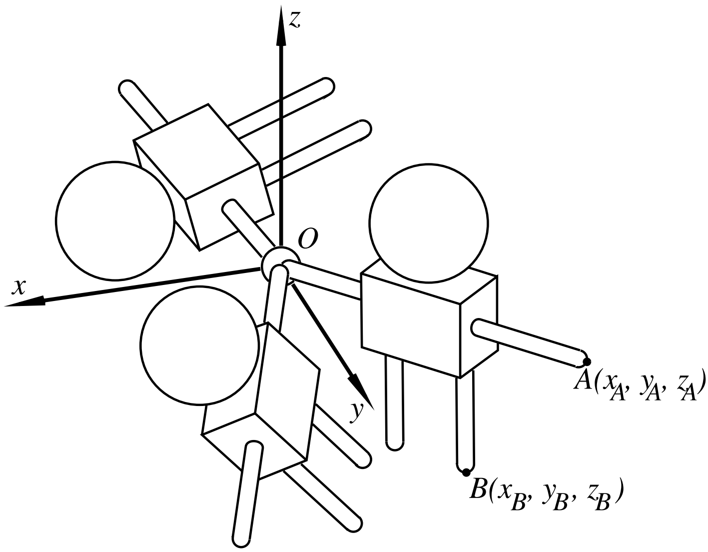


Figure 1.1. Rotating solid.

determine the body's position by the coordinates (x_A, y_A, z_A), (x_B, y_B, z_B) of two of its points A, B. But this is obviously not an economical choice of parameters: It is intuitively clear that only *three* real parameters are required, at least when the solid is not displaced too greatly from its initial position OA_0B_0. Indeed, two parameters determine the direction of OA (e.g., x_A, y_A; see Figure 1.1), and one more is needed to show how the solid is turned about the OA axis (e.g., the angle $\varphi_B = B_0'OB$, where AB_0' is parallel to A_0B_0).

It should be noted that these are not ordinary Euclidean coordinates; the positions of the solid do not correspond bijectively in any natural way to ordinary three-dimensional space $\mathbb{R}^3$. Indeed, if we rotate AB through the angle $\varphi = 2\pi$, the solid does not acquire a new position; it returns to the position OAB; besides, *two* positions of OA correspond to the coordinates (x_A, y_A): For the second one, A is *below* the Oxy plane. However, *locally,* say near the initial position OA_0B_0, there is a bijective correspondence between the position of the solid and a neighborhood of the origin in 3-space $\mathbb{R}^3$, given by the map $OAB \mapsto (x_A, y_A, \varphi_B)$. Thus the configuration space Rot(3) of a rotating solid is an object that can be described locally by three Euclidean coordinates, but *globally* has a more complicated structure.

1.2. An algebraic surface V. In nine-dimensional Euclidean space $\mathbb{R}^9$ consider the set of points satisfying the following system of six algebraic equations:

$$\begin{cases} x_1^2+x_2^2+x_3^2=1; & x_1x_4+x_2x_5+x_3x_6=0; \\ x_4^2+x_5^2+x_6^2=1; & x_1x_7+x_2x_8+x_3x_9=0; \\ x_7^2+x_8^2+x_9^2=1; & x_4x_7+x_5x_8+x_6x_9=0. \end{cases}$$

This happens to be a nice three-dimensional surface in $\mathbb{R}^9$ $(3 = 9 - 6)$. It is not difficult (try!) to describe a bijective map of a neighborhood of any point (say $(1,0,0,0,1,0,0,0,1)$) of the surface onto a neighborhood of the origin of Euclidean 3-space. But this map cannot be extended to cover the entire surface, which is compact (why?). Thus again we have an example of an object V locally like 3-space, but with a different global structure.

It should perhaps be pointed out that the solution set of six algebraic equations with nine unknowns chosen at random will not always have such a simple local structure; it may have self-intersections and other *singularities*. (This is one of the reasons why algebraic geometry, which studies such *algebraic varieties,* as they are called, is not a part of smooth manifold theory.)

1.3. Three-dimensional projective space $\mathbb{R}P^3$. In four-dimensional Euclidean space $\mathbb{R}^4$ consider the set of all straight lines passing through the origin. We want to view this set as a "space" whose "points" are the lines. Each "point" of this space — called *projective space* $\mathbb{R}P^3$ by nineteenth century geometers — is determined by the line's directing vector (a_1, a_2, a_3, a_4), $\sum a_i^2 \neq 0$, i.e., a quadruple of real numbers. Since proportional quadruples define the same line, each point of $\mathbb{R}P^3$ is an equivalence class of proportional quadruples of numbers, denoted by $P = (a_1 : a_2 : a_3 : a_4)$, where (a_1, a_2, a_3, a_4) is any representative of the class. In the vicinity of each point, $\mathbb{R}P^3$ is like $\mathbb{R}^3$. Indeed, if we are given a point $P_0 = \left(a_1^0 : a_2^0 : a_3^0 : a_4^0\right)$ for which $a_4^0 \neq 0$, it can be written in the form $P_0 = \left(a_1^0/a_4^0 : a_2^0/a_4^0 : a_3^0/a_4^0 : 1\right)$ and the three ratios viewed as its three coordinates. If we consider all the points P for which $a_4 \neq 0$ and take $x_1 = a_1/a_4$; $x_2 = a_2/a_4$; $x_3 = a_3/a_4$ to be their coordinates, we obtain a bijection of a neighborhood of P_0 onto $\mathbb{R}^3$. This neighborhood, together with three similar neighborhoods (for $a_1 \neq 0$, $a_2 \neq 0$, $a_3 \neq 0$), covers all the points of $\mathbb{R}P^3$. But points belonging to more than one neighborhood are assigned to different triples of coordinates (e.g., the point $(6 : 12 : 2 : 3)$ will have the coordinates $\left(2, 4, \frac{2}{3}\right)$ in one system of coordinates and $\left(3, 6, \frac{3}{2}\right)$ in another). Thus the overall structure of $\mathbb{R}P^3$ is not that of $\mathbb{R}^3$.

1.4. The special orthogonal group SO(3)**.** Consider the group SO(3) of orientation-preserving isometries of $\mathbb{R}^3$. In a fixed orthonormal basis, each element $A \in \mathrm{SO}(3)$ is defined by an orthogonal positive definite matrix, thus by nine real numbers $(9 = 3 \times 3)$. But of course, fewer than 9 numbers are needed to determine A. In canonical form, the matrix of A

will be

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{pmatrix},$$

and A is defined if we know φ and are given the eigenvector corresponding to the eigenvalue $\lambda = 1$ (two real coordinates a, b are needed for that, since eigenvectors are defined up to a scalar multiplier). Thus again three coordinates (φ, a, b) determine elements of SO(3), and they are Euclidean coordinates only locally.

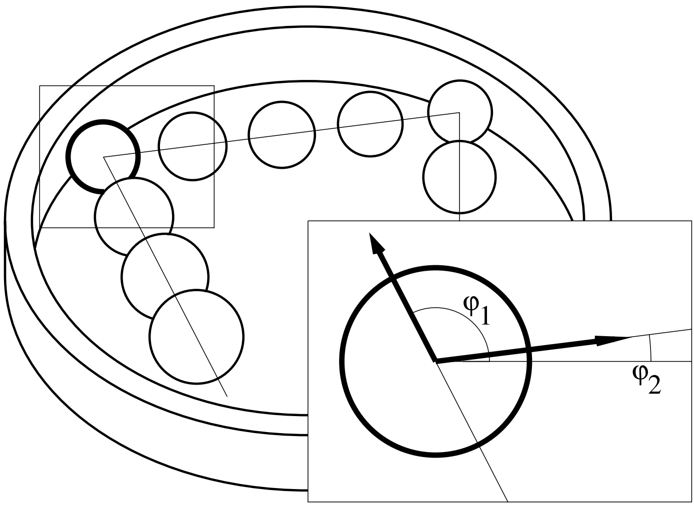


Figure 1.2.

1.5. The phase space of billiards on a disk $\mathcal{B}(D^2)$. A tiny billiard ball P moves with unit velocity in a closed disk D^2, bouncing off its circular boundary C in the natural way (angle of incidence = angle of reflection). We want to describe the *phase space* $\mathcal{B}(D^2)$ of this mechanical system, whose "points" are all the possible *states of the system* (each state being defined by the position of P and the direction of its velocity vector). Since each state is determined by three coordinates $(x, y; \varphi)$ (Figure 1.2), it would seem that as a set, $\mathcal{B}(D^2)$ is $D^2 \times S^1$, where S^1 is the unit circle ($S^1 = \mathbb{R}$ mod 2π). But this is not the case, because at the moment of collision with the boundary, say at (x_0, y_0), the direction of the velocity vector jumps from φ_1 to φ_2 (see Figure 1.2), so that we must identify the states

$$(x_0, y_0, \varphi_1) \equiv (x_0, y_0, \varphi_2). \tag{1.1}$$

Thus $\mathcal{B}(D^2) = (D^2 \times S^1)/\sim$, where $/\sim$ denotes the factorization defined by the equivalence relation of all the identifications (1.1) due to all possible collisions with the boundary C.

Since the identifications take place only on C, all the points of

$$\mathcal{B}^0(D^2) = \operatorname{Int} D^2 \times S^1 = (\operatorname{Int} D^2 \times S^1)/\sim ,$$

where $\operatorname{Int} D^2 = D^2 \setminus C$ is the interior of D^2, have neighborhoods with a structure like that of open sets in $\mathbb{R}^3$ (with coordinates $(x, y; \varphi)$). It is a rather nice fact (not obvious to the beginner) that after identifications the "boundary states" $(x, y; \varphi)$, $(x, y) \in C$, also have such neighborhoods, so that again $\mathcal{B}(D^2)$ is locally like $\mathbb{R}^3$, but not like $\mathbb{R}^3$ globally (as we shall later show).

As a more sophisticated example, the advanced reader might try to describe the phase space of billiards in a right triangle with an acute angle of (a) $\pi/6$; (b) $\sqrt{2}\pi/4$.

1.6. The five examples of three-dimensional manifolds described above all come from different sources: classical mechanics 1.1, algebraic geometry 1.2, classical geometry 1.3, linear algebra 1.4, and mechanics 1.5. The advanced reader has not failed to notice that 1.1–1.4 are actually examples of *one and the same manifold* (appearing in different garb):

$$\operatorname{Rot}(3) = V = \mathbb{R}P^3 = \operatorname{SO}(3).$$

To be more precise, the first four manifolds are all "diffeomorphic," i.e., equivalent as smooth manifolds (the definition is given in Section 6.7). As for Example 1.5, $\mathcal{B}(D^2)$ differs from (i.e., is not diffeomorphic to) the other manifolds, because it happens to be diffeomorphic to the three-dimensional sphere S^3 (the beginner should not be discouraged if he fails to see this; it is not obvious).

What is the moral of the story? The history of mathematics teaches us that if the same object appears in different guises in various branches of mathematics and its applications, and plays an important role there, then it should be studied intrinsically, as a separate concept. That was what happened to such fundamental concepts as *group* and *linear space,* and is true of the no less important concept of *smooth manifold.*

1.7. The examples show us that a manifold M is a point set locally like Euclidean space $\mathbb{R}^n$ with global structure not necessarily that of $\mathbb{R}^n$. How does one go about studying such an object? Since there are Euclidean coordinates near each point, we can try to cover M with *coordinate neighborhoods* (or *charts,* or *local coordinate systems*, as they are also called). A family of charts covering M is called an *atlas.* The term is evocative; indeed, a geographical atlas is a set of charts or maps of the manifold S^2 (the Earth's surface) in that sense.

In order to use the separate charts to study the overall structure of M, we must know how to move from one chart to the next, thus "gluing together"

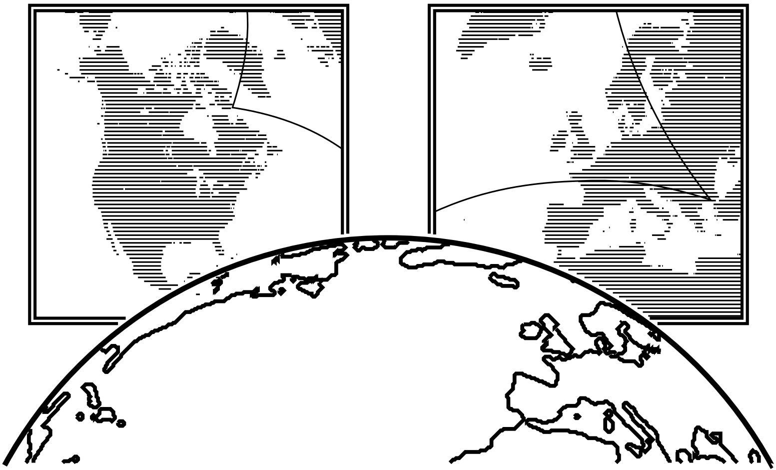


Figure 1.3.

the charts along their common parts, so as to recover M (see Figure 1.3). In less intuitive language, we must be in possession of *coordinate transformations,* expressing the coordinates of points of any chart in terms of those of a neighboring chart. (The industrious reader might profit by actually writing out these transformations for the case of the four-charts atlas of $\mathbb{R}P^3$ described in 1.3.)

If we wish to obtain a *smooth* manifold in this way, we must require that the coordinate transformations be "nice" functions (in a certain sense). We then arrive at the *coordinate* or *classical approach* to smooth manifolds. It is developed in detail in Chapter 5.

1.8. Perhaps more important is the *algebraic approach* to the study of manifolds. In it we forget about charts and coordinate transformations and work only with the $\mathbb{R}$-*algebra* $\mathcal{F}_M$ of *smooth functions* $f\colon\ M \to \mathbb{R}$ on the manifold M. It turns out that $\mathcal{F}_M$ entirely determines M and is a convenient object to work with.

An attempt to give the reader an intuitive understanding of the natural philosophy underlying the algebraic approach is undertaken in the next sections.

1.9. In the description of a classical *physical system* or process, the key notion is the *state* of the system. Thus, in classical mechanics, the state of a moving point is described by its position and velocity at the given moment of time. The state of a given gas from the point of view of thermodynamics is described by its temperature, volume, and pressure, etc. In order to actually assess the state of a given system, the experimentalist must use various *measuring devices* whose *readings* describe the state.

Suppose M is the set of all states of the classical physical system S. Then to each measuring device D there corresponds a function f_D on the set M, assigning to each state $s \in M$ the reading $f_D(s)$ (a real number) that the device D yields in that state. From the physical point of view, we are interested only in those characteristics of each state that can be measured in principle, so that the set M of all states is described by the collection Φ_S of all functions f_D, where the D's are measuring devices (possibly imaginary ones, since it is not necessary — nor indeed practically possible — to construct all possible measuring devices). Thus, theoretically, *a physical system S is nothing more that the collection Φ_S of all functions determined by adequate measuring devices* (*real or imagined*) *on S.*

1.10. Now, if the functions $f_1, \dots, f_k$ correspond to the measuring devices $D_1, \dots, D_k$ of the physical system S, and $\varphi(x_1, \dots, x_k)$ is any "nice" real-valued function in k real variables, then in principle it is possible to construct a device D such that the corresponding function f_D is the composite function $\varphi(f_1, \dots, f_k)$. Indeed, such a device may be obtained by constructing an auxiliary device, synthesizing the value $\varphi(x_1, \dots, x_k)$ from input entries $x_1, \dots, x_k$ (this can always be done if φ is nice enough), and then "plugging in" the outputs $(f_1, \dots, f_k)$ of the devices $D_1, \dots, D_k$ into the inputs $(x_1, \dots, x_k)$ of the auxiliary device. Let us denote this device D by $\varphi(D_1, \dots, D_k)$.

In particular, if we take $\varphi(x_1, x_2) = x_1 + x_2$ (or $\varphi(x) = \lambda x$, $\lambda \in \mathbb{R}$, or $\varphi(x_1, x_2) = x_1 x_2$), we can construct the devices $D_1 + D_2$ (or λD_i, or $D_1 D_2$) from any given devices D_1, D_2. In other words, if $f_i = f_{D_i} \in \Phi_S$, then the functions $f_1 + f_2$, λf_i, $f_1 f_2$ also belong to Φ_S.

Thus the set Φ_S of all functions $f = f_D$ describing the system S has the structure of an *algebra over* $\mathbb{R}$ (or $\mathbb{R}$-*algebra*).

1.11. Actually, the set Φ_S of *all* functions $f_D : M_S \to \mathbb{R}$ is much too large and cumbersome for most classical problems. Systems (and processes) described in classical physics are usually continuous or smooth in some sense. Discontinuous functions f_D are irrelevant to their description; only "smoothly working" measuring devices D are needed. Moreover, the problems of classical physics are usually set in terms of differential equations, so that we must be able to take derivatives of the relevant functions from Φ_S as many times as we wish. Thus we are led to consider, rather than Φ_S, the smaller set $\mathcal{F}_S$ of *smooth functions* $f_D \colon M_S \to \mathbb{R}$.

The set $\mathcal{F}_S$ inherits an $\mathbb{R}$-algebra structure from the inclusion $\mathcal{F}_S \subset \Phi_S$, but from now on we shall forget about Φ, since the *smooth* $\mathbb{R}$-*algebra* $\mathcal{F}_S$ will be our main object of study.

1.12. Let us describe in more detail what the algebra $\mathcal{F}_S$ might be like in classical situations. For example, from the point of view of classical mechanics, a system S of N points in space is adequately described by the positions and velocities of the points, so that we need $6N$ measuring devices D_i to record them. Then the algebra $\mathcal{F}_S$ consists of all elements of the form $\varphi(f_1, \dots, f_{6N})$, where the f_i are the "basic functions" determined by the devices D_i, while $\varphi \colon \mathbb{R}^{6N} \to \mathbb{R}$ is any nice (smooth) function.

In more complicated situations, certain *relations* among the basis functions f_i may arise. For example, if we are studying a system of two mass points joined by a rigid rod of negligible mass, we have the relation

$$\sum_{i=1}^{3}(f_i - f_{i+3})^2 = r^2,$$

where r is the length of the rod and the functions f_i (respectively f_{i+3}) measure the ith coordinate of the first (respectively second) mass point. (There is another relation for the velocity components, which the reader might want to write out explicitly.)

Generalizing, we can say that there usually exists a basis system of devices $D_1, \ldots, D_k$ adequately describing the system S (from the chosen point of view). Then the $\mathbb{R}$-algebra $\mathcal{F}_S$ consists of all elements of the form $\varphi(f_1, \ldots, f_k)$, where $\varphi\colon \mathbb{R}^k \to \mathbb{R}$ is a nice function and the $f_i = f_{D_i}$ are the relevant measurements (given by the devices D_i) that may be involved in relations of the form $F(f_1, \ldots, f_k) \equiv 0$.

Then $\mathcal{F}_S$ may be described as follows. Let $\mathbb{R}^k$ be Euclidean space with coordinates $f_1, \ldots, f_k$ and $U = \{(f_1, \ldots, f_k) \mid a_i < f_i < b_i\}$, where the open intervals $]a_i, b_i[$ contain all the possible readings given by the devices D_i. The relations $F_j(f_1, \ldots, f_k) = 0$ between the basis variables $f_1, \ldots, f_k$ determine a surface M in U. Then $\mathcal{F}_S$ *is the* $\mathbb{R}$*-algebra of all smooth functions on the surface* M.

1.13. Example (thermodynamics of an ideal gas). Consider a certain volume of ideal gas. From the point of view of thermodynamics, we are interested in the following measurements: the volume V, the pressure p, and the absolute temperature T of the gas. These parameters, as is well known, satisfy the relation $pV = cT$, where c is a certain constant. Since $0 < p < \infty$, $0 < V < \infty$, and $0 < T < \infty$, the domain U is the first octant in the space $\mathbb{R}^3 \ni (V, p, T)$, and the hypersurface M in this domain is given by the equation $pV = cT$. The relevant $\mathbb{R}$-algebra $\mathcal{F}$ consists of all smooth functions on M.

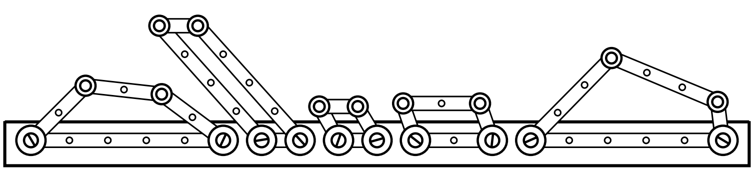


Figure 1.4. Hinge mechanisms $(5; 2, 2, 2)$, $(1; 4, 1, 4)$, $(1; 1, 1, 1)$, $(2; 1, 2, 1)$, $(5; 3, 3, 1)$.

1.14. Example (plane hinge mechanisms). Such a mechanism (see Figure 1.4) consists of $n > 3$ ideal rods in the plane of lengths, say, $(l_1; l_2, \ldots, l_n)$; the rods are joined in cyclic order to each other by ideal hinges at their endpoints; the hinges of the first rod (and hence the rod itself) are fixed to the plane; the other hinges and rods move freely (insofar as the configuration allows them to); the rods can sweep freely over ("through") each other. Obviously, the configuration space of a hinge mechanism is determined completely by the sequence of lengths of its rods. So, one can refer to a concrete mechanism just by indicating the corresponding sequence, for instance, $(5; 2, 3, 2)$. The reader is invited to solve the following problems in the process of reading the book. The first of them she/he can attack even now.

Exercise. Describe the configuration spaces of the following hinge mechanisms:

1. *Quadrilaterals*: $(5; 2, 2, 2)$; $(1; 4, 1, 4)$; $(1; 1, 1, 1)$; $(2; 1, 2, 1)$; $(5; 3, 3, 1)$.
2. *Pentagons*: $(3.9; 1, 1, 1, 1)$; $(1; 4, 1, 1, 4)$; $(6; 6, 2, 2, 6)$; $(1; 1, 1, 1, 1)$.

The reader will enjoy discovering that the configuration space of $(1; 1, 1, 1, 1)$ is the sphere with four handles.

Exercise. Show that the configuration space of a pentagon depends only on the set of lengths of the rods and not on the order in which the rods are joined to each other.

Exercise. Show that the configuration space of the hinge mechanism $(n - \alpha; 1, \dots, 1)$ consisting of $n + 1$ rods is:

1. The sphere S^{n-2} if $\alpha = \frac{1}{2}$.
2. The $(n - 2)$-dimensional torus $T^{n-2} = S^1 \times \cdots \times S^1$ if $\alpha = \frac{3}{2}$.

1.15. So far we have not said anything to explain what a state $s \in M_S$ of our physical system S really is, relying on the reader's physical intuition. But once the set of relevant functions $\mathcal{F}_S$ has been specified, this can easily be done in a mathematically rigorous and physically meaningful way.

The methodological basis of physical considerations is measurement. Therefore, two states of our system must be considered identical if and only if all the relevant measuring devices yield the same readings. Hence each state $s \in M_S$ is entirely determined by the readings in this state on all the relevant measuring devices, i.e., by the correspondence $\mathcal{F}_S \to \mathbb{R}$ assigning to each $f_D \in \mathcal{F}_S$ its reading (in the state s) $f_D(s) \in \mathbb{R}$. This assignment will clearly be an $\mathbb{R}$-algebra homomorphism. Thus we can say, by definition, that *any state s of our system is simply an $\mathbb{R}$-algebra homomorphism $s\colon \mathcal{F}_S \to \mathbb{R}$*. The set of all $\mathbb{R}$-algebra homomorphisms $\mathcal{F}_S \to \mathbb{R}$ will be denoted by $|\mathcal{F}_S|$; it should coincide with the set M_S of all states of the system.

1.16. Summarizing Sections 1.9–1.15, we can say that *any classical physical system is described by an appropriate collection of measuring devices,* *each state of the system being the collection of readings that this state determines on the measuring devices.*

The sentence in italics may be translated into mathematical language by means of the following dictionary:

- physical system = manifold, M;
- state of the system = point of the manifold, $x \in M$;
- measuring device = function on M, $f \in \mathcal{F}$;
- adequate collection of measuring devices = smooth $\mathbb{R}$-algebra, $\mathcal{F}$;
- reading on a device = value of the function, $f(x)$;
- collection of readings in the given state = $\mathbb{R}$-algebra homomorphism

$$x \colon \mathcal{F} \to \mathbb{R}, \quad f \mapsto f(x).$$

The resulting translation reads: *Any manifold M is determined by the smooth $\mathbb{R}$-algebra $\mathcal{F}$ of functions on it, each point x on M being the $\mathbb{R}$-algebra homomorphism $\mathcal{F} \to \mathbb{R}$ that assigns to every function $f \in \mathcal{F}$ its value $f(x)$ at the point x.*

1.17. Mathematically, the crucial idea in the previous sentence is the identification of points $x \in M$ of a manifold and $\mathbb{R}$-algebra homomorphisms $x \colon \mathcal{F} \to \mathbb{R}$ of its $\mathbb{R}$-algebra of functions $\mathcal{F}$, governed by the formula

$$x(f) = f(x). \tag{1.2}$$

This formula, read from left to right, defines the homomorphism $x \colon \mathcal{F} \to \mathbb{R}$ when the functions $f \in \mathcal{F}$ are given. Read from right to left, it defines the functions $f \colon M \to \mathbb{R}$, when the homomorphisms $x \in M$ are known.

Thus formula (1.2) is right in the middle of the important duality relationship existing between points of a manifold and functions on it, a duality similar to, but much more delicate than, the one between vectors and covectors in linear algebra.

1.18. In the general mathematical situation, the identification $M \leftrightarrow |\mathcal{F}|$ between the set M on which the functions $f \in \mathcal{F}$ are defined and the family of all $\mathbb{R}$-algebra homomorphisms $\mathcal{F} \to \mathbb{R}$ cannot be correctly carried out. This is because, first of all, $|\mathcal{F}|$ may turn out to be “much smaller” than M (an example is given in Section 3.6) or “bigger” than M, as we can see from the following example:

Example. Suppose M is the set $\mathbb{N}$ of natural numbers and $\mathcal{F}$ is the set of all functions on $\mathbb{N}$ (i.e., sequences $\{a(k)\}$) such that the limit $\lim_{k\to\infty} a(k)$ exists and is finite. Then the homomorphism

$$\alpha \colon \mathcal{F} \to \mathbb{R}, \quad \{a(k)\} \mapsto \lim_{k\to\infty} a(k),$$

does not correspond to any point of $M = \mathbb{N}$.

◀ Indeed, if α did correspond to some point $n \in \mathbb{N}$, we would have by (1.2)

$$n(a(\cdot)) = a(n),$$

so that

$$\lim_{k\to\infty} a(k) = \alpha(a(\cdot)) = n(a(\cdot)) = a(n)$$

for any sequence $\{a(k)\}$. But this is not the case, say, for the sequence $a_i = 0$, $i \leqslant n$, $a_i = 1$, $i > n$. Thus $|\mathcal{F}|$ is bigger than M, at least by the homomorphism α. ▶

However, we can always add to $\mathbb{N}$ the "point at infinity" ∞ and extend the sequences (elements of $\mathcal{F}$) by putting $a(\infty) = \lim_{k\to\infty} a(k)$, thus viewing the sequences in $\mathcal{F}$ as functions on $\mathbb{N} \cup \{\infty\}$. Then obviously the homomorphism above corresponds to the "point" ∞.

This trick of adding *points at infinity* (or imaginary points, improper points, points of the absolute, etc.) is extremely useful and will be exploited to great advantage in Chapter 8.

1.19. In our mathematical development of the algebraic approach (Chapter 3) we shall start from an $\mathbb{R}$-algebra $\mathcal{F}$ of abstract elements called "functions." Of course, $\mathcal{F}$ will not be just any algebra; it must meet certain "smoothness" requirements. Roughly speaking, the algebra $\mathcal{F}$ must be smooth in the sense that locally (the meaning of that word must be defined in abstract algebraic terms!) it is like the $\mathbb{R}$-algebra $C^\infty(\mathbb{R}^n)$ of infinitely differentiable functions in $\mathbb{R}^n$. This will be the algebraic way of saying that the manifold M is locally like $\mathbb{R}^n$; it will be explained rigorously and in detail in Chapter 3. When the smoothness requirements are met, it will turn out that $\mathcal{F}$ entirely determines the manifold M as the set $|\mathcal{F}|$ of all $\mathbb{R}$-algebra homomorphisms of $\mathcal{F}$ into $\mathbb{R}$, and $\mathcal{F}$ can be identified with the $\mathbb{R}$-algebra of smooth functions on M. The algebraic definition of smooth manifold appears in the first section of Chapter 4.

1.20. Smoothness requirements are also needed in the classical coordinate approach, developed in detail below (see Chapter 5). In particular, coordinate transformations must be infinitely differentiable. The rigorous coordinate definition of a smooth manifold appears in Section 5.8.

1.21. The two definitions of smooth manifold (in which the algebraic approach and the coordinate approach result) are of course equivalent. This is proved in Chapter 7 below. Essentially, this book is a detailed exposition of these two approaches to the notion of smooth manifold and their equivalence, involving many examples, including a more rigorous treatment of the examples given in Sections 1.1–1.5 above.

2

Cutoff and Other Special Smooth Functions on $\mathbb{R}^n$

2.1. This chapter is an auxiliary one and can be omitted on first reading. In it we show how to construct certain specific infinitely differentiable functions on $\mathbb{R}^n$ (the $\mathbb{R}$-algebra of all such functions is denoted by $C^\infty(\mathbb{R}^n)$) that vanish (or do not vanish) on subsets of $\mathbb{R}^n$ of special form. These functions will be useful further on in the proof of many statements, especially in the very important Chapter 3.

2.2 Proposition. *There exists a function $f \in C^\infty(\mathbb{R})$ that vanishes for all negative values of the variable and is strictly positive for its positive values.*

◀ We claim that such is the function

$$f(x) = \begin{cases} 0 & \text{for } x \leqslant 0, \\ e^{-1/x} & \text{for } x > 0 \end{cases} \tag{2.1}$$

(see Figure 2.1 in the background). The only thing that must be checked is that f is smooth, i.e., $f \in C^\infty(\mathbb{R})$.

By induction over n, we shall show that the nth derivative of f is of the form

$$f^{(n)}(x) = \begin{cases} 0 & \text{for } x \leqslant 0, \\ e^{-1/x} P_n(x) x^{-2n} & \text{for } x > 0, \end{cases} \tag{2.2}$$

where $P_n(x)$ is a polynomial, and that $f^{(n)}$ is continuous.

For $n = 0$ this is obvious, since $\lim_{x \to +0} e^{-1/x} = 0$.

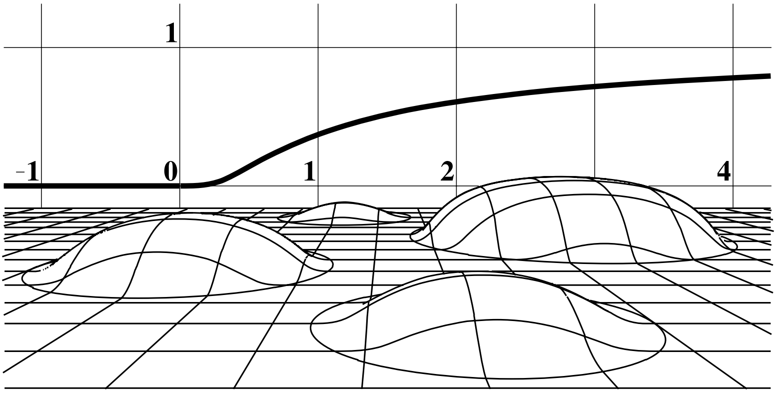


Figure 2.1. Special functions for Proposition 2.2 and Corollary 2.3.

If (2.2) is established for some $n \geqslant 0$, then obviously $f^{(n+1)}(x) = 0$ when $x < 0$, while if $x > 0$, we have

$$f^{(n+1)}(x) = e^{-1/x}\big(P_n(x) + x^2 P_n'(x) - 2nxP_n(x)\big)x^{-2n-2},$$

which shows that $f^{(n+1)}$ is of the form (2.2).

To show that it is continuous, note that $\lim_{x\to+0} e^{-1/x}x^\alpha = 0$ by L'Hospital's rule for any real α. Hence $\lim_{x\to+0} f^{(n+1)}(x) = 0$ and (again by L'Hospital's rule)

$$f^{(n+1)}(0) = \lim_{x\to 0} \frac{f^{(n)}(x) - f^{(n)}(0)}{x} = \lim_{x\to 0} \frac{f^{(n+1)}(x) - 0}{1},$$

so that $f^{(n+1)}$ equals 0 for $x \leqslant 0$ and is continuous for all x. ▶

Exercise. Let f be the function defined in (2.1) and let $c_k = \max\left|f^{(k)}\right|$.

1. Prove that $c_k < \infty$ for all k.
2. Investigate the behavior of the sequence $\{c_k\}$ when $k \to \infty$.

2.3 Corollary. *For any $r > 0$ and $a \in \mathbb{R}^n$ there exists a function $g \in C^\infty(\mathbb{R}^n)$ that vanishes for all $x \in \mathbb{R}^n$ satisfying $\|x - a\| \geqslant r$ and is positive for all other $x \in \mathbb{R}^n$.*

◀ Such is, for example, the function

$$g(x) = f\left(r^2 - \|x - a\|^2\right),$$

where f is the function (2.1) from Section 2.2 (see Figure 2.1). ▶

2.4 Proposition. *For any open set $U \subset \mathbb{R}^n$ there exists a function $f \in C^\infty(\mathbb{R}^n)$ such that*

$$\begin{cases} f(x) = 0, & \textit{if } x \notin U, \\ f(x) > 0, & \textit{if } x \in U. \end{cases}$$

◀ If $U = \mathbb{R}^n$, take $f \equiv 1$; if $U = \varnothing$, take $f \equiv 0$. Now suppose $U \neq \mathbb{R}^n$, $U \neq \varnothing$, and let $\{U_k\}$ be a covering of U by a countable collection of open balls (e.g., all the balls of rational radius centered at the points with rational coordinates and contained in U). By Corollary 2.3, there exist smooth functions $f_k \in C^\infty(\mathbb{R}^n)$ such that $f_k(x) > 0$ if $x \in U_k$ and $f_k(x) = 0$ if $x \notin U_k$. Put

$$M_k = \sup_{\substack{0 \leqslant p \leqslant k \\ p_1+\cdots+p_n=p \\ x\in\mathbb{R}^n}} \left| \frac{\partial^p f_k}{\partial^{p_1} x_1 \cdots \partial^{p_n} x_n}(x) \right|.$$

Note that $M_k < \infty$, since outside the compact set $\overline{U}_k$ (the bar denotes closure) the function f_k and all its derivatives vanish.

Further, the series

$$\sum_{k=1}^{\infty} \frac{f_k}{2^k M_k}$$

converges to a smooth function f, since for all $p_1, \ldots, p_n$ the series

$$\sum_{k=1}^{\infty} \frac{f_k}{2^k M_k} \frac{\partial^{p_1+\cdots+p_n} f_k}{\partial x_1^{p_1} \cdots \partial x_n^{p_n}}$$

converges uniformly (because whenever $k \geqslant p_1+\cdots+p_n$, the absolute value of the kth term is no greater than 2^{-k}).

Clearly, the function f possesses the required properties. ▶

2.5 Corollary. *For any two nonintersecting closed sets A, $B \subset \mathbb{R}^n$ there exists a function $f \in C^\infty(\mathbb{R}^n)$ such that*

$$\begin{cases} f(x) = 0, & \textit{when } x \in A; \\ f(x) = 1, & \textit{when } x \in B; \\ 0 < f(x) < 1, & \textit{for all other } x \in \mathbb{R}^n. \end{cases}$$

◀ Using Proposition 2.4, choose a function f_A that vanishes on A and is positive outside A and a similar function f_B for B. Then for f we can take the function

$$f = \frac{f_A}{f_A + f_B}$$

(see Figure 2.2). ▶

2.6 Corollary. *Suppose $U \subset \mathbb{R}^n$ is an open set and $f \in C^\infty(U)$. Then for any point $x \in U$ there exists a neighborhood $V \subset U$ and a function $g \in C^\infty(\mathbb{R}^n)$ such that $f\big|_V \equiv g\big|_V$.*

◀ Suppose W is an open ball centered at x whose closure is contained in U. Let V be a smaller concentric ball. The required function g can be defined

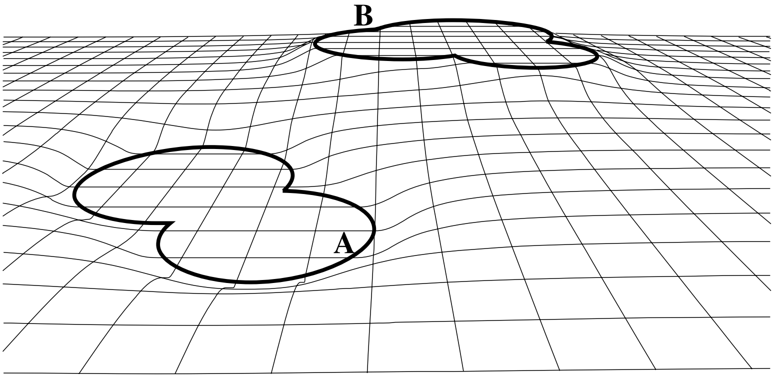


Figure 2.2. Smooth function separating two sets.

as

$$g(y) = \begin{cases} h(y) \cdot f(y), & \text{when } y \in U, \\ 0, & \text{when } y \in \mathbb{R}^n \setminus U, \end{cases}$$

where the function $h \in C^\infty(\mathbb{R}^n)$ is obtained from Corollary 2.3 and satisfies

$$h\big|_{\overline{V}} \equiv 1, \ h\big|_{\mathbb{R}^n \setminus W} \equiv 0. \ ▶$$

2.7 Proposition. *On any nonempty open set $U \subset \mathbb{R}^n$ there exists a smooth function with compact level surfaces, i.e., a function $f \in C^\infty(U)$ such that for any $\lambda \in \mathbb{R}$ the set $f^{-1}(\lambda)$ is compact.*

◀ Denote by A_k the set of points $x \in U$ satisfying both of the following conditions:

(i) $\|x\| \leqslant k$,

(ii) the distance from x to the boundary of U is not less than $1/k$ (if $U = \mathbb{R}^n$, then condition (ii) can be omitted).

Obviously, all points of A_k are interior points of A_{k+1}. Hence A_k and the complement in $\mathbb{R}^n$ to the interior of the set A_{k+1} are two closed nonintersecting sets. By Corollary 2.5 there exists a function $f_k \in C^\infty(\mathbb{R}^n)$ such that

$$\begin{cases} f_k(x) = 0 & \text{if } x \in A_k, \\ f_k(x) = 1 & \text{if } x \notin A_{k+1}, \\ 0 < f_k(x) < 1 & \text{otherwise.} \end{cases}$$

Since any point $x \in U$ belongs to the interior of the set A_k for all sufficiently large k, the sum

$$f = \sum_{k=1}^{\infty} f_k$$

is well defined, and f is smooth on U (locally it is a finite sum of smooth functions).

Consider a point $x \in U \setminus A_k$. Since all the functions f_i are nonnegative, and for $i < k$ we have $f_i(x) = 1$, it follows that $f(x) \geqslant k - 1$. Hence, for any $\lambda \in \mathbb{R}$ the set $f^{-1}(\lambda)$ is a closed subset of the compact set A_k, where k is an integer such that $\lambda < k - 1$. A closed subset of a compact set is always compact, so that f is the required function. ▶

Let us fix a coordinate system $x_1, \ldots, x_n$ in a neighborhood U of a point z. Recall that a domain U is called *starlike* with respect to z if together with any point $y \in U$ it contains the whole interval (z, y).

2.8 Hadamard's lemma. *Any smooth function f in a starlike neighborhood of a point z is representable in the form*

$$f(x) = f(z) + \sum_{i=1}^{n} (x_i - z_i) g_i(x), \tag{2.3}$$

where g_i are smooth functions.

◀ In fact, consider the function

$$\varphi(t) = f(z + (x - z)t).$$

Then $\varphi(0) = f(z)$ and $\varphi(1) = f(x)$, and by Newton–Leibniz formula,

$$\begin{aligned} \varphi(1) - \varphi(0) &= \int_0^1 \frac{d\varphi}{dt} dt = \int_0^1 \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(z + (x_i - z_i)t)(x_i - z_i) dt \\ &= \sum_{i=1}^{n} (x_i - z_i) \int_0^1 \frac{\partial f}{\partial x_i}(z + (x_i - z_i)t) dt. \end{aligned}$$

Since the functions

$$g_i(x) = \int_0^1 \frac{\partial f}{\partial x_i}(z + (x_i - z_i)t) dt$$

are smooth, this concludes the proof of Hadamard's lemma. ▶

Let, as before, $x_1, \ldots, x_n$ be a fixed coordinate system of a point z in a neighborhood U and let $\tau = (i_1, \ldots, i_n)$ be a multi-index. Set

$$|\tau| = i_1 + \cdots + i_n, \qquad \frac{\partial^{|\tau|}}{\partial x^\tau} = \frac{\partial^{|\tau|}}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}},$$
$$(x - z)^\tau = (x_1 - z_1)^{i_1} \cdots (x_n - z_n)^{i_n}, \qquad \tau! = i_1! \cdots i_n!.$$

2.9 Corollary. (Taylor expansion in Hadamard's form.) *Any smooth function f in a starlike neighborhood U of a point z is representable in the form*

$$f(x) = \sum_{|\tau|=0}^{n} \frac{1}{\tau!} (x - z)^\tau \frac{\partial^{|\tau|} f}{\partial x^\tau}(z) + \sum_{|\sigma|=n+1} (x - z)^\sigma g_\sigma(x), \tag{2.4}$$

where $g_\sigma \in C^\infty(U)$.

◀ In fact, using Hadamard's lemma for each function g_i in the decomposition (2.3), the function f can be represented in the form

$$f(x) = f(z) + \sum_{i=1}^{n}(x_i - z_i)g_i(z) + \sum_{i,j=1}^{n}(x_i - z_i)(x_j - z_j)g_{ij}(x).$$

Repeating this procedure for the functions g_{ij}, etc., we shall obtain the decomposition

$$f(x) = f(z) + \sum_{|\tau|=1}^{n}(x - z)^\tau \alpha_\tau + \sum_{|\sigma|=n+1}(x - z)^\sigma g_\sigma(x),$$

where α_τ are constants and $g_\sigma \in C^\infty(U)$. Applying to this equality all kinds of operators of the form $\partial^{|\tau|}/\partial x^\tau(z)$, $|\tau| \leqslant n$, we see that

$$\alpha_\tau = \frac{1}{\tau!}\frac{\partial^{|\tau|} f}{\partial x^\tau}(z). \qquad ▶$$

2.10 Corollary. *Let* $f(x) \in C^\infty(\mathbb{R})$ *and* $f(0) = 0$. *Then we have* $f(x)/x \in C^\infty(\mathbb{R})$.

◀ In fact, by Hadamard's lemma, any smooth function $f(x) \in C^\infty(\mathbb{R})$ is representable in the form $f(x) = f(0) + xg(x)$, where $g(x) \in C^\infty(\mathbb{R})$. If, in addition, $f(0) = 0$, then $f(x)/x = g(x)$. ▶

2.11 Lemma. *If* $f \in C^\infty(\mathbb{R}^n)$ *and* $f(z) = f(y) = 0$, *where* z *and* y *are two different points of the space* $\mathbb{R}^n$, *then the function* f *can be represented as a sum of products* $g_i h_i$, *where* $g_i(z) = 0$, $h_i(y) = 0$.

◀ By a linear coordinate change, the problem can be reduced to the case $z = (0, \ldots, 0, 0)$, $y = (0, \ldots, 0, 1)$. By Hadamard's lemma 2.8, any function f satisfying $f(z) = 0$ is representable in the form

$$f(x) = \sum_{i=1}^{n} x_i g_i(x).$$

Let us represent the function $g_i(x)$ in the form $g_i(x) = \big(g_i(x) - \alpha_i\big) + \alpha_i$, where $\alpha_i = g_i(y)$. Since in the product $x_i\big(g_i(x) - \alpha_i\big)$ the first factor vanishes at the point z, while the second one vanishes at the point y, the problem reduces to the case of a linear function

$$f(x) = \sum_{i=1}^{n} \alpha_i x_i.$$

The condition $f(y) = 0$ means that $\alpha_n = 0$. Representing now x_i, $i < n$, in the form $x_i = x_n x_i - x_i(x_n - 1)$, we conclude the proof of the lemma. ▶

Exercises. 1. Show that any function $f(x, y) \in C^\infty\left(\mathbb{R}^2\right)$ vanishing on the coordinate cross $\mathbf{K} = \{x = 0\} \cup \{y = 0\}$ is of the form

$$f = xy\, g(x, y), \quad g(x, y) \in C^\infty\left(\mathbb{R}^2\right).$$

2. Does a similar result hold if the cross is replaced by the union of the x-axis and the parabola $y = x^2$?

3
Algebras and Points

3.1. This chapter is a mathematical exposition of the algebraic approach to manifolds, which was sketched in intuitive terms in Chapter 1. Here we give a detailed answer to the following fundamental question: Given an *abstract* $\mathbb{R}$-algebra $\mathcal{F}$, find a set (smooth manifold) M whose $\mathbb{R}$-algebra of (smooth) functions can be identified with $\mathcal{F}$.

Further, $\mathcal{F}$ will always be a *commutative*, *associative algebra with unit over* $\mathbb{R}$, or briefly, an $\mathbb{R}$-*algebra*. All $\mathbb{R}$-*algebra homomorphisms* $\alpha\colon \mathcal{F}_1 \to \mathcal{F}_2$, i.e., maps of $\mathcal{F}_1$ into $\mathcal{F}_2$ preserving the operations

$$\alpha(f+g) = \alpha(f) + \alpha(g), \quad \alpha(f \cdot g) = \alpha(f) \cdot \alpha(g), \quad \alpha(\lambda f) = \lambda\alpha(f),$$

are assumed *unital* (i.e., α sends the unit in $\mathcal{F}_1$ into the one in $\mathcal{F}_2$).

We stress that the elements of $\mathcal{F}$, also called "functions," are not really functions at all; they are abstract objects of an unspecified nature. The point is to turn these objects into real functions on a manifold. In order to succeed in this undertaking, we shall successively impose certain conditions on $\mathcal{F}$. The key terms will be *geometric* (Section 3.7), *complete* (Section 3.27), and *smooth* (Section 4.1) $\mathbb{R}$-algebras.

We begin with simple illustrations of how an abstractly defined $\mathbb{R}$-algebra can acquire substance and become a genuine algebra of nice functions on a certain set.

3.2. Example. Suppose $\mathcal{F}$ is the $\mathbb{R}$-algebra of all infinite sequences of real numbers $\{a_i\} = (a_0, a_1, a_2, \ldots)$ such that $a_i = 0$ for all i, except perhaps a finite number. The sum operation and multiplication by elements of $\mathbb{R}$ is defined term by term ($\lambda\{a_i\} = \{\lambda a_i\}$, etc.). The product $\{c_i\}$ of two sequences $\{a_i\}$ and $\{b_i\}$ is defined by the formula

$$c_i = \sum_{k+l=i} a_k b_l.$$

Can this algebra $\mathcal{F}$ be realized as an algebra of nice functions on some set M?

We hope the reader has guessed the answer. By putting

$$\{a_i\} \mapsto \sum_{i \geqslant 0} a_i x^i$$

(this sum is always finite), we obtain an $\mathbb{R}$-algebra isomorphism $\mathcal{F} \to \mathbb{R}[x]$ of $\mathcal{F}$ onto the $\mathbb{R}$-algebra of polynomials in x, $\mathbb{R}[x]$. Thus any sequence $\{a_i\} \in \mathcal{F}$ may be viewed as the function on $M = \mathbb{R}$ given by $x \mapsto \sum_{i \geqslant 0} a_i x^i$.

3.3. Exercise. Suppose that the $\mathbb{R}$-algebras $\mathcal{F}_1$ and $\mathcal{F}_2$, as linear spaces, are isomorphic to the plane $\mathbb{R}^2 = \{(x, y)\}$. Let the multiplication in $\mathcal{F}_1$ and $\mathcal{F}_2$ be respectively given by

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1 x_2, y_1 y_2),$$
$$(x_1, y_1) \cdot (x_2, y_2) = (x_1 x_2 + y_1 y_2, x_1 y_2 + x_2 y_1).$$

Find the set (manifold) M_i for which the algebra $\mathcal{F}_i$, $i = 1, 2$, is the algebra of smooth functions, explicitly indicating what function on M_i corresponds to the element $(x, y) \in \mathcal{F}_i$. Are the algebras $\mathcal{F}_1$ and $\mathcal{F}_2$ isomorphic?

3.4. We now return to our given abstract $\mathbb{R}$-algebra $\mathcal{F}$. Recalling the philosophy of Section 1.16 (a point of a manifold or state of a physical system is determined by all the relevant measurements), we introduce the following notations and definitions.

Denote by $M = |\mathcal{F}|$ the set of all $\mathbb{R}$-algebra homomorphisms of $\mathcal{F}$ onto $\mathbb{R}$:

$$M \ni x \colon \mathcal{F} \to \mathbb{R}, \quad f \mapsto x(f).$$

The elements of M will sometimes be called $\mathbb{R}$-*points* for the algebra $\mathcal{F}$ (they will indeed be the points of our future manifold), and $|\mathcal{F}|$ the *dual space* of $\mathbb{R}$-points. Further, set

$$\widetilde{\mathcal{F}} = \left\{ \tilde{f} \colon M \to \mathbb{R} \mid \tilde{f}(x) = x(f), \quad f \in \mathcal{F} \right\}. \tag{3.1}$$

The set $\widetilde{\mathcal{F}}$ has a natural $\mathbb{R}$-algebra structure given by

$$\begin{aligned} \left(\tilde{f} + \tilde{g}\right)(x) &= \tilde{f}(x) + \tilde{g}(x) = x(f) + x(g), \\ \left(\tilde{f} \cdot \tilde{g}\right)(x) &= \tilde{f}(x) \cdot \tilde{g}(x) = x(f) \cdot x(g), \\ \left(\lambda \tilde{f}\right)(x) &= \lambda \tilde{f}(x) = \lambda x(f). \end{aligned} \tag{3.2}$$

There is a natural map

$$\tau \colon \mathcal{F} \to \widetilde{\mathcal{F}}, \quad f \mapsto \tilde{f}.$$

We would like this map to be an isomorphism: Then we could view $\widetilde{\mathcal{F}}$ as a realization of $\mathcal{F}$ in the form of an $\mathbb{R}$-algebra of functions on the dual space $M = |\mathcal{F}|$. But is this the case?

3.5. First we note that $\tau\colon \mathcal{F} \to \widetilde{\mathcal{F}},\ f \mapsto \tilde{f}$, *is a homomorphism.*
◀ Indeed, by definition of $\widetilde{\mathcal{F}}$ (and because any $x \in M$ is a homomorphism),

$$\left(\widetilde{f+g}\right)(x) = x(f+g) = x(f) + x(g) = \tilde{f}(x) + \tilde{g}(x) = \left(\tilde{f} + \tilde{g}\right)(x).$$

The other two verifications are similar, and we leave them to the industrious reader. ▶

It is also obvious that τ *is surjective.*

Thus it remains to show that τ is injective. Unfortunately, this is not so in the general case.

3.6. Example. Suppose $\mathcal{F}$ is the $\mathbb{R}$-algebra isomorphic (as a linear space) to the plane $\mathbb{R}^2 = \{(x,y) \mid x, y \in \mathbb{R}\}$ with the product

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1x_2, x_1y_2 + x_2y_1).$$

We shall show that the dual space $M = |\mathcal{F}|$ consists of a single point. This implies that τ is not injective, since $\widetilde{\mathcal{F}}$ is then isomorphic to $\mathbb{R}$, while $\mathcal{F}$ is not ($\mathcal{F} \supset \{(x,0),\ x \in \mathbb{R}\} \cong \mathbb{R}$).

The element $(1,0)$ is obviously the unit of the algebra $\mathcal{F}$, and any element (x,y) has an inverse if $x \neq 0$, namely $(x,y)^{-1} = \left(x^{-1}, -yx^{-2}\right)$. Hence the only ideal of the algebra $\mathcal{F}$, other than $\{0\}$ and $\mathcal{F}$, is the *ideal* $\mathcal{I} = \{(0,y) \mid y \in \mathbb{R}\}$. The quotient algebra $\mathcal{F}/\mathcal{I}$ is naturally isomorphic to $\mathbb{R}$, the quotient map $q\colon \mathcal{F} \to \mathcal{F}/\mathcal{I} = \mathbb{R}$ being the projection $(x,y) \mapsto x$. This map is the only surjective $\mathbb{R}$-algebra homomorphism $\mathcal{F} \to \mathbb{R}$, so that $M = \{q\}$.

3.7. In order to be able to assert that $\tau\colon \mathcal{F} \to \widetilde{\mathcal{F}}$ is injective (and hence an isomorphism), certain conditions must be imposed on $\mathcal{F}$. Note that τ *will be injective iff the ideal* $\mathcal{I}(\mathcal{F}) = \bigcap_{x\in M} \operatorname{Ker} x$ *is trivial*.
◀ Indeed,

$$\begin{aligned} f \in \operatorname{Ker}\tau &\iff \tau(f) = \tilde{f} = 0 \\ &\iff \tilde{f}(x) = x(f) = 0 \quad \forall x \in M \\ &\iff f \in \bigcap_{x\in M} \operatorname{Ker} x = \mathcal{I}(\mathcal{F}), \end{aligned}$$

and therefore $\operatorname{Ker}\tau = 0 \iff \mathcal{I}(\mathcal{F}) = 0$. ▶

This motivates (mathematically) the following definition:

Definition. An $\mathbb{R}$-algebra $\mathcal{F}$ is called *geometric* if

$$\mathcal{I}(\mathcal{F}) = \bigcap_{x\in|\mathcal{F}|} \operatorname{Ker} x = 0.$$

(In the previous example, $\mathcal{I}(\mathcal{F}) = \mathcal{I} = \{(0,y) \mid y \in \mathbb{R}\} \neq 0$).

It is worth noticing that an algebra with empty dual space is not geometric.

Exercises. 1. Prove that the polynomial algebra $\mathbb{R}[x_1, \ldots, x_n]$ is geometric.

2. Let V be a finite-dimensional vector space over $\mathbb{R}$ and let $G: V \to V$ be a linear operator. Consider the algebra $\mathcal{F}_G$ generated by G^k, $k = 0, 1, \ldots,$ as a vector space. Characterize the operators G for which $\mathcal{F}_G$ is geometric.

3.8. In Sections 3.5, 3.7, we have in fact proved the following theorem:

Theorem. *Any geometric $\mathbb{R}$-algebra $\mathcal{F}$ is canonically isomorphic to the $\mathbb{R}$-algebra $\widetilde{\mathcal{F}}$ of functions defined on the dual space $M = |\mathcal{F}|$ of $\mathbb{R}$-points ($M \ni x: \mathcal{F} \to \mathbb{R}$) by the rule $\widetilde{f}(x) = x(f)$.*

Having this isomorphism in mind, we shall identify our abstract algebra $\mathcal{F}$ (which will usually be assumed geometric) with the $\mathbb{R}$-algebra $\widetilde{\mathcal{F}}$ of functions on the dual space $M = |\mathcal{F}|$ once and for all. The notation $\widetilde{\mathcal{F}}$ will be abandoned; the elements $f \in \mathcal{F}$ will often be viewed as functions $M \to \mathbb{R}$.

3.9 Exercises. Check which of the following algebras are geometric:

1. The formal series algebra $\mathbb{R}[[x_1, \ldots, x_n]]$.
2. The quotient algebra
$$\mathbb{R}[x_1, \ldots, x_n]/f^k\mathbb{R}[x_1, \ldots, x_n], \ f \in \mathbb{R}[x_1, \ldots, x_n].$$
3. The algebra of germs of smooth functions at $0 \in \mathbb{R}^n$.
4. The algebra of all smooth bounded functions.
5. The algebra of all smooth periodic functions (of period 1) on $\mathbb{R}$ (see Section 3.18).
6. The subalgebra of the previous algebra consisting of all even functions.
7. The algebras $\mathcal{F}_1$ and $\mathcal{F}_2$ described in Exercise 3.3.
8. The algebra of all differential operators in $\mathbb{R}^n$ with constant coefficients (multiplication in this algebra is the composition of operators).

3.10. The algebra $\mathcal{F}_S$ of functions corresponding to measuring devices of a classical physical system S (see Sections 1.9, 1.15) is always geometric. This property is the mathematical formulation of the classical physical postulate asserting that if all the readings of two different devices for all the states of the system S are the same, then these two devices measure the same physical parameter (i.e., only one of the devices is needed).

3.11 Proposition. *For an arbitrary $\mathbb{R}$-algebra $\mathcal{F}$, the quotient $\mathbb{R}$-algebra*
$$\mathcal{F}/\mathcal{I}(\mathcal{F}), \quad \textit{where} \quad \mathcal{I}(\mathcal{F}) = \bigcap_{p \in |\mathcal{F}|} \operatorname{Ker} p,$$

is geometric and $|\mathcal{F}| = |\mathcal{F}/\mathcal{I}(\mathcal{F})|$.

◀ Define the map $\varphi\colon |\mathcal{F}/\mathcal{I}(\mathcal{F})| \to |\mathcal{F}|$ by assigning to each homomorphism $b\colon \mathcal{F}/\mathcal{I}(\mathcal{F}) \to \mathbb{R}$ the homomorphism $\varphi(b) = a = b \circ \mathrm{pr}$, where pr is the quotient map $\mathrm{pr}\colon \mathcal{F} \to \mathcal{F}/\mathcal{I}(\mathcal{F})$.

We claim that φ is bijective. Obviously, $b_1 \neq b_2$ implies $a_1 \neq a_2$, so φ is injective. Now suppose $a \in |\mathcal{F}|$. Then $\operatorname{Ker} a \supset \mathcal{I}(\mathcal{F})$. Hence the element $b([f]) = a(f)$, where $[f]$ is the coset of the element f modulo $\mathcal{I}(\mathcal{F})$, is well defined and determines a homomorphism $b\colon \mathcal{F}/\mathcal{I}(\mathcal{F}) \to \mathbb{R}$. Clearly, $a = \varphi(b)$, i.e., the map φ is surjective, so that φ identifies $|\mathcal{F}|$ with $|\mathcal{F}/\mathcal{I}(\mathcal{F})|$.

Suppose further that $b \in |\mathcal{F}/\mathcal{I}(\mathcal{F})|$ and $a = \varphi(b) = b \circ \mathrm{pr}$. Then $\operatorname{Ker} b = \operatorname{Ker} a/\mathcal{I}(\mathcal{F})$. Hence

$$\mathcal{I}(\mathcal{F}/\mathcal{I}(\mathcal{F})) = \bigcap_{b \in |\mathcal{F}/\mathcal{I}(\mathcal{F})|} \operatorname{Ker} b = \bigcap_{a \in \mathcal{F}} \big(\operatorname{Ker} a/\mathcal{I}(\mathcal{F})\big)$$
$$= \Big(\bigcap_{a \in \mathcal{F}} \operatorname{Ker} a\Big) \Big/ \mathcal{I}(\mathcal{F}) = \mathcal{I}(\mathcal{F})/\mathcal{I}(\mathcal{F}) = \{0\}. \quad ▶$$

3.12. Given a geometric $\mathbb{R}$-algebra $\mathcal{F}$, we intend to introduce a topology in the dual set $M = |\mathcal{F}|$ of $\mathbb{R}$-points.

From the physical point of view, two states s_1, s_2 of a classical system S (two $\mathbb{R}$-points) are near each other if all the readings of the relevant measuring devices are close, i.e., for all measuring devices D we must have

$$f_D(s_2) \in \left]f_D(s_1) - \varepsilon,\ f_D(s_1) + \varepsilon\right[.$$

Mathematically, we express this by saying that the topology in M is given by the basis of open sets of the form $f^{-1}(V)$, where $V \subset \mathbb{R}$ is open and $f \in \mathcal{F}$. (The reader should recall at this point that the expression f^{-1} is meaningful only because we have identified $\mathcal{F}$ with an algebra of functions $f\colon M \to \mathbb{R}$.)

Another way of saying this is the following: *The topology in the dual space* $M = |\mathcal{F}|$ *is the weakest for which all the functions in* $\mathcal{F}$ *are continuous.*

3.13 Proposition. *The topology introduced in* Section 3.12 *in the dual space* $M = |\mathcal{F}|$ *is that of a Hausdorff space.*

◀ Suppose x and y are distinct points of $|\mathcal{F}|$, i.e., different homomorphisms of $\mathcal{F}$ into $\mathbb{R}$. This means there is an $f \in \mathcal{F}$ for which $f(x) \neq f(y)$, say $f(x) < f(y)$. Then the sets

$$f^{-1}\left(\left]-\infty, \frac{f(x)+f(y)}{2}\right[\right), \quad f^{-1}\left(\left]\frac{f(x)+f(y)}{2}, +\infty\right[\right)$$

are nonintersecting neighborhoods of the points x and y. ▶

When speaking of the "space" $M = |\mathcal{F}|$, it is this topological (Hausdorff) structure that will always be understood.

3.14. In this section we assume that $\mathcal{F}_0$ is any $\mathbb{R}$-algebra of functions on a given set M_0. Then there is a natural map $\theta\colon M_0 \to |\mathcal{F}_0|$ assigning to each point $a \in M_0$ the homomorphism $f \mapsto f(a)$. In other words, $\theta(a)(f) = f(a)$, $a \in M_0$, and therefore if an element $f \in \mathcal{F}_0$, viewed as a function on the dual space $|\mathcal{F}_0|$, vanishes on $\theta(M_0)$, then f is the zero element of $\mathcal{F}_0$. In particular, the algebra $\mathcal{F}_0$ will be geometric, and we have the following result:

Proposition. *If $\mathcal{F}_0$ is a subalgebra of the $\mathbb{R}$-algebra of continuous functions on the topological space M_0, then the map $\theta\colon M_0 \to |\mathcal{F}_0|$, $a \mapsto \big(f \mapsto f(a)\big)$, is continuous.*

◀ Suppose $U = f^{-1}(V)$ is a basis open set in $|\mathcal{F}_0|$. By definition U consists of all the homomorphisms $\mathcal{F}_0 \to \mathbb{R}$ that send the (fixed) function $f \in \mathcal{F}$ to some point of the open set $V \subset \mathbb{R}$. Then the inverse image $\theta^{-1}(U)$ consists of all points $a \in M_0$ such that $f(a) \in V$ and is therefore an open subset of M. ▶

It should be noted that in our general situation (when $\mathcal{F}_0$ is any geometric $\mathbb{R}$-algebra, $M_0 = |\mathcal{F}_0|$), M_0 is a topological space and the elements of $\mathcal{F}_0$ are continuous functions (see Section 3.12), so that the proposition proved above applies.

Exercise. Describe the dual space for each of the following algebras:

1. $\mathbb{R}[x, y]/xy\mathbb{R}[x, y]$;
2. $\mathbb{R}[x, y, z]/\left(x^2 + y^2 + z^2 - 1\right)\mathbb{R}[x, y, z]$.

3.15. Example. Suppose $\mathcal{F} = \mathbb{R}[x_1, \ldots, x_n]$ is the $\mathbb{R}$-algebra of polynomials in n variables. Every homomorphism $a\colon \mathcal{F} \to \mathbb{R}$ is determined by the "vector" $(\lambda_1, \ldots, \lambda_n)$, where $\lambda_i = a(x_i)$, since

$$a\left(\sum_{k_1,\ldots,k_n} c_{k_1\ldots k_n} x_1^{k_1} \cdots x_n^{k_n}\right) = \sum_{k_1,\ldots,k_n} c_{k_1\ldots k_n} \left(a(x_1)\right)^{k_1} \cdots \left(a(x_n)\right)^{k_n}$$

$$= \sum_{k_1,\ldots,k_n} c_{k_1\ldots k_n} \lambda_1^{k_1} \cdots \lambda_n^{k_n}.$$

Moreover, the map

$$\mathcal{F} \ni \sum_{k_1,\ldots,k_n} c_{k_1\ldots k_n} x_1^{k_1} \cdots x_n^{k_n} \longmapsto \sum_{k_1,\ldots,k_n} c_{k_1\ldots k_n} \lambda_1^{k_1} \cdots \lambda_n^{k_n} \in \mathbb{R}$$

is a homomorphism of the algebra $\mathcal{F}$ into $\mathbb{R}$ for all $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. Thus the dual space $|\mathcal{F}|$ in this case is naturally identified with $\mathbb{R}^n = \{(\lambda_1, \ldots, \lambda_n)\}$.

The topology defined in $|\mathcal{F}|$ (see Section 3.12) coincides with the usual topology of $\mathbb{R}^n$.

◀ Indeed, the sets $f^{-1}(V)$, where f is a polynomial and $V \subset \mathbb{R}$ is open, are open in $\mathbb{R}^n = |\mathcal{F}|$, since polynomials are continuous functions. Moreover, a ball of radius r with center $(b_1, \dots, b_n)$ in $\mathbb{R}^n = |\mathcal{F}|$ is of the form $f^{-1}(\mathbb{R}_+)$, where $\mathbb{R}_+$ is the positive half of $\mathbb{R}$, if we take f to be

$$f(x_1, \dots, x_n) = r^2 - \sum_{i=1}^{n} (b_i - x_i)^2.$$

Since such balls constitute a basis for the usual topology in $\mathbb{R}^n$, the two topologies coincide. ▶

3.16. Example. Suppose $\mathcal{F} = C^\infty(U)$ is the $\mathbb{R}$-algebra of infinitely differentiable real-valued functions on an open subset U of $\mathbb{R}^n$. Consider the map

$$\theta\colon U \to |\mathcal{F}|, \quad x \mapsto (f \mapsto f(x)).$$

We claim that the *map* θ *is a homeomorphism*, so that the dual space $|C^\infty(U)|$ is homeomorphic to U.

◀ Since injectivity is obvious (elements of $\mathcal{F}$ being functions on U), we first prove the surjectivity of θ. Suppose $p \in |\mathcal{F}|$, i.e., $p\colon \mathcal{F} \to \mathbb{R}$ is an $\mathbb{R}$-algebra homomorphism onto $\mathbb{R}$. Choose a smooth function $f_c \in C^\infty(U)$ all of whose level surfaces are compact (such a functions exists by Proposition 2.7). Then, in particular, the set $L = f_c^{-1}(\lambda)$, where $\lambda = p(f)$, is compact. Assume that $p \in |\mathcal{F}|$ does not correspond to any point of U. Then for any point $a \in U$ there exists a function $f_a \in \mathcal{F}$ for which $f_a(a) \neq p(f_a)$. The sets

$$U_a = \{x \in U \mid f_a(x) \neq p(f_a)\}, \quad a \in L,$$

constitute an open covering of L. Since L is compact, we can choose a finite subcovering $U_{a_1}, \dots, U_{a_m}$. Consider the function

$$g = (f - p(f))^2 + \sum_{i=1}^{m} (f_{a_i} - p(f_{a_i}))^2.$$

This is a smooth nonvanishing function on U, so that $1/g \in \mathcal{F}$. Since p is a (unital!) $\mathbb{R}$-algebra homomorphism, we must have

$$p(1) = p(g \cdot (1/g)) = p(g) \cdot p(1/g) = 1. \tag{3.3}$$

But by the definition of g,

$$p(g) = (p(f) - p(f))^2 + \sum (p(f_{a_i}) - p(f_{a_i}))^2 = 0,$$

which contradicts (3.3), proving the surjectivity of θ.

The fact that θ is a homeomorphism is an immediate consequence of Proposition 3.14. ▶

In particular, we have proved that $|C^\infty(\mathbb{R}^n)| = \mathbb{R}^n$.

3.17 Exercises. Describe the dual space for each of the following algebras:

1. $C^\infty(\mathbb{R}^3) / (x^2 + y^2 + z^2 - 1)\, C^\infty(\mathbb{R}^3)$.

2. $C^\infty(\mathbb{R}^3) / (x^2+y^2-z^2) C^\infty(\mathbb{R}^3)$.
3. Smooth even functions on the real line.
4. Smooth even functions of period 1 on the real line.
5. Smooth functions of rational period (not necessarily the same) on the real line.
6. Functions defined on the real line as the ratio of two polynomials $p(x)/q(x)$, where $q(x) \neq 0$ for all $x \in \mathbb{R}$.
7. The same functions as before, but with the additional requirement $\deg p(x) \leqslant \deg q(x)$.
8. Functions defined on the real line as the ratio of two polynomials $p(x)/q(x)$, where $q(x)$ is not identically zero (i.e., defined as rational functions).
9. The subalgebra $\{f \in C^\infty(\mathbb{R}^2) \mid f(x+1,y) = f(x,y)\}$ of $C^\infty(\mathbb{R}^2)$.
10. The subalgebra $\{f \in C^\infty(\mathbb{R}^2) \mid f(x+1,-y) = f(x,y)\}$ of $C^\infty(\mathbb{R}^2)$.
11. The subalgebra $\{f \in C^\infty(\mathbb{R}^2) \mid f(x,y+1) = f(x,y) = f(x+1,y)\}$ of $C^\infty(\mathbb{R}^2)$.
12. The subalgebra
$$\{f \in C^\infty(\mathbb{R}^3 \setminus 0) \mid f(x,y,z) = f(\lambda x, \lambda y, \lambda z), \quad \forall\, \lambda \neq 0\}$$
of $C^\infty(\mathbb{R}^3 \setminus 0)$.
13. The subalgebra $\{f \in C^\infty(\mathbb{R}^2) \mid f(x+1,-y) = f(x,y) = f(x,y+1)\}$ of $C^\infty(\mathbb{R}^2)$.
14. The subalgebra
$$\{f \in C^\infty(\mathbb{R}^3 \setminus 0) \mid f(x,y,z) = f(\lambda x, \lambda y, \lambda z), \quad \forall \lambda \in \mathbb{R}_+\}$$
of $C^\infty(\mathbb{R}^3 \setminus 0)$.

3.18. Example. Suppose $\mathcal{F}$ consists of all periodic smooth functions of period 1 on the line $\mathbb{R}$. Then, as usual, each point $a \in \mathbb{R}$ determines the homomorphism $\mathcal{F} \to \mathbb{R}$, $f \mapsto f(a)$. But different points can give rise to the same homomorphism; this happens iff the distance between the points is an integer.

We claim that there are no homomorphisms other than the ones determined by the points $a \in \mathbb{R}$.

◀ The proof is similar to the one in the previous section. Namely, if $p\colon \mathcal{F} \to \mathbb{R}$ is not determined by any point, then for any $a \in \mathbb{R}$ there exists a function $f_a \in \mathcal{F}$ such that $p(f_a) \neq f_a(a)$. From the open covering of the closed interval $[0,1]$ by sets of the form
$$U_a = \{x \in \mathbb{R} \mid f_a(x) \neq p(f_a)\}, \quad a \in \mathbb{R},$$

choose a finite subcovering $U_{a_1}, \dots, U_{a_n}$. The function

$$g = \sum_{i=1}^{m} (f_{a_i} - p(f_{a_i}))^2$$

does not vanish anywhere on $[0, 1]$ and, by periodicity, anywhere on $\mathbb{R}$. Hence $1/g \in \mathcal{F}$, etc., just as in Section 3.16. ▶

Thus in our case $|\mathcal{F}|$ can be identified with the quotient space $\mathbb{R}/\mathbb{Z}$, where $\mathbb{Z}$ is the subgroup of integers in $\mathbb{R}$. Of course, $\mathbb{R}/\mathbb{Z} = S^1$ is the circle. Thus we have shown rigorously that *smooth periodic functions of period* 1 *on* $\mathbb{R}$ *are actually functions on the circle*, which is in accord with our intuitive understanding of such functions.

3.19. Now suppose $\mathcal{F}_1$ and $\mathcal{F}_2$ are two geometric $\mathbb{R}$-algebras and $\varphi\colon \mathcal{F}_1 \to \mathcal{F}_2$ is an $\mathbb{R}$-algebra homomorphism. Then for the dual spaces of $\mathbb{R}$-points $|\mathcal{F}_1|$ and $|\mathcal{F}_2|$ the dual map $|\varphi|$ arises:

$$|\varphi|\colon |\mathcal{F}_2| \to |\mathcal{F}_1|, \quad x \mapsto x \circ \varphi.$$

We claim that *the map* $|\varphi|$ *is continuous.*

◀ Indeed, take a basis open set $U = f^{-1}(V) \subset |\mathcal{F}_1|$, where $f \in \mathcal{F}_1$ and $V \subset \mathbb{R}$ is open. Then U consists of all points $x \in |\mathcal{F}_1|$ such that $f(x) \in V$. The inverse image of U by $|\varphi|$ consists of all points $y \in |\mathcal{F}_2|$ such that $|\varphi|(y) \in U$, i.e., $f(|\varphi|(y)) \in V$. But

$$f(|\varphi|(y)) = f(y \circ \varphi) = (y \circ \varphi)(f) = y(\varphi(f)) = \varphi(f)(y)$$

(the reader should check each of these relations!). Therefore, the set $|\varphi|^{-1}(U)$ consists of all points $y \in |\mathcal{F}_2|$ for which $\varphi(f)(y) \in V$; thus the set $|\varphi|^{-1}(U)$ is open. ▶

3.20. If $\varphi_1\colon \mathcal{F}_1 \to \mathcal{F}_2$, $\varphi_2\colon \mathcal{F}_2 \to \mathcal{F}_3$ are $\mathbb{R}$-algebra homomorphisms of geometric $\mathbb{R}$-algebras $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$, then obviously $|\varphi_2 \circ \varphi_1| = |\varphi_1| \circ |\varphi_2|$ and $|\operatorname{id}_{\mathcal{F}_i}| = \operatorname{id}_{|\mathcal{F}_i|}$. Further, if $\varphi\colon \mathcal{F}_1 \to \mathcal{F}_2$ has an inverse homomorphism φ^{-1}, then

$$\left|\varphi^{-1}\right| = |\varphi|^{-1}.$$

In particular, if φ is an isomorphism, then $|\varphi|$ is a homeomorphism.

3.21. Having started from an abstract geometric $\mathbb{R}$-algebra $\mathcal{F}$, we have constructed the (Hausdorff) topological space $M = |\mathcal{F}|$ (the dual space of $\mathbb{R}$-points), for which $\mathcal{F}$ is a subalgebra of the algebra of all continuous functions. It might now seem that we need only postulate that $\mathcal{F}$ be locally isomorphic to $C^\infty(\mathbb{R}^n)$ (i.e., cover M with a family of sets E_i, $M = \bigcup E_i$, such that the restriction of $\mathcal{F}$ to each E_i is isomorphic to $C^\infty(\mathbb{R}^n)$), and our program of defining a manifold in terms of its $\mathbb{R}$-algebra of functions will be carried out. Unfortunately, things are not as simple as they appear at first glance: Certain technical difficulties, related to the notion of *restriction*, must be overcome before we succeed in implementing our program.

3.22. Example. Suppose $\mathcal{F} = C^\infty(\mathbb{R})$ and $\mathbb{R}_+ \subset \mathbb{R}$ is the set of positive real numbers. We would like to obtain the algebra of smooth functions on $\mathbb{R}_+$ as a "restriction" of the algebra $\mathcal{F}$. But consider the function $x \mapsto 1/x$ on R_+; it is certainly a smooth function on $\mathbb{R}_+$, but clearly is not the restriction of any function $f \in \mathcal{F} = C^\infty(\mathbb{R})$. How can such functions be obtained from $\mathcal{F}$?

3.23. Definition. Suppose $\mathcal{F}$ is a geometric $\mathbb{R}$-algebra and $A \subset |\mathcal{F}|$ is any subset of its dual space $|\mathcal{F}|$; the *restriction* $\mathcal{F}\big|_A$ of $\mathcal{F}$ to A is the set of all functions $f\colon A \to \mathbb{R}$ such that for any point $a \in A$ there exists a neighborhood $U \subset A$ and an element $\bar{f} \in \mathcal{F}$ such that the (ordinary) restriction of f to U coincides with the restriction of $\bar{f}$ (understood as a function on $|\mathcal{F}|$) to U.

Obviously, $\mathcal{F}\big|_A$ is an $\mathbb{R}$-algebra.

Now we can return to Example 3.22. We claim that the function $x \mapsto 1/x$ belongs to $C^\infty(\mathbb{R})\big|_{\mathbb{R}_+}$.

◀ Indeed, for any point $a > 0$ there exists (see Section 2.5) a function $\alpha \in C^\infty(\mathbb{R})$ that vanishes when $x \leqslant a/3$ and equals 1 whenever $x \geqslant 2a/3$. For $\bar{f}$ take the function that vanishes when $x \leqslant 0$ and equals $\alpha(x)/x$ when $x > 0$. Obviously, $\bar{f}$ is smooth and coincides with the function $x \mapsto 1/x$ in the neighborhood $]2a/3, 4a/3[$ of the point a. ▶

In a similar way we can show that any smooth function on $\mathbb{R}_+$ belongs to $C^\infty(\mathbb{R})\big|_{\mathbb{R}_+}$; i.e., we have

$$C^\infty(\mathbb{R})\big|_{\mathbb{R}_+} = C^\infty(\mathbb{R}_+).$$

This statement has the following generalization.

3.24 Proposition. *If $\mathcal{F} = C^\infty(U)$, where $U \subset \mathbb{R}^n$ is not empty and open, while V is open in $U = |\mathcal{F}|$, then $\mathcal{F}\big|_V = C^\infty(V)$.*

◀ The identification $U = |\mathcal{F}|$ in the statement of the proposition was established in Section 3.16. Suppose $f \in C^\infty(V)$ and $x \in V$. By 2.6 there exists a neighborhood W of the point x such that $W \subset V$ and a function $g \in C^\infty(\mathbb{R}^n)$ such that $g\big|_W = f\big|_W$. If $\bar{g} = g\big|_U$, then we also have $\bar{g}\big|_W = f\big|_W$. Thus

$$f \in C^\infty(U)\big|_V; \text{ i.e., } C^\infty(V) \subset C^\infty(U)\big|_V.$$

The inverse inclusion immediately follows from the definition of the algebra $C^\infty(U)\big|_V$. ▶

Exercise. For the subsets $A = \{1/n \mid n = 1, 2, 3, \dots\}$ and $B = A \cup \{0\}$ in $\mathbb{R}$ describe the restrictions $\mathbb{R}[x]\big|_A$ and $\mathbb{R}[x]\big|_B$.

3.25. In the general case, in which $\mathcal{F}$ is any geometric $\mathbb{R}$-algebra and A is a subset of the dual space $|\mathcal{F}|$, we can assign to every function $f \in \mathcal{F}$ its restriction to $A \subset |\mathcal{F}|$, which obviously belongs to $\mathcal{F}\big|_A$. Thus we obtain the *restriction homomorphism*

$$\rho_A : \mathcal{F} \to \mathcal{F}\big|_A, \quad f \mapsto f\big|_A.$$

(Here as usual the element $f \in \mathcal{F}$ is viewed as a function on the dual space $|\mathcal{F}|$.)

Proposition. *Suppose* $i\colon \mathcal{F}_1 \to \mathcal{F}_2$ *is an isomorphism of two geometric algebras,* $A_2 \subset |\mathcal{F}_2|$, $A_1 = |i|(A_2)$. *Then the map*

$$\mathcal{F}_1\big|_{A_1} \to \mathcal{F}_2\big|_{A_2}, \quad f \mapsto f\left(|i|\big|_{A_2}\right)$$

is an isomorphism.

◀ The proof is a straightforward verification of definitions. ▶

3.26. Now, in the most important particular case $A = |\mathcal{F}|$, we can consider the restriction homomorphism $\rho\colon \mathcal{F} \to \mathcal{F}\big|_{|\mathcal{F}|}$. Since $\mathcal{F}$ is assumed geometric (different elements $f \in \mathcal{F}$ are identified with different functions on $|\mathcal{F}|$), ρ *is injective.* Surjectivity, surprisingly enough, is *not* obvious: By the definition given in Section 3.23, $\mathcal{F}\big|_{|\mathcal{F}|}$ consists of all functions that are locally like those of $\mathcal{F}$, but it is not clear why all such functions belong to $\mathcal{F}$. Indeed, this is not always the case.

3.27. Example. Suppose $\mathcal{F}$ is the subalgebra of the algebra $C^\infty(\mathbb{R}^n)$ consisting of functions each of which is less in absolute value than some polynomial. Then the dual space $|\mathcal{F}|$ is homeomorphic to $\mathbb{R}^n$.

◀ The proof is similar to the one given in Section 3.16, except that the function f with compact level surfaces must be chosen so that it belongs to $\mathcal{F}$ but $f(x) \to \infty$ as $\|x\| \to \infty$; e.g., we can take $f : x \to \|x\|^2 + 1$. Then $1/g(x) \to 0$ as $\|x\| \to \infty$, so that $1/g$ also belongs to $\mathcal{F}$, and the proof proceeds as in Section 3.16. ▶

Thus $\mathcal{F}\big|_{|\mathcal{F}|} = \mathcal{F}\big|_{\mathbb{R}^n}$ coincides with the algebra $C^\infty(\mathbb{R}^n)$ of *all* smooth functions on $\mathbb{R}^n$, since any function $f \in C^\infty(\mathbb{R}^n)$ in a neighborhood of some point $a \in \mathbb{R}^n$ coincides with the function $f\theta$, where θ is a smooth function that vanishes outside the ball of radius 2 and center a and equals 1 inside the concentric ball of radius 1 (see Section 2.5) and, obviously, $f\theta \in \mathcal{F}$. Hence $\rho\colon \mathcal{F} \to \mathcal{F}\big|_{|\mathcal{F}|} = C^\infty(\mathbb{R}^n)$ cannot be surjective.

3.28. Definition. A geometric $\mathbb{R}$-algebra $\mathcal{F}$ is said to be *complete* if the restriction homomorphism $\rho\colon \mathcal{F} \to \mathcal{F}\big|_{|\mathcal{F}|}$ is surjective (and is therefore an isomorphism), i.e., if any function $|\mathcal{F}| \to \mathbb{R}$ locally coinciding with elements of $\mathcal{F}$ is itself an element of $\mathcal{F}$.

It is clear that the algebras $C^\infty(U)$, where $U \subset \mathbb{R}^n$ is open, are complete (see Section 3.24). The algebra in the previous example (Section 3.27) is not complete.

Exercise. Determine which of the following algebras are complete.

1. $\mathcal{F} = \mathbb{R}[x]$.

2. The algebra of all smooth bounded functions.
3. The algebra of all smooth periodic functions (of period 1) on $\mathbb{R}$ (see Section 3.18).

3.29. We now return to the general situation in which A is a subset of the dual space $|\mathcal{F}|$ of a geometric $\mathbb{R}$-algebra $\mathcal{F}$. It is natural to ask the following question: Can the set $\left|\mathcal{F}\big|_A\right|$ be identified with $A \subset \mathcal{F}$?

Proposition. *Suppose $\mathcal{F}$ is a geometric $\mathbb{R}$-algebra and $A \subset |\mathcal{F}|$. Then the map*

$$\mu\colon A \to \left|\mathcal{F}\big|_A\right|, \quad (\mu(a))\,(f) = f(a),$$

is a homeomorphism onto a subset of the space $\left|\mathcal{F}\big|_A\right|$.

◀ Since all the elements of $\mathcal{F}$, understood as functions on the dual space $|\mathcal{F}|$, are continuous, $\mathcal{F}\big|_A$ is a subalgebra of the algebra of continuous functions on A, and therefore μ is continuous by Proposition 3.14. Further, μ is injective: If a_1 and a_2 are distinct points of A, there is a function $f_0 \in \mathcal{F}$ taking different values at these points; but then $f_0\big|_A$, which belongs to the algebra $\mathcal{F}\big|_A$, has the same values as f_0 at a_1 and a_2, and hence these points determine different homomorphisms $\mathcal{F}\big|_A \to \mathbb{R}$.

To prove that the inverse map $\mu^{-1}\colon \mu(A) \to A$ is continuous, consider a basis open set in A of the form $A \cap f^{-1}(V)$, where $f \in \mathcal{F}$ and $V \subset \mathbb{R}$ is open. It is mapped onto the set $\mu(A) \cap (f\big|_A)^{-1}(V)$, i.e., onto an open subset of $\mu(A)$. ▶

This proposition immediately implies that

$$A \subset B \subset |\mathcal{F}| \Rightarrow \left(\mathcal{F}\big|_B\right)\big|_A = \mathcal{F}\big|_A.$$

Should $\mu(A)$ coincide with $|\mathcal{F}\big|_A|$, we would have a positive answer to the question put at the beginning of this section. Proposition 3.24 implies that this is true for algebras $\mathcal{F} = C^\infty(U)$, where $U \subset \mathbb{R}^n$ is open, when $A \subset |\mathcal{F}| = U$ is also open. However, this is false in the general case.

3.30. Example. Suppose $\mathcal{F} = \mathbb{R}[x]$. Let $A = \mathbb{R}_+ \subset \mathbb{R}$ be the positive reals. Then the restriction homomorphism $\rho_A\colon \mathbb{R}[x] \to \mathbb{R}[x]\big|_{\mathbb{R}_+}$ is an isomorphism, since any nth degree polynomial is determined by its values at $(n+1)$ points, hence by its values on $\mathbb{R}_+$. On the other hand, the map $\mu\colon \mathbb{R}_+ \to \left|\mathbb{R}[x]\big|_{\mathbb{R}_+}\right|$ is the inclusion of $\mathbb{R}_+$ into $\mathbb{R} = \left|\mathbb{R}[x]\big|_{\mathbb{R}_+}\right| = \left|\mathbb{R}[x]\right|$, so that here $A = \mathbb{R}_+$ cannot be identified with $\left|\mathcal{F}\big|_A\right| = \mathbb{R}$.

3.31 Exercise. Describe an $\mathbb{R}$-algebra whose dual space is the configuration space of one of the given hinge mechanisms (see Section 1.14).

3.32. In order to avoid situations like the one in Example 3.30, we need a condition that would guarantee the bijectivity of the map

$$\mu\colon A \to \left|\mathcal{F}\big|_A\right|, \quad a \mapsto (f \mapsto f(a)).$$

Definition. A geometric $\mathbb{R}$-algebra $\mathcal{F}$ said to be *closed with respect to smooth composition, or C^∞-closed*, if for any finite collection of its elements $f_1, \dots, f_k \in \mathcal{F}$ and any function $g \in C^\infty\left(\mathbb{R}^k\right)$ there exists an element $f \in \mathcal{F}$ such that

$$f(a) = g(f_1(a), \dots, f_k(a)) \quad \text{for all } a \in |\mathcal{F}|. \tag{3.4}$$

Note that the function $f \in \mathcal{F}$ appearing in this definition is uniquely determined (since $\mathcal{F}$ is geometric).

For the case in which $\mathcal{F}_S$ is the algebra of functions determined by measuring devices of a physical system S, the algebra $\mathcal{F}_S$ is always C^∞-closed. This is because the composite function (3.4) may be constructed by means of a device synthesizing the function $g(f_1, \dots, f_k)$; see Section 1.10.

Exercise. Determine which of the algebras listed in Exercise 3.28 are closed.

3.33. We shall now show that *the map* $\mu\colon A \to \left|\mathcal{F}\big|_A\right|$ (see Section 3.29) *is surjective* (*and therefore a homeomorphism*) *for C^∞-closed algebras $\mathcal{F}$* in the case of any basis open set A:

$$A = \{a \in |\mathcal{F}| \mid \alpha < h(a) < \beta\}, \quad \alpha, \beta \in \mathbb{R}, \quad h \in \mathcal{F}.$$

(We shall not require this fact in more general form in the sequel.)
◀ By Corollary 2.3, there exists a function $g \in C^\infty(\mathbb{R})$ such that $g \equiv 0$ on $\mathbb{R} \setminus\,]\alpha, \beta[$ and $g > 0$ on $]\alpha, \beta[$. Since $\mathcal{F}$ is C^∞-closed, there is a function $f \in \mathcal{F}$ such that $f(a) = g(h(a))$ for all points $a \in |\mathcal{F}|$. Then $f(a) > 0$ whenever $a \in A$, so that $f\big|_A$ is an invertible element of the algebra $\mathcal{F}\big|_A$.

Further, suppose $b' \in |\mathcal{F}|$ is the image of some point $b \in \left|\mathcal{F}\big|_A\right|$ under the natural map $\left|\mathcal{F}\big|_A\right| \to |\mathcal{F}|$. If $b' \notin A$, then

$$0 = f\left(b'\right) = \left(f\big|_A\right)(b),$$

which contradicts the fact that $f\big|_A$ is invertible. Thus

$$\mu(A) = \left|\mathcal{F}\big|_A\right|. \quad ▶$$

3.34. Having in mind the results of Section 3.33, we would like to modify a given geometric $\mathbb{R}$-algebra $\mathcal{F}$ so as to obtain a C^∞-closed algebra $\overline{\mathcal{F}}$. The most direct way to do that is the following. Identifying $\mathcal{F}$ with the corresponding algebra of functions on $|\mathcal{F}|$, consider the set $\overline{\mathcal{F}}$ of functions on $|\mathcal{F}|$ that can be represented in the form

$$g(f_1, \dots, f_l), \quad \text{where} \quad l \in \mathbb{N}, \quad f_1, \dots, f_l \in \mathcal{F}, \quad g \in C^\infty\left(\mathbb{R}^l\right).$$

The set $\overline{\mathcal{F}}$ has an obvious $\mathbb{R}$-algebra structure, and $\mathcal{F}$ is a subalgebra of $\overline{\mathcal{F}}$. Denote the natural inclusion $\mathcal{F} \subset \overline{\mathcal{F}}$ by $i_{\mathcal{F}}$. Since the composition of smooth functions is smooth, the algebra $\overline{\mathcal{F}}$ is C^∞-closed. It is also geometric, being the algebra of certain functions on a set (see Section 3.14). Thus we have constructed a natural inclusion map

$$i_{\mathcal{F}} : \mathcal{F} \hookrightarrow \overline{\mathcal{F}}$$

for any geometric $\mathbb{R}$-algebra $\mathcal{F}$ into a C^∞-closed $\mathbb{R}$-algebra $\overline{\mathcal{F}}$, which we (temporarily) call the C^∞-closure of $\mathcal{F}$. This algebra possesses the following remarkable property.

3.35 Proposition. *Any homomorphism $\alpha \colon \mathcal{F} \to \mathcal{F}'$ of a geometric $\mathbb{R}$-algebra $\mathcal{F}$ into a C^∞-closed $\mathbb{R}$-algebra $\mathcal{F}'$ can be uniquely extended to a homomorphism $\overline{\alpha} \colon \overline{\mathcal{F}} \to \mathcal{F}'$ of its C^∞-closure $\overline{\mathcal{F}}$.*

◀ Assume that the required extension $\overline{\alpha}$ exists, i.e., that $\alpha = \overline{\alpha} \circ i_{\mathcal{F}}$, where $i_{\mathcal{F}} : \mathcal{F} \hookrightarrow \overline{\mathcal{F}}$ is the natural inclusion. Then, by Section 3.20, we have $|\alpha| = |i_{\mathcal{F}}| \circ |\overline{\alpha}|$. Here $|\alpha|$ denotes the dual map (see 3.19). Further, for any point $a \in |\mathcal{F}'|$,

$$\begin{aligned}\overline{\alpha}(g(f_1, \dots, f_l))(a) &= g(f_1, \dots, f_l)(|\overline{\alpha}|(a)) = g(i_{\mathcal{F}}(f_1), \dots, i_{\mathcal{F}}(f_l))(|\overline{\alpha}|(a)) \\ &= g(f_1, \dots, f_l)(|i_{\mathcal{F}}|(|\overline{\alpha}|(a))) = g(f_1, \dots, f_l)(|\alpha|(a)) \\ &= g(f_1(|\alpha|(a)), \dots, f_l(|\alpha|(a))) = g(\alpha(f_1), \dots, \alpha(f_l))(a).\end{aligned}$$

Since $\mathcal{F}'$ is geometric, this implies

$$\overline{\alpha}(g(f_1, \dots, f_l)) = g(\alpha(f_1), \dots, \alpha(f_l)).$$

If $\overline{\alpha}$ exists, this last formula proves its uniqueness. To prove existence, we can use this formula as the definition of $\overline{\alpha}$, if we establish that the right-hand side is well defined, i.e., if we show that $g(f_1, \dots, f_l) = g'(f'_1, \dots, f'_{l'})$ implies

$$g(\alpha(f_1), \dots, \alpha(f_l)) = g'\left(\alpha\left(f'_1\right), \dots, \alpha\left(f'_{l'}\right)\right).$$

Since $\mathcal{F}'$ is geometric, it suffices to prove this at an arbitrary point $a' \in \mathcal{F}'$. But

$$\begin{aligned}g(\alpha(f_1), \dots, \alpha(f_l))(a') &= g\left(\alpha(f_1)\left(a'\right), \dots, \alpha(f_l)\left(a'\right)\right) \\ &= g(f_1(a), \dots, f_l(a)) = g(f_1, \dots, f_l)(a),\end{aligned}$$

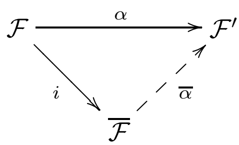
where $a = |\alpha|(a')$. Similarly,

$$g'\left(\alpha\left(f'_1\right), \dots, \alpha\left(f'_{l'}\right)\right)\left(a'\right) = g'\left(f'_1, \dots, f'_{l'}\right)(a).$$

Comparing the last two formulas, we see that $\overline{\alpha}$ is well defined, concluding our proof. ▶

3.36. It is remarkable that Proposition 3.35 entirely characterizes the C^∞-closure $\overline{\mathcal{F}}$ of a geometric $\mathbb{R}$-algebra $\mathcal{F}$. To explain this in adequate terms, we need the following definition:

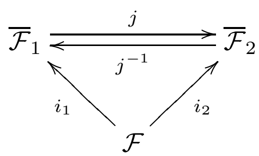
Definition. A C^∞-closed geometric $\mathbb{R}$-algebra $\overline{\mathcal{F}}$ together with a homomorphism $i\colon \mathcal{F} \to \overline{\mathcal{F}}$ is called the *smooth envelope* of the $\mathbb{R}$-algebra $\mathcal{F}$ if for any homomorphism $\alpha\colon \mathcal{F} \to \mathcal{F}'$ of $\mathcal{F}$ into a geometric C^∞-closed $\mathbb{R}$-algebra $\mathcal{F}'$ there exists a unique homomorphism $\overline{\alpha}\colon \overline{\mathcal{F}} \to \mathcal{F}'$ extending α (i.e., such that $\alpha = \overline{\alpha} \circ i$). In other words, under the above assumptions, the diagram



can always be uniquely completed (by the dotted arrow $\overline{\alpha}$) to a commutative one.

It now follows from Proposition 3.35 that the C^∞-closure (see 3.34) is a smooth envelope of $\mathcal{F}$.

3.37 Proposition. *The smooth envelope of any $\mathbb{R}$-algebra $\mathcal{F}$ is unique up to isomorphism. More precisely, if the pairs $\big(i_k, \overline{\mathcal{F}}_k\big)$, $k = 1, 2$, are smooth envelopes of $\mathcal{F}$, there exists a unique isomorphism $j\colon \overline{\mathcal{F}}_1 \to \overline{\mathcal{F}}_2$ such that $i_2 = j \circ i_1$. In other words, we have the following commutative diagram:*



◀ First we note that for any given smooth envelope $\big(i, \overline{\mathcal{F}}\big)$ of $\mathcal{F}$ any homomorphism $\alpha\colon \overline{\mathcal{F}} \to \overline{\mathcal{F}}$ satisfying $\alpha \circ i = i$ is the identity, $\alpha = \mathrm{id}_{\mathcal{F}}$. (Indeed, by Definition 3.36, the "solution" of the "equation" $\alpha \circ i = i$ is unique, but this equation has the obvious solution $\mathrm{id}_{\overline{\mathcal{F}}}$.)

Further, according to the same definition for $(i_1, \mathcal{F}_1)$, the homomorphism $i_2\colon \mathcal{F} \to \overline{\mathcal{F}}_2$ can be uniquely represented in the form $i_2 = j_1 \circ i_1$, where $j_1\colon \overline{\mathcal{F}}_1 \to \overline{\mathcal{F}}_2$ is a homomorphism. Similarly, $i_1 = j_2 \circ i_2$, where $j_2\colon \overline{\mathcal{F}}_2 \to \overline{\mathcal{F}}_1$ is a homomorphism. Hence

$$i_2 = j_1 \circ i_1 = j_1 \circ (j_2 \circ i_2) = (j_1 \circ j_2) \circ i_2.$$

By the remark at the beginning of the proof, this implies $j_1 \circ j_2 = \mathrm{id}_{\overline{\mathcal{F}}_2}$. Similarly, $j_2 \circ j_1 = \mathrm{id}_{\overline{\mathcal{F}}_1}$. Thus j_1 and j_2 are isomorphisms inverse to each other, and we can put $j = j_1$ to establish the proposition. (The uniqueness of the isomorphism j follows from the definition of smooth envelopes.) ▶

A direct consequence of this proposition is that the temporary term "C^∞-closure" (see 3.34) is characterized by its universal property expressed in Proposition 3.35 and therefore coincides with the term "smooth envelope." It is the latter term that will be used from now on.

3.38. In accordance with its definition (see Section 3.36), the smooth envelope $\overline{\mathcal{F}}$ of a geometric algebra $\mathcal{F}$ is an object that plays a universal role in its interactions (i.e., isomorphisms) with the "world" of C^∞-closed geometric algebras. We can say that the smooth envelope is the "ambassador plenipotentiary" of the algebra $\mathcal{F}$ in this "world," and that $\mathcal{F}$ interacts with the latter exclusively via this ambassador.

The reader may have noticed that the arguments used in Section 3.37 are very general in nature. The art of finding and using such arguments is one of the main facets of *category theory*, familiarly known as *abstract nonsense.* We feel that one has to get used to it before learning it mathematically, so we shall not develop the theory, but use many of its standard arguments and tricks (e.g., see Sections: 6.4, 6.6, 6.16, 6.17).

In the set-theoretic approach to mathematics, one studies the inner nature of mathematical objects, i.e., point sets supplied with certain structures. It is a biology of species. On the other hand, the categorical approach is a kind of sociology: One is no longer interested in the properties of individual objects, but in their relationships (called "morphisms" in the theory) with other objects of the same or similar type. One can also say that the categorical approach is similar to the experimental method in the natural sciences, when objects are not studied per se, but are analyzed in terms of their interaction with other objects.

3.39 Exercises. Find the smooth envelope of

1. the algebra $\mathbb{R}[x_1, \dots, x_n]$;
2. the algebra of functions on the line $\mathbb{R}$ of the form

$$f(x) = \sum_{k=0}^{n} a_k(x)|x|^k, \quad x \in \mathbb{R},$$

   where $a_i(x) \in C^\infty(\mathbb{R})$.

4

Smooth Manifolds (Algebraic Definition)

4.1. A complete (Section 3.27) geometric (Section 3.7) $\mathbb{R}$-algebra $\mathcal{F}$ is called *smooth* if there exists a finite or countable open covering $\{U_k\}$ of the dual space $|\mathcal{F}|$ such that all the algebras $\mathcal{F}\big|_{U_k}$ (Section 3.23) are isomorphic to the algebra $C^\infty(\mathbb{R}^n)$ of smooth functions in Euclidean space. The (fixed positive) integer n is said to be the *dimension* of the algebra $\mathcal{F}$.

Smooth n-dimensional algebras are our main object of study; they can be viewed as $\mathbb{R}$-algebras of smooth functions on n-dimensional smooth manifolds. From the viewpoint of formal mathematics, the $\mathbb{R}$-algebra $\mathcal{F}$ entirely determines the corresponding manifold M as the dual space $M = |\mathcal{F}|$ of its $\mathbb{R}$-points (Section 3.8) and is most convenient to work with: All of differential mathematics applies neatly to $\mathcal{F}$, so that the space M is not formally required. Nevertheless, in order to be able to visualize M as a geometric object, we must learn to work simultaneously with the smooth algebra $\mathcal{F}$ and the space $M = |\mathcal{F}|$ of its $\mathbb{R}$-points.

Learning this will be the main goal of the present chapter.

Considering $\mathcal{F}$ and $M = |\mathcal{F}|$ simultaneously, we say that we are dealing with a *smooth manifold.* Although the second object in this pair is determined by the first, we make a concession to our geometric intuition (and to traditional terminology) and say that $\mathcal{F}$ *is the algebra of smooth functions on the manifold* M $(= |\mathcal{F}|)$.

4.2. A somewhat more general concept is that of a *smooth algebra with boundary.* In this case, for each element U_k of the covering $\{U_k\}$ we require the algebra $\mathcal{F}\big|_{U_k}$ to be isomorphic either to $C^\infty(\mathbb{R}^n)$ or to $C^\infty(\mathbb{R}^n_H)$, where

$$\mathbb{R}^n_H = \{(r_1, \ldots, r_2) \in \mathbb{R}^n \mid r_1 \geqslant 0\},$$

and $C^\infty(\mathbb{R}^n_H)$ consists of the restrictions (in the usual sense) of all functions from $C^\infty(\mathbb{R}^n)$ to the set $\mathbb{R}^n_H$.

Exercise. Prove that the algebras $C^\infty(\mathbb{R}^n)$ and $C^\infty(\mathbb{R}^n_H)$ are not isomorphic.

The points of the space $|\mathcal{F}|$ that correspond to the boundary of the half-space $\mathbb{R}^n_H$ in the identification $U_k = \mathbb{R}^n_H$ are called *boundary points.*

Exercise. Prove that the set of boundary points $\partial|\mathcal{F}|$ has a natural structure of a smooth manifold (without boundary).

As above, emphasizing the geometric viewpoint on this concept, we shall say that *$\mathcal{F}$ is the algebra of smooth functions on the manifold M $(= |\mathcal{F}|)$ with boundary.*

4.3 Lemma. *If a geometric $\mathbb{R}$-algebra $\mathcal{F}$ is isomorphic to $C^\infty(\mathbb{R}^n)$ or $C^\infty(\mathbb{R}^n_H)$, then it is C^∞-closed* (see Section 3.32).

◀ The lemma immediately follows from the following stronger statement. If $i\colon \mathcal{F}_1 \to \mathcal{F}_2$ is an isomorphism of geometric $\mathbb{R}$-algebras, and $\mathcal{F}_1$ is C^∞-closed, then so is $\mathcal{F}_2$.

The verification of this statement is quite similar to the uniqueness proof of $\overline{\alpha}$ in Section 3.35, and the reader should have no difficulty in carrying it out. ▶

4.4 Proposition. *Smooth algebras are C^∞-closed.* (*The same is true for smooth algebras with boundary.*)

◀ Let $\mathcal{F}$ be a smooth $\mathbb{R}$-algebra (possibly with boundary), $l \in \mathbb{N}$, $g \in C^\infty(\mathbb{R}^l)$, $f_1, \dots, f_l \in \mathcal{F}$, and let $\{U_k\}$ be the covering that appears in Definition 4.1 (or 4.2).

Consider the function

$$h\colon |\mathcal{F}| \to \mathbb{R}, \quad h(a) = g(f_1(a), \dots, f_l(a)).$$

By Lemma 4.3, for any k there exists an $h_k \in \mathcal{F}\big|_{U_k}$ such that

$$\forall a \in U_k, \quad h_k = g(f_1(a), \dots, f_l(a)).$$

Thus in a neighborhood of each point the function coincides with a function from $\mathcal{F}$.

Since by 4.1 (or 4.2) $\mathcal{F}$ is complete (Section 3.28), it follows that $h \in \mathcal{F}$. ▶

4.5. Example. Suppose $\mathcal{F}$ is the algebra of smooth periodic functions on the line $\mathbb{R}$ of period 1:

$$\mathcal{F} = \{f \in C^\infty(\mathbb{R}) \mid f(r+1) = f(r), \quad \forall r \in \mathbb{R}\}.$$

Being a subalgebra of the geometric algebra $C^\infty(\mathbb{R})$, the algebra $\mathcal{F}$ is itself geometric (see Section 3.19). It is not difficult to prove that $\mathcal{F}$ is complete. (In Section 4.22 we shall present a general argument implying the completeness of all the algebras considered in Examples 4.5–4.8.)

It was shown in 3.18 that the space $|\mathcal{F}|$ is the circle S^1. Now consider the functions $g_1,\, g_2 \in \mathcal{F}$,

$$g_1(r) = \sin^2 \pi r, \quad g_2(r) = \cos^2 \pi r,$$

and the open covering of the circle $|\mathcal{F}|$ by the sets

$$U_i = \{r \in |\mathcal{F}| \mid g_i(x) \neq 0\}, \quad i = 1, 2.$$

It is easy to establish bijections $U_i \leftrightarrow\ ]0,1[$ that correspond to the isomorphisms

$$\mathcal{F}\big|_{U_i} \cong C^\infty\big(\,]0,1[\,\big) \quad (\cong C^\infty(\mathbb{R})).$$

Thus the algebra $\mathcal{F}$ is a smooth algebra of dimension 1, and the manifold it determines is the *circle* $S^1 = |\mathcal{F}|$.

In a similar way one establishes that the $\mathbb{R}$-algebra

$$\mathcal{F} = \big\{f \in C^\infty(\mathbb{R})^2 \mid f(r_1+1, r_2) = f(r_1, r_2) \quad \forall (r_1, r_2) \in \mathbb{R}^2\big\}$$

is a smooth algebra of dimension 2. In this case the space $M = |\mathcal{F}|$ is homeomorphic to the *cylinder*.

4.6 Exercise. Carefully review the previous example and find the mistake in the following argument: Since a 1-periodic smooth function takes the same finite value at the end points of the closed interval [0,1], while the algebra $C^\infty\big(\,]0,1[\,\big)$ contains unbounded functions as well as functions that have different limits at the points 0 and 1, the algebra $\mathcal{F}\big|_{]0,1[}$ (where $\mathcal{F}$ is the algebra of 1-periodic smooth functions on $\mathbb{R}$) cannot be isomorphic to $C^\infty\big(\,]0,1[\,\big)$.

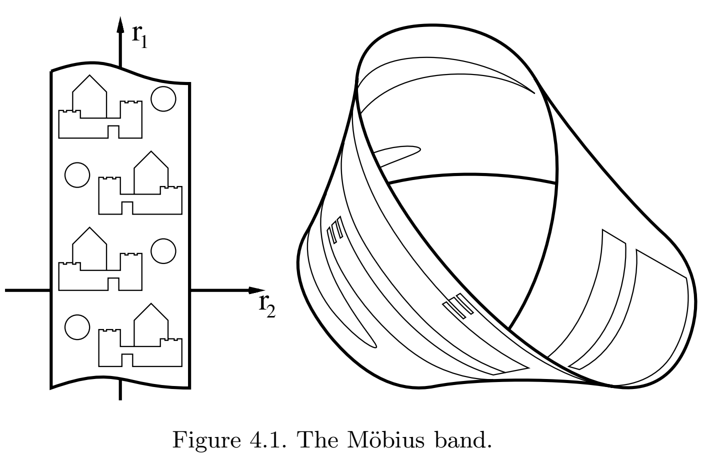


Figure 4.1. The Möbius band.

4.7. Examples. I. $\mathcal{F} = \{f \in C^\infty(\mathbb{R}^2) \mid f(r_1, r_2) = f(r_1 + 1, -r_2)$ for all $(r_1, r_2) \in \mathbb{R}^2\}$.

The space $|\mathcal{F}|$ is called the *open Möbius band.*

II. $\mathcal{F} = \{f \in C^\infty(\Pi) \mid f(r_1, r_2) = f(r_1 + 1, -r_2)\ \forall (r_1, r_2) \in \Pi\}$, where Π is the strip $\{(r_1, r_2) \in \mathbb{R}^r \mid -1 \leqslant r_2 \leqslant 1\}$. This is a smooth algebra with boundary, and the space $|\mathcal{F}|$ is known as the (*closed*) *Möbius band* (see Figure 4.1).

III. $\mathcal{F} = \{f \in C^\infty(\mathbb{R}^2) \mid f(r_1, r_2) = f(r_1 + 1, -r_2) = f(r_1, r_2 + 1)\ \forall (r_1, r_2) \in \mathbb{R}^2\}$.

Using the functions

$$g_1(r_1, r_2) = \sin^2 \pi r_1, \qquad h_1(r_1, r_2) = \sin^2 \pi r_2,$$
$$g_2(r_1, r_2) = \cos^2 \pi r_1, \qquad h_2(r_1, r_2) = \cos^2 \pi r_2,$$

we can cover the space $|\mathcal{F}|$ by the four open sets

$$U_{ik} = \{(r_1, r_2) \in |\mathcal{F}| \mid g_i(r_1, r_2) \neq 0,\ h_k(r_1, r_2) \neq 0\},\ i, k = 1, 2.$$

For each of these sets one can immediately construct a homeomorphism on the open square corresponding to the isomorphism of the $\mathbb{R}$-algebra $\mathcal{F}\big|_{U_{ik}}$ on the algebra of smooth functions on the open square. Therefore, $\mathcal{F}\big|_{U_{ik}} \cong C^\infty(\mathbb{R}^2)$, and $\mathcal{F}$ is a smooth $\mathbb{R}$-algebra of dimension 2. The space $|\mathcal{F}|$ is known as the *Klein bottle.*

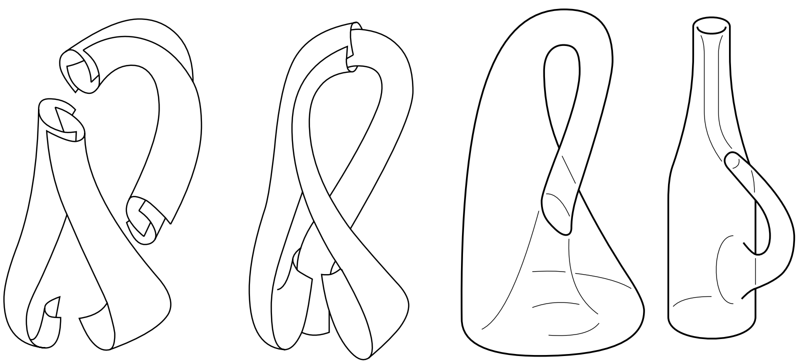


Figure 4.2. The Klein bottle.

It is useful to visualize how the squares U_{ik} are "glued together" when they are embedded into $|\mathcal{F}|$. The beginning of this process is pictured on the left-hand side of Figure 4.2. What happens if the process is continued in 3-space is shown on the right-hand side of the same figure (the little circular self-intersection does not really occur in the Klein bottle; it is due to the fact that the latter does not fit into 3-space).

4.8 Exercise. Prove that the $\mathbb{R}$-algebra

$$\mathcal{F} = \left\{ f \in C^\infty\left(\mathbb{R}^2\right) \mid f(r_1, r_2) = f(r_1 + 1, -r_2) = f(-r_1, r_2 + 1), \quad (r_1, r_2) \in \mathbb{R}^2 \right\}$$

is smooth. Find as many geometric descriptions of the topological space $|\mathcal{F}|$ as you can. Prove, for example, that $|\mathcal{F}|$ is homeomorphic to the space whose "points" are the straight lines of $\mathbb{R}^3$ passing through the origin $(0, 0, 0)$; see Section 1.3.

4.9. Suppose $\mathcal{F}$ is the algebra of smooth functions on a manifold $M = |\mathcal{F}|$ with boundary. Recalling Definition 4.2, we say that a point $p \in M$ is a *boundary point* of M if it corresponds to a boundary point of $\mathbb{R}^n_H$ in the identification $U_k = \mathbb{R}^n_H$. The set of all boundary points is denoted by ∂M and called the *boundary* of M.

Exercise.

1. Prove that *if the boundary ∂M of the n-dimensional manifold $M = |\mathcal{F}|$ is nonempty, then $\mathcal{F}\big|_{\partial M}$ is a smooth algebra of dimension $(n-1)$ and ∂M has no boundary points* (see the second exercise in Section 4.2).
2. Check algebraically that the manifold $\partial|\mathcal{F}|$ in Example 4.7, II, can be identified with the circle (cf. Sections 4.5 and 6.9).

4.10. Remark. It is far from obvious that the dimension of a smooth algebra is well defined, i.e., that it does not depend on the choice of the covering $\{U_k\}$ and of the isomorphisms $\mathcal{F}\big|_{U_k} \cong C^\infty(\mathbb{R}^n)$ (see 4.1). This almost immediately follows from the fact that *the algebras $C^\infty(\mathbb{R}^n)$ and $C^\infty(\mathbb{R}^m)$ are not isomorphic if $n \neq m$.*

The reader who has industriously worked his way through the previous examples undoubtedly feels that this is true. A more experienced reader will probably have no trouble in proving this fact by using "Sard's theorem on singular points of smooth maps" (advanced calculus). As for us, we shall prove this result in Chapter 9. Until then, our skeptical readers may consider dimension to be an invariant of the covering $\{U_k\}$ rather than that of the algebra $\mathcal{F}$ itself.

4.11. Definition. Suppose $\mathcal{F}$ is the algebra of smooth functions on the manifold M and $N \subset M = |\mathcal{F}|$ is a subset. If the algebra $\mathcal{F}_N = \mathcal{F}\big|_N$ is a smooth $\mathbb{R}$-algebra, then we say that N is a *smooth submanifold* of the smooth manifold M and that $\mathcal{F}_N$ is the *algebra of smooth functions on the submanifold N*.

If the restriction homomorphism $i\colon \mathcal{F} \to \mathcal{F}_N$ is surjective, the smooth submanifold $N \subset M = |\mathcal{F}|$ is called *closed*.

4.12. Let N be a closed submanifold of M. Were we being consistent in 4.11 when we spoke of $\mathcal{F}_N$ as "the algebra of smooth functions on the manifold N"? The answer to that question is given by the following result:

Proposition. *Suppose $\mathcal{F}$ is the algebra of smooth functions on the manifold M and $N \subset M = |\mathcal{F}|$ is a closed smooth submanifold. Then*

(i) *N is closed as a subset of the topological space M;*

(ii) *$N = |\mathcal{F}_N|$.*

◀ (i) Let $a \in M \smallsetminus N$ be a limit point of N, and $U \subset M$ a neighborhood of a such that $\mathcal{F}\big|_U \cong C^\infty(\mathbb{R}^n)$. This isomorphism may be chosen so that the elements of $\mathcal{F}\big|_U$ corresponding to the coordinate functions $r_1, \ldots, r_n \in C^\infty(\mathbb{R}^n)$ vanish at the point a. Consider the function on $\mathcal{F} \smallsetminus a$ corresponding to $1/\left(r_1^2 + \cdots + r_n^2\right)$. This function may be extended from the punctured neighborhood $U \smallsetminus a$ of a to a smooth function g on the submanifold $M \smallsetminus a$.

It is obvious that the restriction $g\big|_N$ belongs to the algebra $\mathcal{F}\big|_N = \mathcal{F}_N$ but does not belong to the image of the restriction homomorphism $i\colon \mathcal{F} \to \mathcal{F}\big|_N$. This contradiction proves that $a \in N$.

(ii) Consider the $\mathbb{R}$-point $b\colon \mathcal{F}_N \to \mathbb{R}$ and take the composition $c = b \circ i\colon \mathcal{F} \to \mathbb{R}$. Assume that $c \notin N$. Generalizing Proposition 2.5, let us construct a function $f \in \mathcal{F}$ such that $f\big|_N \equiv 0$, $f(c) \neq 0$. But then $i(f) = 0$ and $f(c) = 0$: a contradiction. Therefore, the map $b \mapsto c = b \circ i$ is a surjection of $|\mathcal{F}_N|$ onto N. Together with Proposition 3.29 this gives the result. ▶

4.13. Example. In $\mathbb{R}^2$ consider the set of points S^1 given by the equation

$$r_1^2 + r_2^2 - 1 = 0.$$

Let us check that the $\mathbb{R}$-algebra $\mathcal{F}_{S^1} = C^\infty\left(\mathbb{R}^2\right)\big|_{S^1}$ is isomorphic to the algebra of smooth periodic functions of period 1 on the line $\mathbb{R}$ (see Example 4.5).

◀ First, note that $\mathcal{F}_{S^1} = C^\infty\left(\mathbb{R}^2\right)|_{S^1} = C^\infty(\mathbb{R}^2 \setminus \{0\})|_{S^1}$ and consider the map

$$w\colon \mathbb{R} \to S^1 \subset \mathbb{R}^2, \quad r \mapsto (\cos 2\pi r, \sin 2\pi r).$$

Clearly, the corresponding $\mathbb{R}$-algebra homomorphism

$$|w|\colon \mathcal{F}_{S^1} \to C^\infty(\mathbb{R}),$$
$$|w|(f)(r) = f(w(r)) = f(\cos 2\pi r, \sin 2\pi r), \quad r \in \mathbb{R},$$

is injective and its image is contained in the subalgebra $C^\infty_{\mathrm{per}}(\mathbb{R})$ of smooth 1-periodic functions on $\mathbb{R}$. To prove that $|w|$ is surjective, consider the homomorphism $\iota\colon C^\infty_{\mathrm{per}}(\mathbb{R}) \to C^\infty(\mathbb{R}^2 \setminus \{0\})$ defined by

$$\iota(f)(r_1, r_2) = f\left(\frac{\arg z}{2\pi}\right), \quad z = r_1 + ir_2.$$

Obviously, $|w|(\iota(f)|_{S^1}) = f$. ▶

Exercise. 1. Show that any odd 2π-periodic smooth function is of the form $g(x)\sin x$, where $g(x)$ is an even 2π-periodic smooth function.

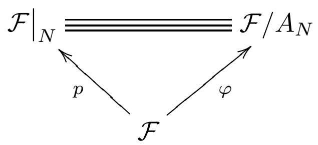
2. Is it true that any even 2π-periodic smooth function can be written as $f(\cos x)$ for $f \in C^\infty(\mathbb{R})$?

4.14. Recall that the restriction of elements of an $\mathbb{R}$-algebra $\mathcal{F}$ to a subset $N \subset |\mathcal{F}|$ was not defined in algebraic terms, but in geometric ones (see Section 3.23). However, if $N \subset |\mathcal{F}|$ is a closed smooth submanifold, then there is a purely algebraic way to find the algebra $\mathcal{F}_N = \mathcal{F}\big|_N$.

Namely, suppose $A_N \subset \mathcal{F}$ is the set of elements of $\mathcal{F}$ that vanish on N, i.e.,

$$A_N = \{f \in \mathcal{F} \mid \forall a \in N,\ f(a) = 0\}.$$

This is obviously an ideal of the algebra $\mathcal{F}$, so we can consider the quotient algebra $\mathcal{F}/A_N$. There exists an obvious identification of $\mathcal{F}/A_N$ with the algebra $\mathcal{F}_N = \mathcal{F}\big|_N$ (see the proof of Proposition 4.12) for which the quotient map $\varphi\colon \mathcal{F} \to \mathcal{F}\big|_{A_N}$ becomes the restriction homomorphism $p\colon \mathcal{F} \to \mathcal{F}_N$:



4.15. Example. If S^1 is the circle from Example 4.13, then A_{S^1} is the principal ideal in $C^\infty(\mathbb{R}^2)$ generated by the function $r_1^2 + r_2^2 - 1$.

◀ Let $f \in A_{S^1}$. Let us prove that

$$f(r_1, r_2) = g(r_1, r_2) \cdot \left(r_1^2 + r_2^2 - 1\right)$$

for a suitable function $g \in C^\infty(\mathbb{R}^2)$. Since the algebra $C^\infty(\mathbb{R}^2)$ is complete, if suffices to construct g in a neighborhood of S^1, say, in $\mathbb{R}^2 \backslash \{0\}$. To this end, introduce the following auxiliary functions:

$$\begin{aligned} u(t, r_1, r_2) &= t + \frac{1 - t}{\sqrt{r_1^2 + r_2^2}}, \\ h(t, r_1, r_2) &= f\left(r_1 \cdot u(t, r_1, r_2),\ r_2 \cdot u(t, r_1, r_2)\right). \end{aligned}$$

Then, taking into account the fact that

$$\begin{aligned} \frac{\partial u}{\partial t} &= 1 - \frac{1}{\sqrt{r_1^2 + r_2^2}} = \frac{r_1^2 + r_2^2 - 1}{r_1^2 + r_2^2 + \sqrt{r_1^2 + r_2^2}}, \\ h(0, r_1, r_2) &= 0 \quad (\text{since } f \in A_{S^1}), \\ h(1, r_1, r_2) &= f(r_1, r_2), \end{aligned}$$

we obtain

$$f(r_1, r_2) = \int_0^1 \frac{\partial h}{\partial t}(t, r_1, r_2)dt$$

$$= \frac{\int_0^1 \left(r_1 \frac{\partial f}{\partial r_1}(r_1 u, r_2 u) + r_2 \frac{\partial f}{\partial r_2}(r_1 u, r_2 u) \right) dt}{r_1^2 + r_2^2 + \sqrt{r_1^2 + r_2^2}} \cdot \left(r_1^2 + r_2^2 - 1 \right).$$

The first factor in the last expression is the desired function $g(r_1, r_2)$. ▶

4.16 Exercise. Let $A = C^\infty \left(\mathbb{R}^2\right)$. Show that the algebra $A/\left(y^2 - x^3\right) A$ is not smooth.

A traditional purely algebraic approach to solving such problems is based on the following two facts:

1. If an algebra is smooth, then its localization at a maximum ideal is isomorphic to the algebra of germs of smooth functions on some $\mathbb{R}^k$ (at the origin), see Example III on page 148;

2. The formal completion of the above local algebra is isomorphic to the algebra of formal power series.

The task becomes much simpler if one uses certain elementary tools of differential calculus over commutative algebras (see Exercise 9.34).

4.17. Lemmas. It is not difficult to generalize the statements in Sections 2.3–2.7 from $\mathbb{R}^n$ to arbitrary manifolds. In particular, for any arbitrary algebra $\mathcal{F}$ of smooth functions on a manifold M, the following statements hold:

(i) *For any open set $U \subset M$ there exists a function $f \in \mathcal{F}$ such that*

$$\begin{cases} f(x) > 0 & \text{for all } x \in U, \\ f(x) = 0 & \text{if } x \notin U. \end{cases}$$

(ii) *For any two nonintersecting closed subsets $A, B \subset M$ there exists a function $f \in \mathcal{F}$ such that*

$$\begin{cases} f(x) = 0 & \text{for all } x \in A, \\ f(x) = 1 & \text{for all } x \in B, \\ 0 < f(x) < 1 & \text{otherwise.} \end{cases}$$

(The reader possibly established a weaker statement when working through Proposition 4.12.)

(iii) *There exists a function $f \in \mathcal{F}$ all of whose level surfaces are compact.*

4.18. In proving Lemma 4.17, the following statement, called the "partition of unity lemma," may be useful: *If* $\{U_\alpha\}$ *is a locally finite open covering of the space* $M = |\mathcal{F}|$, *then there exist functions* $f_\alpha \in \mathcal{F}$ *such that* $f_\alpha(x) = 0$ *if* $x \in M \smallsetminus U_\alpha$ *and*

$$\sum_\alpha f_\alpha(x) \equiv 1.$$

(A *locally finite covering* $\{U_\alpha\}$ is a covering such that for any $x \in M$ there exists a neighborhood $U \subset M$ of x that intersects only a finite number of sets U_α.)

We suggest that the reader try to prove this statement first in the particular case where the covering $\{U_\alpha\}$ is supplied with isomorphisms $\mathcal{F}\big|_{U_\alpha} \cong C^\infty(\mathbb{R}^n)$ (or with diffeomorphisms $U_\alpha \cong \mathbb{R}^n$).

4.19. Suppose $\mathcal{F}$ is the algebra of smooth functions on the manifold M. Consider the *action of a group* on this smooth manifold, i.e., a family Γ of automorphisms $\gamma\colon \mathcal{F} \to \mathcal{F}$ such that

(i) $\gamma_1, \gamma_2 \in \Gamma \Rightarrow \gamma_1 \circ \gamma_2 \in \Gamma$,

(ii) $\gamma \in \Gamma \Rightarrow \gamma^{-1} \in \Gamma$.

Suppose $\mathcal{F}^\Gamma \subset \mathcal{F}$ is the subalgebra of invariant functions for this action, i.e.,

$$\mathcal{F}^\Gamma = \{f \in \mathcal{F} \mid \gamma(f) = f \quad \text{for all} \quad \gamma \in \Gamma\}.$$

4.20 Lemma. *Suppose* $\mathcal{F}^\Gamma$ *is the subalgebra of invariant functions of a group action of* Γ *on* $M = |\mathcal{F}|$. *If* $\widetilde{\mathcal{F}^\Gamma}$ (*see* Section 3.4) *contains a function all of whose level surfaces are compact*, *then the algebra* $\mathcal{F}^\Gamma$ *is geometric.*

◀ The proof will be a repetition, word for word, of 3.16, if we note that the operations applied there preserve Γ-invariance. ▶

In particular, if the set $|\mathcal{F}^\Gamma|$ is compact, then the algebra $\mathcal{F}^\Gamma$ is always geometric.

4.21. In order to learn to visualize the algebra $\mathcal{F}^\Gamma$ of Γ-invariant functions on a manifold $|\mathcal{F}| = M$, let us consider the orbit $\mathcal{O}_a = \{|\gamma|(a) \mid \gamma \in \Gamma\}$ for each point $a \in M$. Denote the set of all orbits by N.

Elements of $\mathcal{F}^\Gamma$ can be understood as functions on N. Indeed, if $b = |\gamma|(a) \in \mathcal{O}_a$ and $f \in \mathcal{F}$ is a Γ-invariant function, then $f(b) = f(a)$. In other words, each "point" of the set N (i.e., each orbit) determines an $\mathbb{R}$-point of the algebra $\mathcal{F}^\Gamma$, so that we have the natural map

$$N \to \left|\mathcal{F}^\Gamma\right|, \quad \mathcal{O}_a \mapsto (f \mapsto f(a)).$$

This map will be bijective if the two following conditions hold:

(i) Any homomorphism $a\colon \mathcal{F}^\Gamma \to \mathbb{R}$ can be extended to a homomorphism $\tilde{a}\colon \mathcal{F} \to \mathbb{R}$ (surjectivity).

(ii) If $b \notin \mathcal{O}_a$, then there exists an $f \in \mathcal{F}^\Gamma$ such that $f(b) \neq f(a)$ (injectivity).

These two conditions are satisfied, for example, if the group Γ is finite. As an exercise to the reader, we leave their verification in Examples 4.24 and 4.25 below.

4.22 Proposition. *The algebra $\mathcal{F}^\Gamma$ of Γ-invariant functions on a smooth manifold $M = |\mathcal{F}|$ is complete if conditions* (i), (ii) *of* 4.21 *and the assumptions of* Lemma 4.20 *hold.*

◀ Each real-valued function $f\colon \left|\mathcal{F}^\Gamma\right| \to \mathbb{R}$ determines, by means of the projection $M \to N = \left|\mathcal{F}^\Gamma\right|$, $a \mapsto \mathcal{O}_a$, the function $\tilde{f}\colon M \to \mathbb{R}$. If f coincides in a neighborhood of each point $b \in \mathcal{O}_a \in \left|\mathcal{F}^\Gamma\right|$ with some function belonging to $\mathcal{F}^\Gamma$, then the Γ-invariant function $\tilde{f}$ coincides with some smooth function (from $\mathcal{F}$) in a neighborhood of each point $a \in |\mathcal{F}|$. Since $\mathcal{F}$ is complete, $\tilde{f} \in \mathcal{F}$, so that $f \in \mathcal{F}^\Gamma$. ▶

4.23. It is clear that the set of orbits N of a group action of Γ on the manifold $M = |\mathcal{F}|$ is the quotient set of M by an equivalence relation (the one identifying points within each orbit).

Definition. Assume that N coincides with $\left|\mathcal{F}^\Gamma\right|$ and the algebra $\mathcal{F}^\Gamma$ is smooth (or smooth with boundary); then we say that $\mathcal{F}^\Gamma$ is the algebra of smooth functions on the *quotient manifold of M by the group action of* Γ.

Following tradition, we shall often denote the quotient manifold by M/Γ, although this is sometimes a misleading notation (just as is the notation $|\mathcal{F}^\Gamma|$).

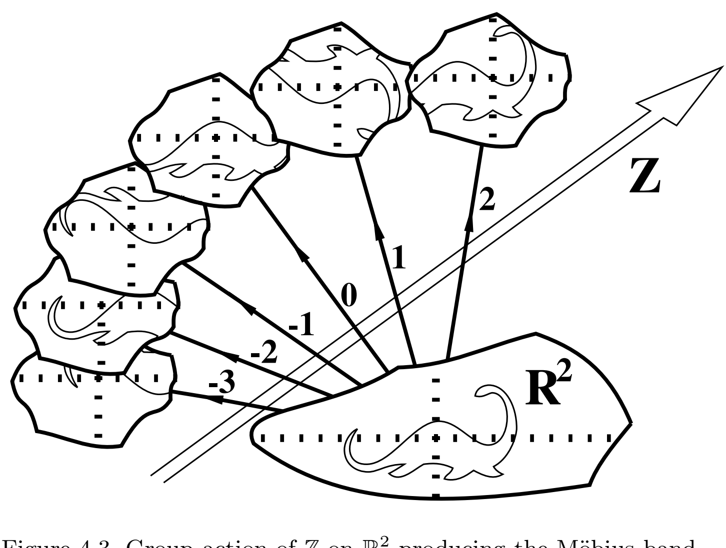


Figure 4.3. Group action of $\mathbb{Z}$ on $\mathbb{R}^2$ producing the Möbius band.

4.24. Examples. I. In Examples 4.5 and 4.7, I, we actually dealt with quotient manifolds of the line $\mathbb{R}$ and of the plane $\mathbb{R}^2$ by the discrete cyclic group $\mathbb{Z}$ of isometries (see Figure 4.3).

II. In Examples 4.7, II, and 4.8 we took the quotient manifolds with respect to the action of isometry groups with two generators. The reader may try to depict how the map $|i|: \mathbb{R}^2 \to |\mathcal{F}|$ winds the plane $\mathbb{R}^2$ about the Klein bottle (see Section 4.7, III).

III. Consider the action on $\mathcal{F} = C^\infty(\mathbb{R}^n)$ of the free abelian group Γ with n generators $\gamma_1, \ldots, \gamma_n$, where γ_i is the parallel translation by the unit vector along the ith coordinate, i.e.,

$$\gamma_i(f)(r_1, \ldots, r_n) = f(r_1, \ldots, r_{i-1}, r_i + 1, r_{i+1}, \ldots, r_n)$$

for all $f \in \mathcal{F}$, $(r_1, \ldots, r_n) \in \mathbb{R}^n$. It is easy to see that $\mathcal{F}^\Gamma$ is the subalgebra of all functions in $C^\infty(\mathbb{R}^n)$ that are 1-periodic with respect to each variable. Generalizing the arguments carried out in Sections 4.5 and 4.7, the reader will easily check that the quotient of $\mathbb{R}^n$ with respect to this action of Γ will be a smooth manifold. This manifold is known as the *n-dimensional torus* and is denoted by T^n. (It is easy to see that $T^1 = S^1$; the most popular case, $n = 2$, was mentioned in the Introduction: It is the surface of the doughnut.)

4.25. Examples. I. Consider the $\mathbb{R}$-algebra $\mathcal{F} = C^\infty(\mathbb{R}^{n+1} \smallsetminus \{0\})$. The multiplicative group $\mathbb{R}^{\boldsymbol{\cdot}}_+$ of positive real numbers acts on $\mathcal{F}$ by the automorphism h_λ:

$$h_\lambda(f)(r_1, \ldots, r_{n+1}) = f(\lambda r_1, \ldots, \lambda r_{n+1})$$
$$\text{for all } f \in \mathcal{F},\ (r_1, \ldots, r_{n+1}) \in \mathbb{R}^{n+1} \smallsetminus \{0\}.$$

It turns out that the corresponding quotient set by this group action is a smooth manifold. Generalizing the argument of Section 4.13, show that the quotient manifold $\mathbb{R}^{n+1}/\mathbb{R}^{\boldsymbol{\cdot}}_+$ can be identified with the closed submanifold in $\mathbb{R}^{n+1}$ whose points satisfy the equation $r_1^2 + \cdots + r_{n+1}^2 = 1$. This is the *$n$-dimensional sphere* S^n. Prove that the ideal $A_{S^n} = \{f \in \mathcal{F} \mid f(a) = 0\ \forall a \in S^n\}$ is the principal ideal generated by the function $(r_1, \ldots, r_{n+1}) \mapsto r_1^2 + \cdots + r_{n+1}^2 - 1$.

II. In the previous example replace the group $\mathbb{R}^{\boldsymbol{\cdot}}_+$ by the multiplicative group $\mathbb{R}^{\boldsymbol{\cdot}}$ of all nonzero real numbers (with the action described by the same formula). Prove that the quotient of $M = |\mathcal{F}|$ by this action is a smooth manifold. In the case $n = 2$ check that it can be identified with the projective plane (cf. Section 4.8 (ii)). In the general case this manifold is known as the *n-dimensional real projective space* and is denoted by $\mathbb{R}P^n$.

III. $\mathbb{R}P^n$ can also be obtained from S^n, $n \geqslant 1$, by taking the quotient with respect to the group action of $\mathbb{Z}_2 = \mathbb{R}^{\boldsymbol{\cdot}}/\mathbb{R}^{\boldsymbol{\cdot}}_+$. Geometrically, this quotient space can be visualized as obtained by "gluing together" all pairs of diametrically opposed points on the sphere.

Nevertheless, note that $\mathbb{R}P^1 \cong S^1$.

4.26 Exercises. 1. Suppose Γ is the automorphism group of the algebra $\mathcal{F} = C^\infty(\mathbb{R}^2)$ with one generator γ:

$$\gamma(f)(r_1, r_2) = f(-r_1, -r_2) \quad \text{for all} \quad (r_1, r_2) \in \mathbb{R}^2, \; f \in \mathcal{F}.$$

Show that the algebra $\mathcal{F}^\Gamma$ of Γ-invariant functions is not smooth (nor smooth with boundary).

2. Suppose Γ is the rotation group of the plane about the origin:

$$|\gamma| = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix},$$

$$\gamma(f)(r_1, r_2) = f(r_1\cos\varphi - r_2\sin\varphi, r_1\sin\varphi + r_2\cos\varphi),$$

for all $(r_1, r_2) \in \mathbb{R}^2$, $f \in \mathcal{F} = C^\infty(\mathbb{R}^2)$.
Show that the space $|\mathcal{F}^\Gamma|$ is the closed half-line, so that $\mathcal{F}^\Gamma$ is an algebra of smooth functions on a manifold with boundary. Why does this algebra not coincide with the whole algebra of smooth functions on the half-line?

4.27. Remarks.

(i) For a number of reasons, the definition of a group action on a manifold given in Section 4.19 is not a very fortunate one; it should be regarded as preliminary. We shall give a satisfactory definition only in Section 6.10.

(ii) We also do not possess any meaningful criterion for the smoothness of the algebra $\mathcal{F}^\Gamma$ of Γ-invariant functions simple enough to mention here. The reader, however, will profit by proving the following: *If for any $a \in M$ there exists a neighborhood $U \subset M$, $U \ni a$, such that for all nontrivial $\gamma \in \Gamma$, $|\gamma|(U) \cap U = \varnothing$, then the algebra $\mathcal{F}^\Gamma$ is smooth.*

(iii) In the next six sections we are also anticipating a bit. Since these sections are relatively difficult, they may be omitted at first reading, which should then be continued from the beginning of Chapter 5.

4.28. If a physical system consists of independent parts, then it is natural to think of any state of the system as being a pair (a_1, a_2), where a_1 and a_2 are the corresponding states of the first and second part. If the states a_i are understood as points of the manifold M_i $(i = 1, 2)$, the algebras $\mathcal{F}_i$ of smooth functions on the M_i being correctly defined and well known, it may be useful to define the manifold M of states of the entire system by using these algebras.

Exercise. Let $\mathbb{R}$-algebras $\mathcal{F}_1$ and $\mathcal{F}_2$ be geometric. Show that their tensor product $\mathcal{F}_1 \otimes_\mathbb{R} \mathcal{F}_2$ is geometric too.

Definition. The smooth envelope $\mathcal{F} = \overline{\mathcal{F}_1 \otimes_\mathbb{R} \mathcal{F}_2}$ of the tensor product $\mathcal{F}_1 \otimes_\mathbb{R} \mathcal{F}_2$ of geometric $\mathbb{R}$-algebras (see Section 3.36) is said to be the *algebra* *of smooth functions of the Cartesian product of the smooth manifolds* M_1 *and* M_2*.*

In the next section this terminology will be justified.

4.29 Proposition. *If* $\mathcal{F}$ *is the algebra of smooth functions on the Cartesian product of the manifolds* M_1 *and* M_2*, then* $|\mathcal{F}|$ *is indeed homeomorphic to the Cartesian product of the topological spaces* $M_1 = |\mathcal{F}_1|$ *and* $M_2 = |\mathcal{F}_2|$*.*

◀ (We suggest that the reader return to this proof after having read Chapter 6.) Our goal is to identify the space $M_1 \times M_2$ with the set of $\mathbb{R}$-points of the algebra $\mathcal{F} = \overline{\mathcal{F}_1 \otimes_{\mathbb{R}} \mathcal{F}_2}$.

To each pair $(a_1, a_2) \in M_1 \times M_2$ let us assign the homomorphism $a_1 \otimes a_2$:

$$\mathcal{F}_1 \underset{\mathbb{R}}{\otimes} \mathcal{F}_2 \to \mathbb{R} \quad (f_1 \otimes f_2 \mapsto f_1(a_1) \cdot f_2(a_2)), \quad f_i \in \mathcal{F}_i \ (i = 1, 2).$$

Note that the algebra $\mathbb{R}$ is C^∞-closed (Section 3.32). Hence by the definition of smooth envelope, $a_1 \otimes a_2$ can be uniquely extended to the homomorphism

$$\overline{a_1 \otimes a_2} \colon \mathcal{F} = \overline{\mathcal{F}_1 \underset{\mathbb{R}}{\otimes} \mathcal{F}_2} \to \mathbb{R},$$

and we have constructed the map

$$\pi \colon M_1 \times M_2 \to |\mathcal{F}| \quad \big((a_1, a_2) \mapsto \overline{a_1 \otimes a_2}\big).$$

The map π is injective: If $\overline{a_1 \otimes a_2}$ coincides with $\overline{a_1' \otimes a_2'}$, then, using the fact that these homomorphisms coincide on elements of the form $f_1 \otimes 1$ and $1 \otimes f_2$, we immediately conclude that

$$f_i(a_i) = f_i(a_i') \text{ for all } f_i \in \mathcal{F}_i, \quad i = 1, 2.$$

Since the algebras $\mathcal{F}_i$ are geometric, this implies $a_i = a_i'$ and hence the injectivity of π. Its surjectivity follows from elementary properties of tensor products. Thus π identifies $M_1 \times M_2$ and $|\mathcal{F}|$ as sets.

It remains to prove that π identifies the standard product topology in $M_1 \times M_2$ with the $\mathbb{R}$-algebra topology (Section 3.12) in $|\mathcal{F}|$. Consider the basis of the topology in $M_1 \times M_2$ consisting of the sets $U_1 \times U_2$ with

$$U_i = \{a \in M_i \mid \alpha_i < f_i(a) < \beta_i\}, \quad f_i \in \mathcal{F}_i, \ \alpha_i, \beta_i \in \mathbb{R}, \ i = 1, 2.$$

Then the sets

$$V_1 = U_1 \times M_2 = \left\{(a_1, a_2) \mid \alpha_1 < \overline{a_1 \otimes a_2}(f_1 \otimes 1) < \beta_1\right\},$$
$$V_2 = M_1 \times U_2 = \left\{(a_1, a_2) \mid \alpha_2 < \overline{a_1 \otimes a_2}(1 \otimes f_2) < \beta_2\right\},$$

are open in the topology induced from $|\mathcal{F}|$ by π in $M_1 \times M_2$. Therefore, $V_1 \cap V_2 = U_1 \times U_2$ is open as well.

Conversely, it follows from the construction of smooth envelopes (Section 3.36) that in order to obtain a basis of the topology in $|\mathcal{F}|$, we can take any subset of functions in $\mathcal{F}$ as long as the subalgebra generated by this subset has a smooth envelope coinciding with $\mathcal{F}$. In the given case it suffices to take the subset of functions of the form $f_1 \otimes f_2$, $f_i \in \mathcal{F}_i$, $i = 1, 2$.

Consider the basis open set

$$V = \{(a_1, a_2) \in M_1 \times M_2 = |\mathcal{F}| \mid \alpha < f_1(a_1) f_2(a_2) < \beta\}, \ \alpha, \beta \in \mathbb{R},$$

corresponding to such a function.

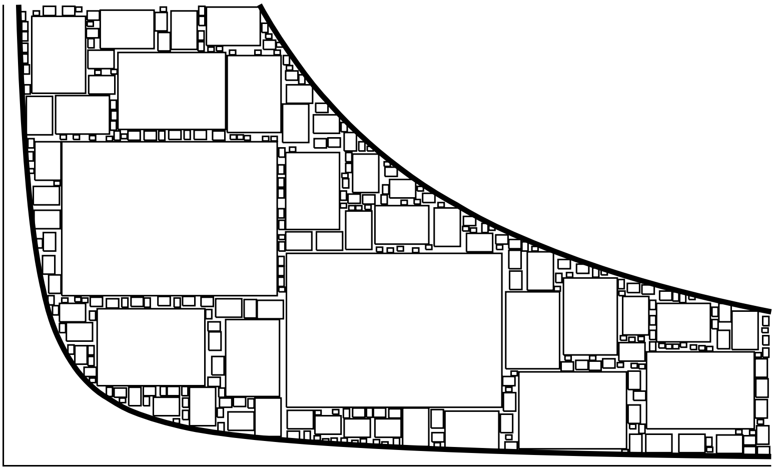


Figure 4.4. A basic open set in the Cartesian product.

The set of points on the plane $\mathbb{R}^2$ satisfying the inequality $\alpha < r_1 r_2 < \beta$ is open in the sense that, together with any of its points, it contains a rectangle $\{(r_1, r_2) \mid \alpha_1 < r_1 < \beta_1, \ \alpha_2 < r_2 < \beta_2\}$ (see Figure 4.4). Hence V is the union of sets of the form

$$\{(a_1, a_2) \in M_1 \times M_2 \mid \alpha_1 < f_1(a_1) < \beta, \ \alpha_2 < f_2(a_2) < \beta_2\}$$

and is therefore open in $M_1 \times M_2$. ►

4.30 Example–Lemma. *The smooth envelope of the* $\mathbb{R}$*-algebra*

$$C^\infty\left(\mathbb{R}^k\right) \otimes_{\mathbb{R}} C^\infty\left(\mathbb{R}^l\right)$$

is isomorphic to the $\mathbb{R}$*-algebra* $C^\infty\left(\mathbb{R}^{k+l}\right)$.

◄ Consider the $\mathbb{R}$-algebra homomorphism

$$i \colon C^\infty\left(\mathbb{R}^k\right) \otimes_{\mathbb{R}} C^\infty\left(\mathbb{R}^l\right) \to C^\infty\left(\mathbb{R}^{k+l}\right),$$
$$i(f \otimes g)(r_1, \ldots, r_{k+l}) = f(r_1, \ldots, r_k) \cdot g(r_{k+1}, \ldots, r_{k+l}).$$

We shall show that i satisfies the definition of smooth envelope (Section 3.36).

Indeed, suppose

$$\Phi \colon C^\infty\left(\mathbb{R}^k\right) \underset{\mathbb{R}}{\otimes} C^\infty\left(\mathbb{R}^l\right) \to \mathcal{F}$$

is a homomorphism into a C^∞-closed (Section 3.32) $\mathbb{R}$-algebra $\mathcal{F}$. A homomorphism

$$\Phi' \colon C^\infty\left(\mathbb{R}^{k+l}\right) \to \mathcal{F}$$

is a prolongation of Φ (i.e., $\Phi = \Phi' \circ i$) if and only if for all $(r_1, \ldots, r_k) \in \mathbb{R}^k$, $(s_1, \ldots, s_l) \in \mathbb{R}^l$, $g \in C^\infty\left(\mathbb{R}^{k+l}\right)$, we have

$$\Phi'(g) = g(\Phi(r_1 \otimes 1), \ldots, \Phi(r_k \otimes 1), \Phi(1 \otimes s_1), \ldots, \Phi(1 \otimes s_l)).$$

(Here, on the right-hand side, we regard the r_i and s_j as functions on $\mathbb{R}^k$ and $\mathbb{R}^l$; the right-hand side is then well defined because $\mathcal{F}$ is C^∞-closed.)

Since the last formula is well defined for any $g \in C^\infty\left(\mathbb{R}^{k+l}\right)$, the required homomorphism exists and is unique. By the uniqueness theorem in Section 3.37, the lemma follows. ▶

4.31 Proposition. *If $\mathcal{F}_1$, $\mathcal{F}_2$ are smooth $\mathbb{R}$-algebras, then so is* $\mathcal{F} = \overline{\mathcal{F}_1 \otimes_{\mathbb{R}} \mathcal{F}_2}$ (see Section 4.28).

◀ Since the proof of this proposition repeats the one given in Section 4.29, we only indicate the main ideas, leaving the details to the industrious reader.

Suppose $a_i \in M_i = |\mathcal{F}|$ and let $U_i \ni a_i$ be neighborhoods such that $\mathcal{F}_i\big|_{U_i} \cong C^\infty\left(\mathbb{R}^n_i\right)$, $i = 1, 2$. We would like to establish the isomorphism

$$\mathcal{F}\big|_{U_1 \times U_2} \cong C^\infty\left(\mathbb{R}^{n_1+n_2}\right).$$

By 4.30, it suffices to show that

$$\mathcal{F}\big|_{U_1 \times U_2} \cong \overline{\mathcal{F}_1\big|_{U_1} \otimes_{\mathbb{R}} \mathcal{F}_2\big|_{U_2}}.$$

But as before, there is a homomorphism

$$i \colon \mathcal{F}_1\big|_{U_1} \otimes_{\mathbb{R}} \mathcal{F}_2\big|_{U_2} \to \mathcal{F}\big|_{U_1 \times U_2},$$

which, as it turns out, satisfies the definition of a smooth envelope. ▶

Exercises. 1. Let $\mathcal{F}_1$ be a smooth algebra with boundary (see Section 4.2) and let $\mathcal{F}_2$ be a smooth algebra. Mimicking the proof of the above proposition, show that the algebra $\mathcal{F} = \overline{\mathcal{F}_1 \otimes_{\mathbb{R}} \mathcal{F}_2}$ is smooth with boundary.

2. Does the previous assertion remain valid if $\mathcal{F}_2$ is also a smooth algebra with boundary?

4.32. Examples. I. The cylinder in Example 4.5, II, is the Cartesian product of S^1 and $\mathbb{R}$.

II. The n-dimensional torus T^n (see 4.24, III) is the Cartesian product of T^{n-1} and S^1. In particular, $T^2 = S^1 \times S^1$.

4.33 Exercise. The reader sufficiently versed in topology will profit a great deal by proving that the Möbius band (Example 4.7, I) is *not* the Cartesian product of $\mathbb{R}$ and S^1.

5
Charts and Atlases

5.1. In this chapter the intuitive idea of "introducing local coordinates" is elaborated into a formal mathematical definition of a differentiable manifold. The definition, of course, turns out to be equivalent to the algebraic one given in the previous chapter, as will be proved in Chapter 7.

The coordinate approach is more traditional, and is certainly more appropriate for practical applications (when something must be computed). However, it is less suitable for developing the theory, since it requires tedious verifications of the fact that the notions and constructions introduced in the theory by means of coordinates are *well defined*, i.e., independent of the specific choice of local coordinates.

In the coordinate approach, a manifold structure on a set is defined by a family of *compatible charts* constituting a *smooth atlas*, much in the same way as the geopolitical structure on the Earth's surface is described by the charts of a geographical atlas. The words in italics above will be given mathematical definition in subsequent sections.

5.2. *A chart* (U, x) on the set M is a bijective map $x\colon U \to \mathbb{R}^n$ of a subset $U \subset M$ onto an open set $x(U)$ of Euclidean space $\mathbb{R}^n$. The integer $n > 0$ is the *dimension* of the chart.

Examples. I. If U is an open set in $\mathbb{R}^n$, then the identity map defines a chart (U, id) on the set $\mathbb{R}^n$.

II. If T^2 is the configuration space of the plane double pendulum (Figure 5.1), then (U, s), where

$$U = \left\{(\varphi, \psi) \in T^2 \mid -\frac{\pi}{4} < \varphi, \psi < \frac{\pi}{4}\right\}$$

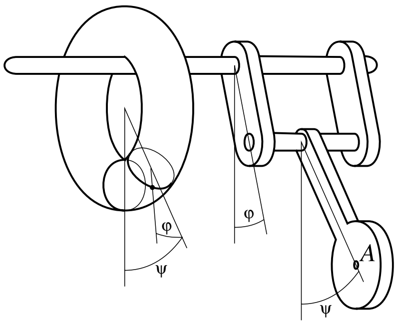


Figure 5.1. Double pendulum.

and s denotes the map

$$U \ni (\varphi, \psi) \overset{s}{\longmapsto} (\sin\varphi, \sin\psi) \in \mathbb{R}^2,$$

is a chart on T^2. (Note that by assigning to each position of the pendulum the coordinates of its end point $A \in \mathbb{R}^2$, we do not obtain a chart, since this assignment is not bijective.)

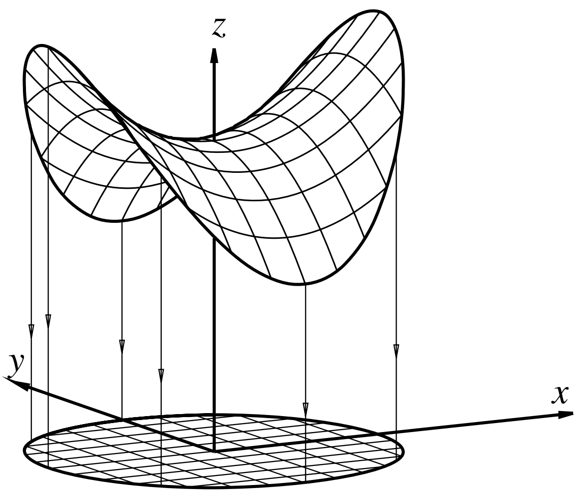


Figure 5.2. Saddle surface.

III. If S is the saddle surface $z = 1 + x^2 - y^2$, then the vertical projection

$$S \ni (x, y, z) \overset{\text{pr}}{\longmapsto} (x, y) \in \mathbb{R}^2$$

of a neighborhood $U \subset S$ of the point $(0,0,1)$ defines a chart (U, pr) on S (Figure 5.2).

5.3. Given a chart (U, x) and a point $a \in U$, note that $x(a)$ is a point of $\mathbb{R}^n$; i.e., we have $x(a) = (r_1, \ldots, r_n) \in \mathbb{R}^n$; the number r_i is called the *ith coordinate* of a, and the corresponding function (sending each $a \in U$ to its ith coordinate) is the *ith coordinate function* (in the chart (U, x)); it is denoted by $x_i \colon U \to \mathbb{R}$. A chart is entirely determined by its coordinate functions; in the literature, the expression *local coordinates* is often used to mean "chart" in this sense.

5.4. Two charts (U, x), (V, y) on the same set M are called *compatible* if the *change of coordinate map*, i.e.,

$$y \circ x^{-1} \colon x(U \cap V) \to y(U \cap V),$$

is a diffeomorphism of open subsets of $\mathbb{R}^n$ (see Figure 5.3) or if $U \cap V = \varnothing$. The compatibility relation is reflexive, symmetric, and transitive (because of appropriate properties of diffeomorphisms in $\mathbb{R}^n$) so that the family of all charts on a given set M splits into equivalence classes (sets of compatible charts).

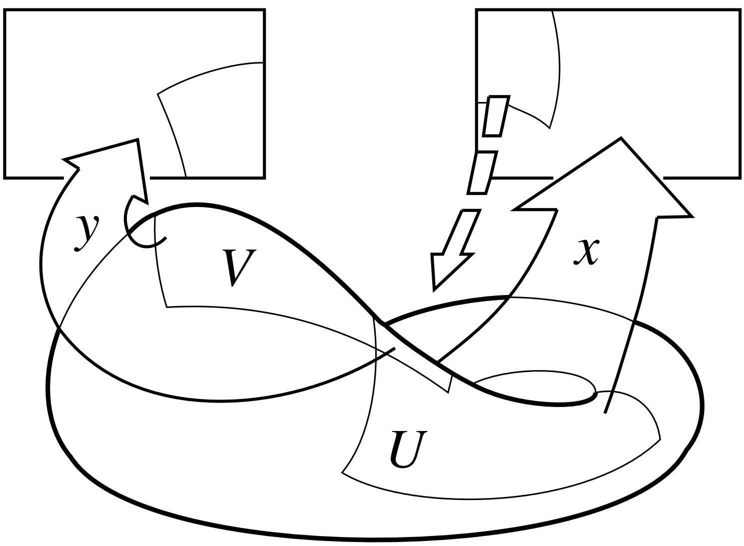


Figure 5.3. Compatible charts.

Examples. I. All the charts (U, id), where U is an open subset of $\mathbb{R}^n$ and n is fixed, are compatible.

II. The chart (U, s) on the double pendulum T^2 described in 5.2, II, is not compatible with the chart (U, c) defined by

$$U \ni (\varphi, \psi) \xrightarrow{c} (\sin\varphi, g(\psi)),$$

where $g(\psi)$ equals $\sin\psi$ for negative ψ and $1-\cos\psi$ for nonnegative ψ. This is because the change of coordinates map $c \circ s^{-1}$ fails to be smooth at the point $(0,0) \in \mathbb{R}^2$.

5.5. A family $\mathcal{A}$ of compatible charts $x_\alpha \colon U_\alpha \to \mathbb{R}^n$ on the set M (where $n \geqslant 0$ is fixed and α ranges over some index set J) is said to be an *atlas* on M if the U_α cover M, i.e., $\bigcup_{\alpha\in J} U_\alpha = M$. The integer n is the *dimension* of $\mathcal{A}$. An atlas is *maximal* if it is not contained in any other atlas. Obviously, *any atlas is contained in a unique maximal atlas*, namely the one consisting of all charts compatible with any of the charts of the given atlas.

Two atlases are said to be *compatible* if any chart of one of them is compatible with any chart of the other. The last condition is equivalent to the fact that the union of these atlases is also an atlas. Note that any two compatible atlases, together with their union, are contained in the same maximal atlas.

This compatibility relation is reflexive, symmetric, and transitive (by the corresponding properties of diffeomorphisms in $\mathbb{R}^n$). Consequently, the family of all atlases on a given set M splits into equivalence classes, and any such a class contains a unique maximal atlas.

Examples. I. $\mathbb{R}^n$ has an atlas consisting of a single chart: $(\mathbb{R}^n, \mathrm{id})$.

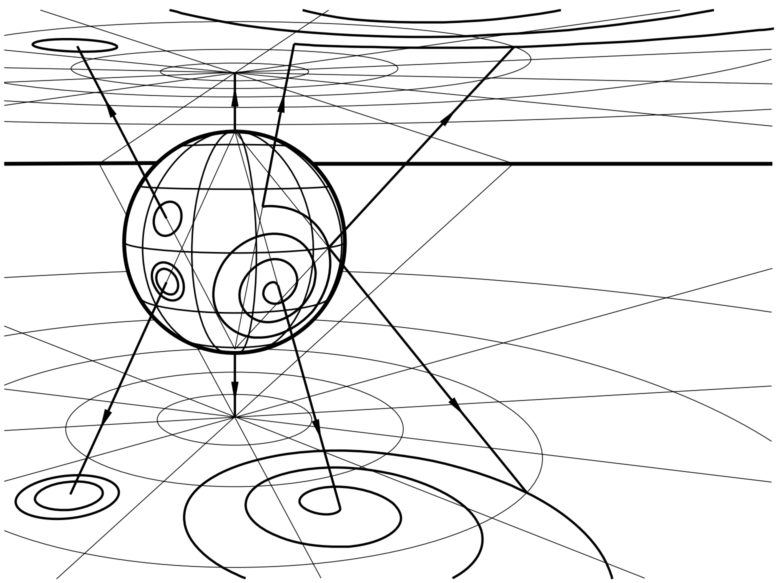


Figure 5.4. Stereographic projections.

II. The sphere S^2 has an atlas consisting of two charts (e.g., stereographic projections from the north and south poles; see Figure 5.4).

III. The double pendulum (see Example 5.2, II) also has two-chart atlases (try to find one).

5.6. The reader is perhaps wondering why we are not defining manifolds as sets supplied with an atlas. Well, we won't be. Because if we did, this excessively general definition would put us under the obligation to bestow the noble title of manifold upon certain ungainly objects. Such as:

I. *The discrete line.* This is the set of points of $\mathbb{R}$ with the discrete atlas consisting of all charts of the form $(\{r\}, v)$, where $r \in \mathbb{R}$ and $v(r) = 0 \in \mathbb{R}^0$.

II. *The long line.* This is the disjoint union

$$\mathcal{R} = \coprod_{\alpha} \mathbb{R}_\alpha$$

of copies $\mathbb{R}_\alpha$ of $\mathbb{R}$, indexed by an ordered uncountable set of indices α (we can take α to range over $\mathbb{R}$ itself). The set $\mathcal{R}$ has a natural order and a natural topology (induced from $\mathbb{R}$ and disjoint union). It also has a natural atlas $\mathcal{A} = \{(\mathbb{R}_\alpha, \mathrm{id}_\alpha) \colon \alpha \in \mathbb{R}\}$, where $\mathrm{id}_\alpha \colon \mathbb{R}_\alpha \to \mathbb{R}^1$ is the identification of each copy $\mathbb{R}_\alpha$ with the original prototype $\mathbb{R}^1 = \mathbb{R}$.

III. *The line with a double point.* This is the line $\mathbb{R}$ to which a point θ is added, while the atlas consists of two charts (U, x) and (V, y), where

$$\begin{aligned} U &= \mathbb{R}, & x &= \mathrm{id}, \\ V &= \{\theta\} \cup \mathbb{R} \smallsetminus \{0\}, & y(\theta) &= 0, \quad y(r) = r, \ r \in \mathbb{R} \smallsetminus \{0\}. \end{aligned}$$

In other words, the line with a double point can be obtained if two copies of the line $\mathbb{R}$ are identified at all points with the same coordinates except for zero.

5.7. The following definitions are needed to exclude pathological atlases (of the types described in Section 5.6) from our considerations. We say that an atlas $\mathcal{A}$ on M satisfies the *countability condition* if it consists of a finite or a countable number of charts or if all its charts are compatible with those of such an atlas. An atlas $\mathcal{A}$ on M satisfies the *Hausdorff condition* if for any two points $a, b \in M$ there exist nonintersecting charts (U, x), (V, y) containing these points ($a \in U$, $b \in V$, $U \cap V = \varnothing$) and compatible with the charts of $\mathcal{A}$.

Clearly, the discrete and long lines (see Examples 5.6, I, and II) do not satisfy the countability condition, while the line with a double point, 5.6, III, fails to meet the Hausdorff condition.

5.8. Coordinate definition of manifolds. A set supplied with a maximal atlas $\mathcal{A}_{\max} = \{(U_\alpha, x_\alpha)\}$, $x_\alpha \colon U_\alpha \to \mathbb{R}^n$, $n \geqslant 0$, satisfying the countability and Hausdorff conditions is called an *n-dimensional differentiable* (or *smooth*) *manifold.* It will be proved in Chapter 7 that this definition is equivalent to the algebraic one given in Section 4.1.

In order to determine a specific manifold $(M, \mathcal{A}_{\max})$, we shall often indicate some smaller atlas $\mathcal{A} \subset \mathcal{A}_{\max}$, since $\mathcal{A}_{\max}$ is uniquely determined by any of its subatlases (see 5.5). In that case we denote our manifold by

$(M, \mathcal{A})$ and say that $\mathcal{A}$ is a *smooth atlas* on M. Note that the adjective "smooth" implicitly includes the compatibility, Hausdorff, and countability conditions.

5.9. We now show that *any smooth atlas* $\mathcal{A} = \{(U_\alpha, x_\alpha)\}$ *on the manifold* M *determines a topological structure on the set* M, carried over from the Euclidean topology in the sets $x_\alpha(U_\alpha) \subset \mathbb{R}^n$ by the maps x_α^{-1}. To be more precise, a base of open sets in the space M is constituted by all the sets $x_\alpha^{-1}(B_\beta)$, where the B_β's are all the open Euclidean balls contained in all the $x_\alpha(U_\alpha)$. It follows immediately from the definitions that M then becomes a *Hausdorff topological space with countable base.* When the set M is supplied with an atlas, this topological structure is always understood. For example, when we say that the manifold M is *compact* or *connected*, we mean that it is a compact (connected) topological space with respect to the topology described above.

It is easy to see that *any chart* (U, x) *is a homeomorphism* of an open subset $U \subset M$ (in this topology) onto an open subset $x(U) \subset \mathbb{R}^n$.

5.10. Examples of manifolds from geometry. I. The *sphere* $S^n = \{\vec{x} \colon |\vec{x}| = 1\} \subset \mathbb{R}^{n+1}$ has a two-chart atlas given by stereographic projection (similar to the two-dimensional case; see 5.5, II). The sphere S^n is the simplest compact connected n-dimensional manifold.

II. The *hyperboloid* $x^2 + y^2 - z^2 = 1$ in $\mathbb{R}^3$ has a simple four-chart atlas $(U_\pm, p_{zy})$, $(V_\pm, p_{zx})$, where

$$U_\pm = \{(x, y, z) \mid x = \pm\sqrt{1 + z^2 - y^2},\ \pm x > 0\},$$
$$V_\pm = \{(x, y, z) \mid y = \pm\sqrt{1 + z^2 - x^2},\ \pm y > 0\},$$

and the maps p_{zy}, p_{zx} are the projections on the corresponding planes. The corresponding two-dimensional manifold is connected but not compact.

III. *The projective space* $\mathbb{R}P^n$ is the set of all straight lines passing through the origin O of $\mathbb{R}^{n+1}$. For each such line l consider the following chart (U_l, p_l). The set U_l consists of all lines forming an angle of less than (say) 30° with l (Figure 5.5). To define p_l, choose a basis in the n-plane $l^\perp \ni O$ perpendicular to l and fix one of the half-spaces $\mathbb{R}^{n+1} \setminus l^\perp$; to every line $l' \in U_l$ let the map p_l assign the coordinates in $l^\perp$ of the projection on $l^\perp$ of the unit vector pointing into the chosen half-space and determining l'. The set of all such charts (U_l, p_l) constitutes a smooth atlas, endowing $\mathbb{R}P^n$ with the structure of a compact connected n-dimensional manifold.

IV. The *Grassmann space* $G_{n,m}$ is the set of all m-dimensional planes in $\mathbb{R}^n$ passing through the origin O. To construct a chart, let us choose in $\mathbb{R}^n$ a Cartesian system $(x_1, x_2, \ldots, x_n)$ and take for U the set of all m-dimensional planes given in these coordinates by the system of equations

$$x_{m+i} = \sum_{j=1}^{m} a_{ij} x_j, \qquad i = 1, \ldots, n - m.$$

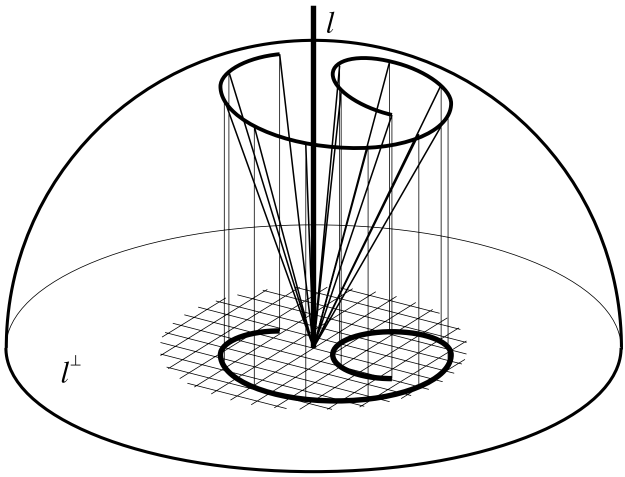


Figure 5.5. Construction of a chart on $\mathbb{R}P^n$.

Let also the map $p\colon U \to \mathbb{R}^{m(n-m)}$ take such a plane to the set of coefficients a_{ij} appearing in the system above. Choosing different Cartesian systems, one can construct different charts covering together the entire space $G_{n,m}$. However, to cover it, it suffices to use only one Cartesian system and change the order of coordinates in it. Compatibility of these charts follows from the smooth dependency of solutions of a linear system on the system's coefficients. Thus we obtain an $m(n-m)$-dimensional manifold that generalizes the previous example, namely $G_{n,1} = \mathbb{R}P^{n-1}$.

One can consider planes not only in $\mathbb{R}^n$ but in any finite-dimensional vector space V. The manifold obtained in this case is denoted by $G_{V,m}$.

Exercise. How many connected charts do you need to obtain an atlas for the Klein bottle (cf. 4.7, III)? Prove that two is enough.

5.11. Examples of manifolds from algebra. I. The *general linear group* $\mathrm{GL}(n)$ of all linear isomorphisms of $\mathbb{R}^n$ has a one-chart atlas of dimension n^2 obtained by assigning to each $g \in \mathrm{GL}(n)$ the n^2 entries of its matrix written column by column in the form of a single column vector with n^2 components. The corresponding manifold is not compact and not connected.

II. The *special orthogonal group* $\mathrm{SO}(n)$ of all positive orthogonal matrices possesses a smooth atlas of $(n(n-1)/2)$-dimensional charts. Its construction is left to the reader, who might profit by referring to Example 5.10, III.

5.12. Examples of manifolds from mechanics. I. The configuration space of the *double pendulum* (see 5.2, II, and 5.4, II), as the reader must have guessed by now, is the two-dimensional *torus* $T^2 = S^1 \times S^1$.

II. The configuration space of a *thin uniform disk whose center is fixed by a hinge* (allowing it be inclined at all angles and directions in three-space) possesses a natural two-dimensional atlas. The reader is urged to find such an atlas and compare it with that of the projective plane $\mathbb{R}P^2$. He/she will also appreciate that if one side of the disk is painted, then the corresponding atlas will be that of S^2.

III. The configuration space (recall Section 1.1) of a *solid freely rotating in the space about a fixed point* is a three-dimensional compact connected manifold. The reader is asked to find an atlas for it and compare it with $\mathbb{R}P^3$ and SO(3).

5.13 Exercises. Consider the following manifolds (already discussed informally in Chapter 1):

1. The projective space $\mathbb{R}P^3$.
2. The sphere S^3 with antipodal points identified.
3. The disk D^3 with antipodal points of its boundary $\partial D^3 = S^2$ identified.
4. The special orthogonal group SO(3).
5. The configuration space of a solid freely rotating about a fixed point in $\mathbb{R}^3$.

Show that they are all diffeomorphic by

1. Constructing atlases and diffeomorphisms between the charts.
2. Constructing isomorphisms of the corresponding smooth $\mathbb{R}$-algebras.

5.14. Previously (see Section 4.2), we introduced the notion of manifold with boundary algebraically. Now we give the corresponding coordinate definition.

This definition is just the same as that of an ordinary manifold (see Section 5.8), except that the notion of chart must be modified (by substituting $\mathbb{R}^n_H$ for $\mathbb{R}^n$). Namely, a *chart with boundary* (U, x) on the set M is a bijective map $x\colon U \to \mathbb{R}^n_H$ of a subset $U \subset M$ onto an open subset $x(U)$ of the Euclidean half-space

$$\mathbb{R}^n_H = \{(r_1, \ldots, r_n) \in \mathbb{R}^n \mid r_n \geqslant 0\}.$$

Note that a chart in $\mathbb{R}^n$ is a particular case of a chart in $\mathbb{R}^n_H$, since $x(U)$ may not intersect the “boundary”

$$\{(r_1, \ldots, r_n) \in \mathbb{R}^n \mid r_n = 0\}$$

of the half-space.

All further definitions (those of dimension, compatibility, atlases, etc.) remain the same, except that $\mathbb{R}^n_H$ must be substituted for $\mathbb{R}^n$ in the appropriate places. Repeating these definitions in this modified form, we obtain the coordinate definition of a *manifold with boundary.*

The industrious reader will gain by actually carrying out these repetitions in detail; this is a good way to check that he/she has in fact mastered the main Definition 5.8.

5.15. If $\mathcal{A} = (U_\alpha, x_\alpha)$, $\alpha \in \mathcal{I}$, consists of charts with boundary, the set

$$\partial M = \{m \in M \mid \exists \alpha \in \mathcal{I},\ m = x_\alpha^{-1}(r_1, \dots, r_n),\ r_n = 0\}$$

of points mapped by coordinate maps on the boundary $(n-1)$-dimensional plane $\{r_n = 0\}$ of the half-space $\mathbb{R}^n_H$ is said to be the *boundary* of M.

When ∂M is empty, we recover the definition of ordinary manifold (the open half space $\{r_n > 0\}$ being homeomorphic to $\mathbb{R}^n$).

Sometimes in the literature the term "manifold" is defined so as to include manifolds with boundary; in that case the expression *closed manifold* is used to mean manifold (with empty boundary).

Proposition. *The boundary ∂M of an n-dimensional manifold with boundary M has a natural $(n-1)$-dimensional manifold structure.*

◀ Hint of the proof. Take the intersection of the charts of M with the boundary $(n-1)$-plane of the half-space

$$\mathbb{R}^{n-1} = \{(r_1, \dots, r_n) \in \mathbb{R}^n_H \mid r_n = 0\}$$

to get an atlas on ∂M. ▶

5.16. Examples of manifolds with boundary. I. The *n-dimensional disk* $D^n = \{x \mid \|x\| \leqslant 1\} \subset \mathbb{R}^n$ is a n-dimensional manifold with boundary $\partial D^n = S^{n-1}$.

II. If M is an n-dimensional manifold (without boundary) defined by its atlas $\mathcal{A}$, a manifold with boundary can be obtained from M by "removing an open disk from it." This means that we take any chart $(U, x) \in \mathcal{A}$, choose an open n-dimensional disk V in $x(U) \subset \mathbb{R}^n$, and consider the set $M \setminus x^{-1}(V)$, which has an obvious manifold-with-boundary structure; $M \setminus x^{-1}(V)$ is sometimes called a *punctured manifold.*

III. Figure 5.6 presents the beginning of a complete list of all two-dimensional manifolds (with boundary) whose boundary is the circle S^1. They are called the 2-disk, the punctured torus, the punctured orientable surface of genus 2, …, the punctured orientable surface of genus k, … (upper row), the Möbius band, the punctured Klein bottle,…, the punctured nonorientable surface of genus k, …. The term "orientable," which we do not discuss in the general case here, in the two-dimensional case means "does not contain a Möbius band."

5.17. In the algebraic study of smooth manifolds, the fundamental concept was the $\mathbb{R}$-algebra of smooth functions $\mathcal{F}$. This $\mathbb{R}$-algebra can also be defined for a manifold M given by an atlas $\mathcal{A}$.

Definitions. A function $f\colon M \to \mathbb{R}$ on the manifold M with smooth atlas $\mathcal{A}$ is called *smooth* if for any chart $(U, x) \in \mathcal{A}$ the function $f \circ x^{-1}\colon x(U) \to \mathbb{R}$ defined on the open set $x(U) \subset \mathbb{R}^n$, is smooth (i.e., $f \circ x^{-1} \in C^\infty(x(U))$).

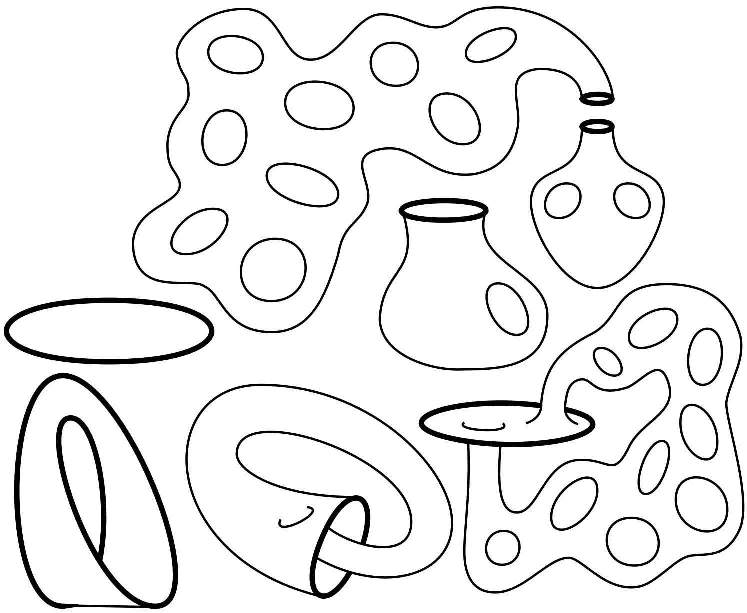


Figure 5.6. Two-dimensional manifolds with S^1 as the boundary.

The set of all smooth functions on M is denoted by $C^\infty(M)$. This set has an obvious $\mathbb{R}$-algebra structure, and will temporarily be called the *$\mathbb{R}$-algebra of smooth functions on M with respect to the atlas $\mathcal{A}$*.

It is easy to establish that $C^\infty(M)$ is the same for any atlas $\mathcal{A}'$ compatible with $\mathcal{A}$. Moreover, we shall see that $C^\infty(M)$ is the same $\mathbb{R}$-algebra as the one in the algebraic approach ($C^\infty(M) = \mathcal{F}$), but this will be proved only in Chapter 7).

Exercise. 1. Describe the smooth function algebra (in the sense of the above definition) of the configuration space for the double pendulum (see 5.12).

2. Same question for the case in which the φ-rod is shorter then the ψ-road, so that the latter is blocked in its rotation when the point A hits the φ-axle (see Figure 5.1). Identify the corresponding smooth manifold.

5.18. In the equivalence proof carried out in Chapter 7, we shall need the following proposition:

Proposition. *If M is a manifold, $\mathcal{A}$ its smooth atlas, and $C^\infty(M)$ the $\mathbb{R}$-algebra of smooth functions on M (with respect to $\mathcal{A}$), then there exists a function $f \in C^\infty(M)$ all of whose level surfaces (i.e., the sets $f^{-1}(\lambda)$, $\lambda \in \mathbb{R}$) are compact subsets of M.*

This proposition generalizes Proposition 2.7 and can be proved by using the latter and the partition of unity lemma, stated in the appropriate form for manifolds with atlases (compare with Section 4.18).

6
Smooth Maps

6.1. Suppose $\mathcal{F}_1$ is the algebra of smooth functions on a manifold M_1, and $\mathcal{F}_2$ is the one on another manifold M_2. The map $f\colon M_1 \to M_2$ is called *smooth* if $f = |\varphi|$, where $\varphi\colon \mathcal{F}_2 \to \mathcal{F}_1$ is an $\mathbb{R}$-algebra homomorphism.

Recall that $M_i = |\mathcal{F}_i|$, $i = 1, 2$, are the dual spaces to $\mathcal{F}_i$ (see Section 3.8), i.e., consist of all $\mathbb{R}$-algebra homomorphisms $x\colon \mathcal{F}_i \to \mathbb{R}$, $i = 1, 2$. Recall also that $|\varphi|\colon |\mathcal{F}_1| \to |\mathcal{F}_2|$ is the dual map defined in 3.19 as $|\varphi|\colon x \mapsto x \circ \varphi$ and that all homomorphisms (see Section 3.1) are unital: $\varphi(1) = 1$.

Exercise. Prove that φ is injective whenever f is surjective. Construct a counterexample to the converse statement (if you do not succeed, try again after reading Example 6.5 below).

6.2. Example. Suppose $\mathcal{F}$ is the algebra of smooth functions on the manifold M, and Γ is a group acting on M. The map

$$p = |i|\colon M \to M/\Gamma$$

dual to the inclusion $i\colon \mathcal{F}^\Gamma \to \mathcal{F}$ of the algebra of Γ-invariant functions (see 4.19) into $\mathcal{F}$ is of course smooth.

To be specific, consider the group $\Gamma = \mathbb{Z}$ acting on $\mathbb{R}$ by identifying points r_1 and r_2 whenever $r_1 - r_2 \in \mathbb{Z}$. Denote by S^1 the set of equivalence classes, and let $\mathcal{F}$ be, as in 4.5, the algebra of smooth 1-periodic functions on the line $\mathbb{R}$. The natural projection $p\colon \mathbb{R} \to S^1$ is a smooth map, since it coincides with $|i|$, where i is the inclusion $\mathcal{F} \subset C^\infty(\mathbb{R})$.

We say that the map p winds the line $\mathbb{R}$ around the circle S^1.

6.3. Example. Let $\mathcal{F}$ be the algebra of smooth functions on the manifold M, and $N \subset M$ a smooth submanifold (Definition 4.11) of M. In this case the inclusion $N \hookrightarrow M$ is a smooth map, since it coincides with $|\rho|$, where

$$\rho\colon \mathcal{F} \to \mathcal{F}_N = \mathcal{F}\big|_N$$

is the restriction homomorphism.

To be specific, suppose that S^1 and $\mathcal{F}$ are the same as in Example 6.2 and let $\mathcal{F}_{\mathbb{C}}$ be the algebra of all real-valued functions of a complex variable $z = x + iy$ smoothly depending on the variables x and y. Consider the inclusion

$$i\colon S^1 \hookrightarrow \mathbb{C} \quad \left([r] \longmapsto e^{2\pi i r}\right),$$

where $[r] \in S^1$ is the equivalence class of the point $r \in \mathbb{R}$. Further, define the homomorphism

$$\beta\colon \mathcal{F}_{\mathbb{C}} \to \mathcal{F} \quad \left(\beta(f)(r) = f\left(e^{2\pi i r}\right),\ f \in \mathcal{F}_{\mathbb{C}},\ r \in \mathbb{R}\right).$$

The inclusion i coincides with $|\beta|$ and is therefore a smooth map of S^1 into $\mathbb{R}^2 = \mathbb{C}$.

Notice that we have already met this inclusion in another coordinate representation; see Example 4.13.

6.4. Examples. Suppose $\mathcal{F}$ is the algebra of smooth functions on the open Möbius band (4.7, I) and $\mathcal{F}(S^1)$ the algebra of 1-periodic smooth functions on the line ("the circle" 3.18).

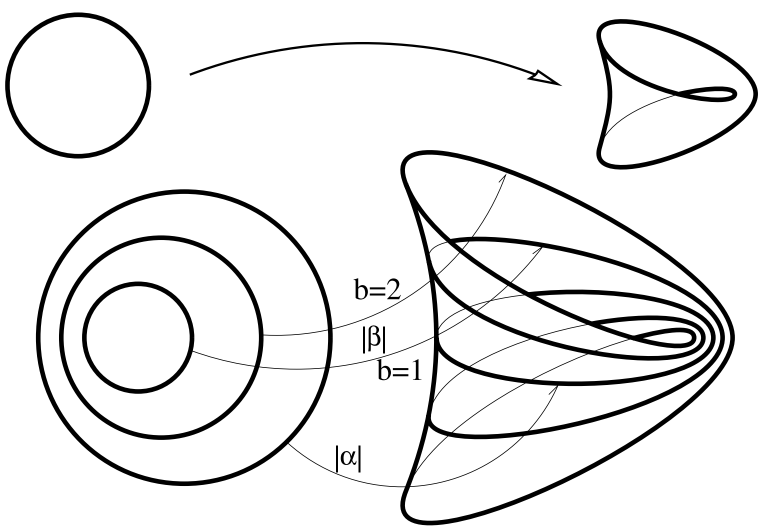


Figure 6.1. Maps from S^1 to the Möbius band.

I. Consider the homomorphisms

$$\alpha, \beta\colon \mathcal{F} \to \mathcal{F}(S^1) \quad (\alpha(f)(r) = f(r, 0),\ \beta(f)(r) = f(2r, b)),$$

where $f \in \mathcal{F}$, $r \in \mathbb{R}$, and $b \neq 0$ is any real number.

Exercise. Why does the formula $\gamma(f)(r) = f(r,b)$ not define a homomorphism $\gamma\colon \mathcal{F} \to \mathcal{F}(S^1)$?

The smooth maps $|\alpha|$ and $|\beta|$ are shown in Figure 6.1. Notice that the image of the map $|\beta|$ is "twice as long" as that of $|\alpha|$.

II. There is a remarkable smooth map of the Möbius band on the circle, namely the map $g = |\xi|$, where

$$\xi\colon \mathcal{F}\left(S^1\right) \to \mathcal{F} \quad (\xi(f)(r_1, r_2) = f(r_1)).$$

Note that $\alpha \circ \xi = \mathrm{id}_{\mathcal{F}(S^1)}$. The map $g = |\xi|$ may be visualized as "collapsing" the Möbius band to its central circle (Figure 6.2).

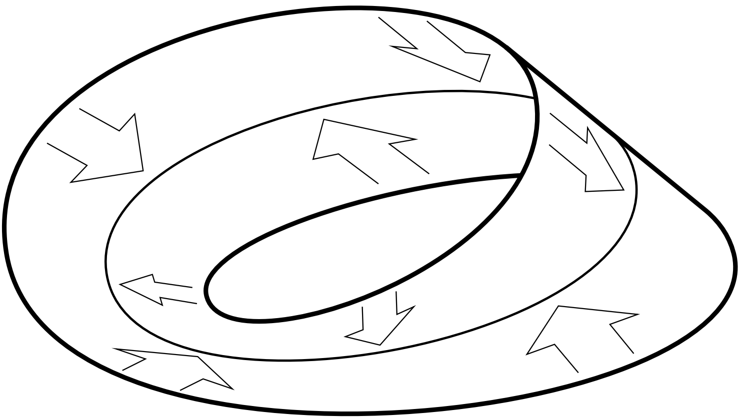


Figure 6.2. "Collapsing" the Möbius band.

6.5. Example. Choose an irrational number $\lambda \in \mathbb{R}$ and consider the map

$$f\colon \mathbb{R}^1 \to \mathbb{R}^2, \quad r \mapsto (r, \lambda r).$$

Then $f = |\varphi|$, where

$$\varphi\colon C^\infty\left(\mathbb{R}^2\right) \to C^\infty\left(\mathbb{R}^1\right) \quad (\varphi(g)(r) = g(r, \lambda r))$$

for all $g \in C^\infty\left(\mathbb{R}^2\right)$, $r \in \mathbb{R}$.

Denote by $\overline{\varphi}$ the restriction of the homomorphism φ to the subalgebra of doubly periodic functions (see Example 4.24, III, with $n = 2$). The image of the smooth map $|\overline{\varphi}|\colon \mathbb{R}^1 \to T^2$ is everywhere dense in the torus T^2 and, therefore the homomorphism $\overline{\varphi}$ is injective.

This example is interesting, since it shows that an algebra of functions "in several variables" may be isomorphic to a subalgebra of $C^\infty\left(\mathbb{R}^1\right)$.

When the number λ is taken to be the rational, the image of $|\overline{\varphi}|$ is compact. A particular case is shown in Figure 6.3. Try to guess what value of λ was taken there.

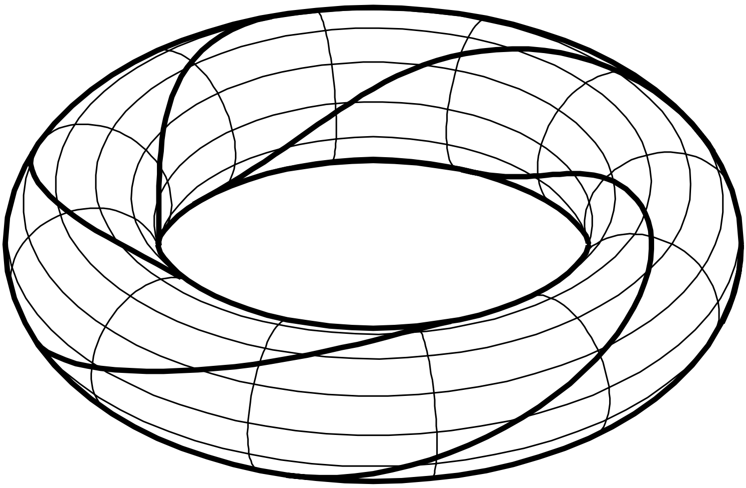


Figure 6.3. A map of $\mathbb{R}$ to the torus.

6.6. Now that smooth maps of manifolds have been introduced, manifolds no longer appear as unrelated, separate objects; they have been brought together into something unified, called a *category*. Other examples of categories are groups and their homomorphisms, topological spaces and continuous maps, $\mathbb{R}$-algebras and $\mathbb{R}$-algebra homomorphisms, linear spaces and linear operators. As we pointed out in Section 3.38, we shall not give any formal definitions from abstract category theory, but will often "think categorically."

In particular, let us point out two fundamental properties of the *category of smooth manifolds and maps*:

(i) if $a = |\alpha|\colon M_1 \to M_2$ and $b = |\beta|\colon M_2 \to M_3$ are smooth maps corresponding to the $\mathbb{R}$-algebra homomorphisms $\alpha\colon \mathcal{F}_2 \to \mathcal{F}_1$ and $\beta\colon \mathcal{F}_3 \to \mathcal{F}_2$, then $b \circ a\colon M_1 \to M_3$ is a smooth map (since it corresponds to the composition $\alpha \circ \beta$, in inverse order, of the homomorphisms α and β);

(ii) the identity map

$$\mathrm{id}\colon M_1 = M \to M_2 = M$$

is smooth, since it corresponds to the identity homomorphism

$$\mathrm{id}\colon \mathcal{F}_2 = \mathcal{F} \to \mathcal{F}_1 = \mathcal{F}.$$

The other categories mentioned above possess similar properties.

Suppose we are given a collection of maps possessing properties (i) and (ii). Then a typical "categorical trick" is to "inverse all arrows"; i.e., whenever a map $A \to B$ belongs to our collection, assume that there is a map $B \to A$ in a new, dual collection of maps, in which composition of maps is written in inverse order. Then we obtain the *dual category*, also satisfying (i), (ii).

We have, in fact, been using that construction in passing from homomorphisms of smooth $\mathbb{R}$-algebras to smooth maps of manifolds. The contents of Chapters 4 and 6 may be summarized as follows: A smooth manifold is a smooth $\mathbb{R}$-algebra, understood as an object of the dual category.

6.7. Suppose M_1 and M_2 are manifolds. The smooth map $a\colon M_1 \to M_2$ is said to be a *diffeomorphism* if there exists a smooth map $b\colon M_2 \to M_1$ such that $b \circ a = \mathrm{id}_{M_1}$, $a \circ b = \mathrm{id}_{M_2}$.

The manifolds M_1 and M_2 are called *diffeomorphic* if there exists a diffeomorphism of one onto the other. Note that two manifolds are diffeomorphic if and only if their algebras of smooth functions are isomorphic.

The relation of being diffeomorphic is an equivalence relation and will be denoted by $\cong$. In Chapter 4, when we spoke of two manifolds being "the same" or "identical," we actually meant that they were diffeomorphic; indeed, from the point of view of the theory, diffeomorphic manifolds are the same manifold presented in different guises.

6.8. Examples. I. The argument in Example 4.13 can be understood as a proof of the fact that the two methods for constructing the circle (as the quotient space of $\mathbb{R}^1$ and as a submanifold of $\mathbb{R}^2$) result in diffeomorphic manifolds.

II. A linear operator $A\colon \mathbb{R}^n \to \mathbb{R}^n$ will be a diffeomorphism if and only if $\det A \neq 0$, i.e., if A is bijective.

III. Suppose $\mathcal{F}_1$ and $\mathcal{F}_2$ are the algebras of smooth functions on manifolds M_1 and M_2. In general, the bijectivity of the smooth map $|\alpha|\colon M_1 \to M_2$ (where $\alpha\colon \mathcal{F}_2 \to \mathcal{F}_1$ is an $\mathbb{R}$-algebra homomorphism) is not sufficient for this map to be a diffeomorphism. As an example, we can take the map

$$\varphi\colon \mathbb{R} \to \mathbb{R}, \quad r \mapsto r^3.$$

In order to establish that various specific bijective smooth maps are in fact diffeomorphisms, one often uses the implicit function theorem 6.22. We shall not dwell on this here.

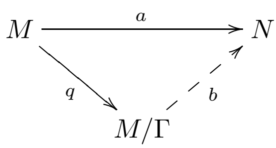
6.9 Exercises. 1. Prove that the boundary of the closed Möbius band (4.7, II) is diffeomorphic to the circle S^1.

2. Following Example 4.13 and I above, construct a diffeomorphism between the two models of the sphere S^n described in 4.25, I.

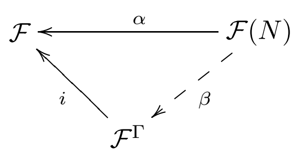
6.10. We now return to the topics of Chapter 4 in order to give, as promised, more satisfactory definitions of group action, quotient manifolds, and Cartesian products. These definitions will be "categorical" in character: They will be based on smooth maps and diagrams.

Suppose $\mathcal{F}$ is the algebra of smooth functions on the manifold M, and Γ is some group of automorphisms of this $\mathbb{R}$-algebra. The smooth map $a\colon M \to N$ of M into the manifold N is called Γ-*invariant* if $a \circ |\gamma| = a$ for all $\gamma \in \Gamma$. Obviously, the quotient map q of M on the quotient manifold M/Γ is Γ-invariant. (Of course, this is true, provided that M/Γ is a smooth manifold, i.e., if the algebra $\mathcal{F}^\Gamma$ of invariant functions is a smooth $\mathbb{R}$-algebra; see Definition 4.1 and Section 4.23.) It turns out that the quotient map is the "universal" Γ-invariant smooth map:

Proposition. *Let $\mathcal{F}(N)$ be the algebra of smooth functions on the smooth manifold N and $a\colon M \to N$ be any Γ-invariant smooth map with respect to an action of Γ on M such that M/Γ is a smooth manifold. Then there exists a unique map $b\colon M/\Gamma \to N$ for which the following diagram is commutative:*



◀ We must prove the existence and uniqueness of an $\mathbb{R}$-algebra homomorphism $\beta\colon \mathcal{F}(N) \to \mathcal{F}^\Gamma$ for which the diagram



where $a = |\alpha|$, is commutative. Clearly, there is no more than one such β. It exists iff $\operatorname{Im}\alpha = \alpha(\mathcal{F}(N))$ consists of Γ-invariant elements. But for a Γ-invariant map a, this is always the case: For all $\gamma \in \Gamma$, $f \in \mathcal{F}(N)$ we have

$$\gamma(\alpha(f)) = \alpha(f) \circ |\gamma| = f \circ (a \circ |\gamma|) = f \circ a = \alpha(f). \quad \blacktriangleright$$

6.11. Remark. The universal property characterizing the quotient map $M \to M/\Gamma$ determines the quotient manifold M/Γ uniquely up to diffeomorphism.

The proof of this statement can be copied over from the uniqueness proof of smooth envelopes (Proposition 3.37), and we suggest that the reader carry it out. The underlying general principle for proofs of this type, which we do not wish to formalize here, is that "any universal property determines an object uniquely."

6.12. Now we return to Cartesian products (Definition 4.28). Let $\mathcal{F}_l$ be the algebra of smooth functions on the manifold M_l, $l = 1, 2$. The *projection maps*

$$p_l\colon M_1 \times M_2 \to M_l, \quad (a_1, a_2) \mapsto a_l, \quad l = 1, 2,$$

are smooth, since $p_l = |\pi_l|$, where the $\mathbb{R}$-algebra homomorphism π is the composition

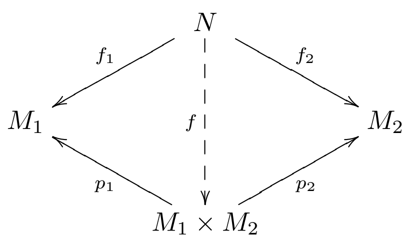
$$\mathcal{F}_l \overset{i_l}{\hookrightarrow} \mathcal{F}_1 \otimes_{\mathbb{R}} \mathcal{F}_2 \overset{\sigma}{\hookrightarrow} \overline{\mathcal{F}_1 \otimes_{\mathbb{R}} \mathcal{F}_2} = \mathcal{F}(M_1 \times M_2);$$

here σ is the smooth envelope homomorphism (3.36), and

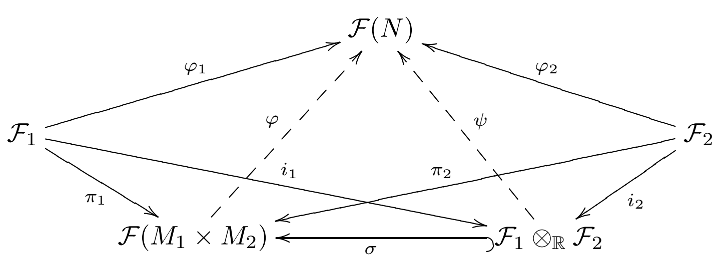
$$i_l(f) = \begin{cases} f \otimes 1, & l = 1, \\ 1 \otimes f, & l = 2, \end{cases} \quad \text{for all } f \in \mathcal{F}_l.$$

The pair of projection maps (p_1, p_2) possesses the following universal property:

Proposition. *For any smooth manifold N and any pair of smooth maps $f_l \colon N \to M_l$, $l = 1, 2$, there exists a unique smooth map $f \colon N \to M_1 \times M_2$ completing the commutative diagram*



◀ Denote the algebra of smooth functions on N by $\mathcal{F}(N)$. Assume that $f_l = |\varphi_l|$, where $\varphi_l \colon \mathcal{F}_l \to \mathcal{F}(N)$ are the dual $\mathbb{R}$-algebra homomorphisms. Our proposition will be proved if we establish the existence and uniqueness of the following diagram:



By the universal property of tensor products, there exists a unique homomorphism ψ shown in the diagram. Since the $\mathbb{R}$-algebra $\mathcal{F}(N)$ is smooth, while the smooth envelope homomorphism has the universal property stated in Proposition 3.37, the homomorphism φ is also well defined and unique. ▶

6.13. Remark. Proposition 6.12 may be used to construct smooth maps from a third manifold M_3 to the product $M_1 \times M_2$ of two given ones. For example, a smooth map of a manifold N to $S^1 \times S^1$ is a pair of smooth maps from N to S^1, i.e., a pair of smooth functions on N defined modulo 1.

6.14. The remainder of this section is a discussion of the notions and constructions developed in 6.1–6.13 carried out in the “coordinate language” introduced in Chapter 5.

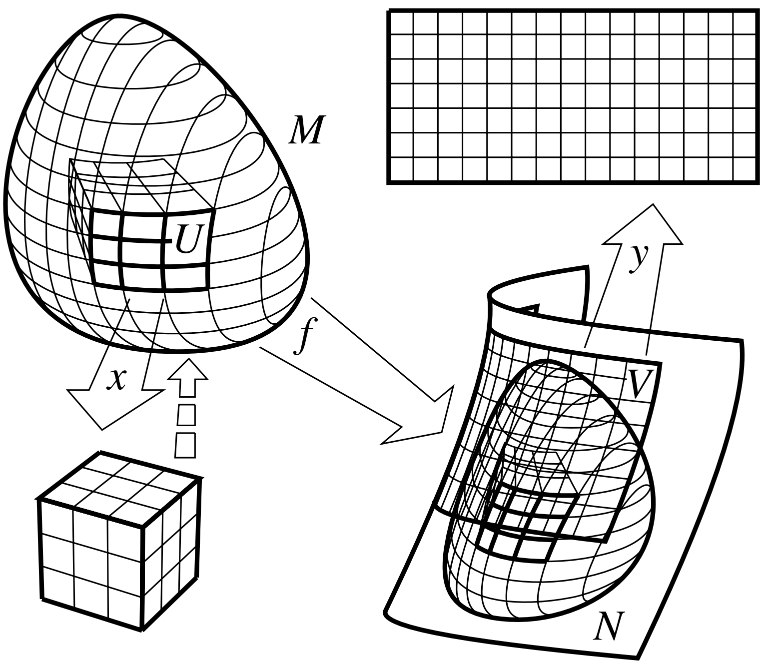


Figure 6.4. Smooth map in the “coordinate language.”

Suppose (M, A) and (N, B) are manifolds with smooth atlases A and B (see Section 5.8). The map $f\colon M \to N$ is called *smooth* (or *differentiable*) *at the point* $a \in M$ if for some (hence for all) pairs of charts (U, x) and (V, y) compatible with the atlases A and B and covering the points a and $f(a)$, respectively, the map $y \circ f \circ x^{-1}$, defined in the neighborhood $x(f^{-1}(V) \cap U)$ of the point $x(a) \in \mathbb{R}^n$, is an infinitely differentiable map of domains in Euclidean space (see Figure 6.4). The map $f\colon M \to N$ is called *smooth* if it is smooth at each point.

In the coordinate language, a smooth bijective map $f\colon M \to N$ is called a *diffeomorphism* if the map f^{-1} is also smooth.

6.15. When working with particular maps of manifolds, we ordinarily use a coordinate representation. Actually, this means that we use the above-mentioned map $y \circ f \circ x^{-1}$, which, being a map of Euclidean spaces, is represented by the functions

$$y_i = f_i(x_1, \ldots, x_m), \quad i = 1, \ldots, n,$$

where $m = \dim M$, $n = \dim N$. For a smooth map (at a point), all the functions f_i, $i = 1, \dots, n$, will also be smooth (respectively, at a point).

It is easy to see that if a map is described by smooth functions in a family of pairs of charts compatible with the corresponding atlases (and the charts cover the entire manifold M), then this map will be described by smooth functions for any pair of charts compatible with the same atlases (of course, charts for which the composition $y \circ f \circ x^{-1}$ is undefined are not taken into account).

The equivalence of these coordinate definitions and the corresponding algebraic ones will be established in Chapter 7.

6.16. Examples. I. The map $a\colon T^2 \to \mathbb{R}^2$ that assigns to each position of the double pendulum (Example 5.2, III) its endpoint A (see Figure 5.1) is smooth.

◀ Choose some fixed position of the double pendulum; then all sufficiently close positions of the pendulum are characterized by two angles x, y, so that we have a chart

$$U \ni \text{(position)} \stackrel{\Phi}{\mapsto} (x, y)$$

that is compatible with the standard atlas of the torus. The manifold $\mathbb{R}^2$ can be covered by a single chart $\mathrm{id}\colon \mathbb{R}^2 \to \mathbb{R}^2$. We can say that *in the chosen local coordinates the map a is described by the formulas*

$$r_1 = R\cos x + r\cos y, \;\; r_2 = R\sin x + r\sin y.$$

The rigorous meaning of the words in quotation marks is that the formulas actually describe the map $(\mathrm{id})^{-1} \circ a \circ \Phi^{-1}$; hence it follows from Definition 6.14 that a is smooth. ▶

II. Choose a fixed unit vector $v \in S^{n-1}$ in $\mathbb{R}^n$ and consider the map

$$f_v\colon \mathrm{SO}(n) \to S^{n-1}, \quad A \mapsto A(v).$$

Verify that f_v is a smooth map by using the atlases described in Examples 5.10, I, and 5.11, II.

We can also consider the map

$$\varphi\colon \mathrm{SO}(n) \times S^{n-1} \to S^{n-1}, \quad (A, v) \mapsto A(v),$$

and attempt to prove its smoothness, working with an atlas on the product $\mathrm{SO}(n) \times S^{n-1}$. The reader will be wise not to take this attempt too seriously, but to prove smoothness of φ eventually by using the algebraic definitions.

6.17. More examples. I. Let us consider four-dimensional Euclidean space $\mathbb{R}^4$ as the algebra of quaternions $\mathbb{H}$, denote by $V \cong \mathbb{R}^3$ the subspace of purely imaginary quaternions $r_1 i + r_2 j + r_3 k$, and introduce the map

$$(\mathbb{H} \setminus \{0\}) \times V \to V, \quad (q, v) \mapsto qvq^{-1}.$$

It is easy to check that for each nonzero quaternion q, the matrix of the linear operator $v \mapsto qvq^{-1}$ in the coordinates r_1, r_2, r_3 is orthogonal, so that this is a map $\mathbb{H} \setminus \{0\} \to \mathrm{SO}(3)$. Two quaternions q_1 and q_2 determine the same transformation of the space V if and only if $q_1 = \lambda q_2$ for some $\lambda \in \mathbb{R} \setminus \{0\}$, so that we have obtained a bijective smooth map

$$\mathbb{R}P^3 \to \mathrm{SO}(3).$$

Recall that $\mathbb{R}P^3$ is precisely the quotient manifold of the punctured space $\mathbb{R}^4 \setminus \{0\} = \mathbb{H} \setminus \{0\}$ by the group $\mathbb{R}^{\cdot}$ of all homotheties with center at the origin (see Example 4.25, II) and that the projection $\mathbb{H} \setminus \{0\} \to \mathbb{R}P^3$ has the universal property (see Proposition 6.10). It is not useless to try to establish the smoothness of this map by using the atlases 5.10, III, and 5.11, II. It is more difficult to prove the fact that it is a diffeomorphism (as mentioned in Section 1.6).

II. The composition

$$S^3 \to \mathbb{R}^4 \setminus \{0\} \to \mathrm{SO}(3) \to S^2,$$

where the first arrow is the inclusion and the others are defined in Examples 6.16, II, and 6.17, I, is called the *Hopf map.* Notice that the composition of the first two arrows may be represented in the form

$$S^3 \to \mathbb{R}P^3 \overset{\cong}{\to} \mathrm{SO}(3),$$

where the map $S^3 \to \mathbb{R}P^3$ was described in Example 4.25, III, as the quotient map $S^3 \to S^3/\mathbb{Z}_2 \cong \mathbb{R}P^3$.

Exercise. Try to show that the inverse image of any point of S^2 under the Hopf map $h\colon S^3 \to S^2$ is a closed submanifold of S^3, diffeomorphic to S^1.

6.18. Example (of a smooth map of a manifold with boundary). Suppose D^3 is the 3-dimensional closed disk with center at the origin $O \in \mathbb{R}^3$ and of radius π. To each point $a \in D^3$ associate the rotation of $\mathbb{R}^3$ about the line joining a to the origin by the angle $\alpha = \|a\|$. Thus we obtain the map $g\colon D^3 \to \mathrm{SO}(3)$.

Clearly, diametrically opposed points on the boundary $\partial D^3 = S^2$ of D^3 determine the same rotation. Thus the manifold $\mathrm{SO}(3) \cong \mathbb{R}P^3$ of orthogonal transformations of $\mathbb{R}^3$ can be represented as the disk D^3 whose diametrically opposed boundary points have been glued together.

The map $g\colon D^3 \to \mathrm{SO}(3)$ is a smooth surjective map of a manifold with boundary onto a manifold (without boundary).

Exercise. Show that the image of the boundary $S^2 = \partial D^3$ under this map is a smooth closed submanifold in $\mathrm{SO}(3)$ diffeomorphic to the manifold $\mathbb{R}P^2$.

6.19 Exercises. 1. Write out the formulas for the orthogonal projection of the unit sphere $S^n \subset \mathbb{R}^{n+1}$ onto the hyperplane $\mathbb{R}^n \subset \mathbb{R}^{n+1}$ in the charts indicated in 5.10 (i) and verify that this projection is a smooth map.

2. Prove that $SO(4) \cong S^3 \times SO(3)$. *Hint*: SO(3) may be understood as the set of orthonormed pairs $\{u, v\}$ in $\mathbb{R}^3$, SO(4) as the set of orthonormed triples $\{u, v, w\}$ in $\mathbb{R}^4$; let $\mathbb{R}^4 = \mathbb{H}$ and $\mathbb{R}^3 = V \subset \mathbb{H}$ as in Example 6.17, I; investigate the map

$$S^3 \times SO(3) \to SO(4), \quad (u, \{v, w\}) \mapsto \{u, uv, uw\}.$$

6.20. The collection of all smooth maps (in the sense of the coordinate Definition 6.13) also possesses the two properties concerning composition and identity maps mentioned in Section 6.6 (which is not surprising, since the coordinate definition is equivalent to the algebraic one; see Section 6.1). The class of all smooth manifolds in the sense of Section 5.8 together with the family of all smooth (Section 6.14) maps constitutes the *category of coordinate manifolds.*

In what follows, we shall need some classical results of multidimensional local calculus, i.e., the theorems on implicit and inverse functions. We formulate them here in a form convenient for the subsequent exposition, without any proofs. The latter may be found in any advanced calculus course.

6.21 The inverse function theorem. *Let a smooth map*

$$f = (f_1, \dots, f_n)\colon \mathbb{R}^n \to \mathbb{R}^n$$

possess a nondegenerate Jacobi matrix in a neighborhood of the origin $0 \in \mathbb{R}^n$:

$$\det f'(0) = \det \left(\frac{\partial f_i}{\partial x_j}(0) \right) \neq 0.$$

Then there exist open sets $U \ni 0$ *and* $V \ni f(0)$ *such that the map* $\varphi \overset{\text{def}}{=} f\big|_U \colon U \to V$ *possesses a smooth inverse* $\varphi^{-1}\colon V \to U$. *The Jacobi matrix of the latter at any point* $y \in V$ *can be computed by the formula*

$$(\varphi^{-1})'(y) = \bigl(\varphi'(\varphi^{-1}(y))\bigr)^{-1}. \quad \blacktriangleright$$

6.22 The implicit function theorem. *Let*

$$f = (f_1, \dots, f_{n+m})\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$$

be a smooth map with $f(a, b) = 0 \in \mathbb{R}^m$ *possessing a nondegenerate* $(m \times m)$ *matrix of partial derivatives with respect to the variables* $x_{n+1}, \dots, x_{n+m}$:

$$\det \left(\frac{\partial f_i}{\partial x_{n+j}}(a) \right)_{1 \leqslant i,j \leqslant m} \neq 0.$$

Then there exist open sets U *and* V, $a \in U \subset \mathbb{R}^n$, $b \in V \subset \mathbb{R}^m$, *and a smooth function* $g\colon U \to V$ *such that* $f(x, g(x)) = 0$ *for all* $x \in U$. $\blacktriangleright$

Finally, we shall need another classical theorem (the theorem on the linearization of a smooth map), which is an amalgam of the previous two results.

6.23 Theorem. *Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$, $m \leqslant n$, be a smooth map such that $f(0) = 0$ and the rank of the matrix*

$$M = \left(\frac{\partial f_i}{\partial x_j}(0)\right)_{\substack{1\leqslant i\leqslant m,\\ 1\leqslant j\leqslant n}}$$

is equal to m. Then there exist neighborhoods U, $V \subset \mathbb{R}^n$ of 0 and a diffeomorphism $\varphi\colon U \to V$ such that

$$(f \circ \varphi)(x_1, \dots, x_n) = (x_{n-m+1}, \dots, x_n). \quad ▶$$

6.24. Remark. The three theorems above are "local": Their assumptions are formulated in Euclidean spaces, while the conclusions are valid for neighborhoods (of Euclidean spaces). Nevertheless, they may be applied to the "global" case of manifolds by considering separate coordinate neighborhoods. By *uniqueness* of the inverse (and implicit) functions in the corresponding neighborhood, the functions constructed in these theorems coincide on the common part of the two neighborhoods. Therefore, we can "glue" them together over the entire manifold. Thus, for example, the conclusion of the implicit function theorem may be formulated as follows: *If $f\colon M \to N$ is a smooth map of manifolds, $m = \dim M > n = \dim N$, and the Jacobian of f has rank n at any point, then $f^{-1}(z) \subset M$ is a submanifold for any $z \in N$.*

6.25 Exercise. A plane hinge mechanism (see Section 1.14) is called *generic* if the configuration with all the hinges positioned along the same straight line is impossible (i.e., there is no linear relation with coefficients ± 1 among the lengths of the rods).

1. Prove that the configuration space of a generic hinge mechanism is a smooth manifold.
2. Show that the configuration space of a generic pentagon (Section 1.14) is diffeomorphic either to the sphere with no more than 4 handles, or to the disjoint union of two tori, or to the disjoint union of two spheres.

7

Equivalence of Coordinate and Algebraic Definitions

7.1. The aim of this chapter is to prove that the two definitions of smooth manifold (4.1 and 5.8) and of smooth maps (Sections 6.1 and 6.13) yield the same concepts, thus showing that the coordinate approach and the algebraic one are equivalent.

The equivalence of the two definitions of smooth manifold will be stated in the form of two theorems (7.2 and 7.7). The equivalence of the definitions of smooth maps is Theorem 7.16.

7.2 Theorem. *Suppose $\mathcal{F} = C^\infty(M)$ is the algebra of smooth functions on a manifold M defined by its smooth atlas A. Then $\mathcal{F}$ is a smooth $\mathbb{R}$-algebra* (*in the sense of* Section 4.1), *and the map*

$$\theta\colon M \to |\mathcal{F}|, \quad \big(\theta(p)\big)(f) = f(p),$$

is a homeomorphism.

◀ The proof will be in four steps. First we shall establish that the map $\theta\colon M \to |\mathcal{F}|$ is bijective (Section 7.3), then that it is a homeomorphism (Section 7.4); then we shall show that $C^\infty(M)$ is geometric and complete (Section 7.5) and finally prove that $C^\infty(M)$ is smooth (Section 7.6).

7.3. *The map $\theta\colon M \to |\mathcal{F}|$ is bijective.*
◁ Injectivity is obvious: If p, $q \in M$ are distinct points, then there exists a function $f \in C^\infty(M)$ such that $f(p) \neq f(q)$ (e.g., any function positive in a small neighborhood of p not containing q and identically zero outside of it; see Proposition 2.4); for this function the values of the homomorphisms

$p, q\colon f \to \mathbb{R}$ differ, since

$$q(f) = f(q) \neq f(p) = p(f).$$

To prove surjectivity, suppose $p\colon \mathcal{F} \to \mathbb{R}$ is any homomorphism; let $f \in C^\infty(M)$ be any function with compact level surfaces (see Proposition 5.17) and $\lambda = p(f)$. Suppose that none of the points of the compact set $L = f^{-1}(\lambda)$ correspond to the homomorphism p. Then there exists a family of functions $\{f_x \mid x \in L\}$ such that $f_x(x) \neq p(f_x)$. Consider the covering of L by the open sets

$$U_x = \{q \in M \mid f_x(q) \neq p(f_x)\}$$

and choose a finite subcovering $U_{x_1}, \ldots, U_{x_m}$. Consider the function

$$g = (f - \lambda)^2 + \sum_{k=1}^{m} (f_{x_k} - p(f_{x_k}))^2.$$

This is a smooth function on M that vanishes nowhere and therefore possesses a smooth inverse $1/g$. Now we easily obtain the standard contradiction, familiar from the examples:

$$p(g) = (p(f) - \lambda)^2 + \sum_{k=1}^{m} (p(f_{x_k}) - p(f_{x_k}))^2 = 0,$$

$$1 = p(1) = p\left(g \cdot \frac{1}{g}\right) = p(g)p\left(\frac{1}{g}\right) = 0.$$

Hence the homomorphism p is given by some point of $L \subset M$, so that θ is surjective. $\triangleright$

7.4. *The map $\theta\colon M \to |\mathcal{F}|$ is a homeomorphism.*
$\triangleleft$ Let U be an open set in $|\mathcal{F}|$. Then, by Definition 3.12, the set U is the union of sets of the form $f^{-1}(V)$, where $V \subset \mathbb{R}$ is open. Since $f \in \mathcal{F} = C^\infty(M)$ is smooth (hence continuous), the sets $f^{-1}(V)$ are open in the topology of M, and so is U.

Conversely, for any open set U in the topology of M there exists a function $f \in \mathcal{F}$ such that $U = f^{-1}(\mathbb{R}_+)$. In fact, Lemma 4.17 (i) remains obviously valid if in its statement M is a manifold in the sense of Section 5.8. But $\mathbb{R}_+ \subset \mathbb{R}$ is open, so that U is open in the topology of $|\mathcal{F}|$. $\triangleright$

7.5. *The algebra $\mathcal{F} = C^\infty(M)$ is geometric and complete.*
$\triangleleft$ The fact that $\mathcal{F}$ is geometric is obvious, since the elements $f \in \mathcal{F}$ are real functions on the set M, so that only the identically zero function vanishes at all points of M. The fact that $\mathcal{F}$ is complete is also obvious: Any function $f\colon M \to \mathbb{R}$ that is smooth in a neighborhood of every point is smooth on the entire manifold, i.e., belongs to $\mathcal{F} = C^\infty(M)$ (see Section 5.17). $\triangleright$

7.6. *$C^\infty(M)$ is a smooth $\mathbb{R}$-algebra.*
$\triangleleft$ To prove this, we shall construct a countable atlas $A_2 = \{(U_k, x_k)\}$ such that $x_k(U_k) = \mathbb{R}^n$. We begin with an arbitrary countable atlas $A_0 =$

$\{(V_l, y_l)\}$ compatible with the given atlas A. Each open set $y_l(V_l) \subset \mathbb{R}^n$ may be represented as the countable union of open balls in $\mathbb{R}^n$; i.e., $y_l(V_l) = \bigcup_{i=1}^{\infty} G_{li}$, where

$$G_{li} = \{r \in \mathbb{R}^n \mid \|r - a_{li}\| < r_{li}\}.$$

The family $A_1 = \left\{ \left(y_l^{-1}(G_{li}), y_l\big|_{y_l^{-1}(G_{li})} \right) \mid l, i \in \mathbb{N} \right\}$ is obviously a countable atlas on M, compatible with A_0 and hence with A. Now note that for any chart (U, x) on M, where $x(U)$ is an open ball in $\mathbb{R}^n$, we can construct the chart $(U, \eta \circ x)$, where $\eta \circ x(U) = \mathbb{R}^n$, by taking η to be any diffeomorphism of the ball $x(U)$ onto $\mathbb{R}^n$. For instance, if $x(U)$ is the ball of radius ρ and center a, we can take η to be the bijective map $\eta \colon \mathbb{R}^n \to x(U)$ given by the formula

$$\eta(s) = a + \frac{s\rho}{\sqrt{\rho^2 + \|s\|^2}}.$$

The map η has the inverse

$$\eta^{-1}(r) = \frac{\rho(r-a)}{\sqrt{\rho^2 - \|r-a\|^2}},$$

and therefore the chart $(U, \eta \circ x)$ is compatible with (U, x).

Carrying out this construction for every chart in A_1, we obtain the required atlas $A_2 = \{(U_k, x_k)\}$.

Consider any chart $(U_k, x_k) \in A_2$. The set U_k is obviously open in the topological space M. Clearly, the restriction of the algebra $\mathcal{F} = C^\infty(M)$ to this set consists of all functions $f \colon U_k \to \mathbb{R}$ such that $f \circ x_k^{-1}$ is smooth on $\mathbb{R}^n$; hence the assignment $f \mapsto f \circ x_k^{-1}$ is an isomorphism of the $\mathbb{R}$-algebra $C^\infty(M)\big|_{U_k}$ onto $C^\infty(\mathbb{R}^n)$. This means that $\mathcal{F} = C^\infty(M)$ is smooth. ▷

This concludes the proof of Theorem 7.2. ▶

7.7 Theorem. *Suppose $\mathcal{F}$ is any smooth $\mathbb{R}$-algebra. Then there exists a smooth atlas A on the dual space $M = |\mathcal{F}|$ such that the map*

$$\mathcal{F} \to C^\infty(M), \quad f \mapsto (p \mapsto p(f)),$$

of the algebra $\mathcal{F}$ onto the algebra $C^\infty(M)$ of smooth functions on M with respect to A (see Section 5.17*) is an isomorphism.*

◀ The proof will require four steps. In the first one (Section 7.8) we construct a chart $x \colon U \to \mathbb{R}^n$ for each of the open sets U of a countable covering of $|\mathcal{F}|$, using a lemma proved in the second step (Section 7.9). In the third step (Section 7.10) we show that any two such charts are compatible. In the fourth and final step (Section 7.11) we show that $f \in \mathcal{F}$ iff $f \in C^\infty(M)$ (where f, an abstract element of $\mathcal{F}$, is identified with the function $f \colon |\mathcal{F}| \to \mathbb{R}$, $p \mapsto p(f)$).

7.8. Construction of a chart $x \colon U \to \mathbb{R}^n$.

◁ By the definition of smooth $\mathbb{R}$-algebras (Section 4.1), there is an open covering of $|\mathcal{F}|$ by open sets U for which there exist isomorphisms $i \colon \mathcal{F}\big|_U \to$

$C^\infty(\mathbb{R}^n)$. Taking any such U, we shall construct the required chart $x\colon U \to \mathbb{R}^n$.

The composition

$$U \xrightarrow{\mu} \left|\mathcal{F}\big|_U\right| \xrightarrow{|\rho|} |\mathcal{F}|,$$

where μ is the inclusion (see Section 3.29) and $\rho\colon \mathcal{F} \to \mathcal{F}\big|_U$ the restriction map (Section 3.25), as can easily be checked, coincides with the inclusion $U \subset |\mathcal{F}|$. Since μ is a homeomorphism *onto* $\left|\mathcal{F}\big|_U\right|$ (this is proved in Lemma 7.9 below) and $|\rho| \circ \mu$ is the inclusion $U \subset |\mathcal{F}|$, it follows that $|\rho|$ must be a homeomorphism onto $U \subset |\mathcal{F}|$.

Now let $h\colon \mathcal{F} \to C^\infty(\mathbb{R}^n)$ be the composition

$$\mathcal{F} \xrightarrow{\rho} \mathcal{F}\big|_U \xrightarrow{i} C^\infty(\mathbb{R}^n), \quad h = i \circ \rho.$$

Consider the dual map $|h| = |\rho| \circ |i|$. Since $|C^\infty(\mathbb{R}^n)| = \mathbb{R}^n$ (by Example 3.16) and i is an isomorphism, it follows from Section 3.20 that $|i|$ is a homeomorphism $|i|\colon \mathbb{R}^n \to \left|\mathcal{F}\big|_U\right|$. But $|\rho|$ is also a homeomorphism (onto $U \subset |\mathcal{F}|$), so the composition $|h| = |\rho| \circ |i|$ is a homeomorphism $|h|\colon \mathbb{R}^n \to U$, where $U \subset |\mathcal{F}|$ is open. Hence we obtain the required chart by putting $x = |h|^{-1}\colon U \to \mathbb{R}^n$. ▷

7.9 Lemma. *The inclusion map* $\mu\colon U \to \left|\mathcal{F}\big|_U\right|$ *is a homeomorphism onto* $\left|\mathcal{F}\big|_U\right|$.

◁ Suppose μ is not surjective; i.e., assume that there exists a point $a \in \left|\mathcal{F}\big|_U\right| \setminus \mu(U)$; set $\overline{a} = |i|^{-1}(a)$, where $i\colon \mathcal{F}\big|_U \to C^\infty(\mathbb{R}^n)$ is the isomorphism appearing in Section 7.8. We consider two cases, depending on whether a belongs to the closure of $\mu(U)$ or not.

First case: $a \in \overline{\mu(U)}$. Consider the function $f\colon x \mapsto 1/\|x - \overline{a}\|$ defined on $\mathbb{R}^n \setminus \{\overline{a}\}$ and the function $g = f \circ |i|^{-1} \circ \mu$ defined on U. We claim that $g \in \mathcal{F}\big|_U$. Indeed, any point $r \in \mathbb{R}^n \setminus \{\overline{a}\}$ possesses a neighborhood on which f coincides with a smooth function defined on all of $\mathbb{R}^n$. Taking the inverse images of such neighborhoods under the map $|i|^{-1} \circ \mu$, we see that any point $q \in U$ possesses a neighborhood on which g coincides with a function from $\mathcal{F}\big|_U$. By the definition of $\mathcal{F}\big|_U$ (Section 3.23), this means that g locally coincides with functions belonging to $\mathcal{F}$; hence $g \in \mathcal{F}\big|_U$, as claimed. But now if we consider the function $i(g)$, which is a smooth function (on the entire space $\mathbb{R}^n$) coinciding with f on the set $|i|^{-1}(\mu(U))$ whose closure contains $\overline{a}$, we obtain a contradiction (since f "becomes infinite" at $\overline{a}$).

Second case: $a \notin \overline{\mu(U)}$. Consider two smooth functions on $\mathbb{R}^n$: the identically zero one, and a function f_0 that vanishes on the closed set $|i|^{-1}(\overline{\mu(U)})$ and equals 1 at $\overline{a}$ (f_0 exists by Corollary 2.5). These are distinct functions, so that their pullbacks by i on $\mathcal{F}\big|_U$ are different elements of this algebra, which is impossible, since both pullbacks vanish on U.

This proves the surjectivity of μ. The fact that it is a homeomorphism follows from Proposition 3.29. $\triangleright$

7.10. *The charts constructed in* Section 7.8 *are compatible.*
$\triangleleft$ Suppose $x\colon U \to \mathbb{R}^n$ and $y\colon V \to \mathbb{R}^n$ are two such charts, while $i\colon \mathcal{F}\big|_U \to C^\infty(\mathbb{R}^n)$ and $j\colon \mathcal{F}\big|_V \to C^\infty(\mathbb{R}^n)$ are the corresponding $\mathbb{R}$-algebra isomorphisms. Let $W = U \cap V \neq \varnothing$ (the case $W = \varnothing$ is trivial). By Proposition 3.25 we have the isomorphisms

$$i\big|_W\colon \mathcal{F}\big|_W \to C^\infty(x(W)), \quad j\big|_W\colon \mathcal{F}\big|_W \to C^\infty(y(W)),$$

which give us the isomorphism $t\colon C^\infty(x(W)) \to C^\infty(y(W))$; this, by Proposition 3.16, shows that $|t|\colon y(W) \to x(W)$, the change of coordinate map, is a diffeomorphism. $\triangleright$

7.11. Final step. $\triangleleft$ Suppose $f \in \mathcal{F}$. To prove that f is smooth in the sense of $C^\infty(M)$ (see Section 5.16), we must show that the function $f \circ x^{-1}$ is a smooth function on $\mathbb{R}^n$ for each of the charts $x\colon U \to \mathbb{R}^n$ constructed in 7.8. But we have (see 7.8)

$$f \circ x^{-1} = f \circ |h| = f \circ |\rho| \circ |i| = i(\rho(f)) \in C^\infty(\mathbb{R}^n),$$

since i is the isomorphism $i\colon \mathcal{F}\big|_U \to C^\infty(\mathbb{R}^n)$. Thus $f \circ x^{-1} \in C^\infty(\mathbb{R}^n)$.

Conversely, let $f \in C^\infty(M)$. This means that for any chart (U, x) the function $f \circ x^{-1} = i(f \circ |\rho|)$ belongs to $C^\infty(\mathbb{R}^n)$ and is the image (by i) of an element of $\mathcal{F}\big|_U$ (namely $f \circ |\rho|$). Thus f locally coincides with elements of $\mathcal{F}\big|_U$ and hence of $\mathcal{F}$. Since $\mathcal{F}$ is complete, $f \in \mathcal{F}$. $\triangleright$ ▶

7.12. Thus we have established the equivalence of the two definitions of smooth manifold: the algebraic one (Definition 4.1) and the coordinate one (Definition 5.8). Theorems similar to 7.2 and 7.7 are valid for manifolds with boundary. The proofs are similar (with obvious modification here and there). The reader who wishes to check that he or she has mastered the contents of Sections 7.1–7.11 will benefit by carrying them out in detail.

7.13. Definition. A *smooth set* is a pair $(W, C^\infty(W))$, where W is a closed subset $W \subset M$ of a smooth manifold M, and $C^\infty(W)$ is the algebra of smooth functions on W defined as follows:

$$C^\infty(W) \stackrel{\text{def}}{=} \left\{ f\big|_W \mid f \in C^\infty(M) \right\}.$$

Exercise. Prove that theorems similar to 7.2 and 7.7 are valid for smooth sets as well.

7.14 Exercise. Describe the algebras $C^\infty(W)$ for the following cases:

1. $W = \mathbf{K}$ is the coordinate cross on the plane:

$$\mathbf{K} = \left\{ (x, y) \subset \mathbb{R}^2 \mid xy = 0 \right\}.$$

2. $W \subset \mathbb{R}^2$ is given by the equation $y = \sqrt{|x|}$.

3. W is a triangle in $\mathbb{R}^2$: $W = T_1 \cup T_2 \cup T_3$, where

$$T_1 = \{(x, y) \mid 0 \leqslant y \leqslant 1,\ x = 0\},$$
$$T_2 = \{(x, y) \mid 0 \leqslant x \leqslant 1,\ y = 0\},$$
$$T_3 = \{(x, y) \mid x + y = 1,\ x, y \geqslant 0\}.$$

4. W is the triangle described in Problem 3 together with its interior part: $W = \{(x, y)\} \mid x + y \leqslant 1,\ x, y \geqslant 0\}$.

5. W is the cone $x^2 + y^2 = z^2$ in $\mathbb{R}^3$.

6. $W = W_i$, $i = 1, 2, 3$, is one of the three one-dimensional pairwise homeomorphic polyhedra depicted in Figure 7.1 (W_1 lives in $\mathbb{R}^2$, while W_2 and W_3 are in $\mathbb{R}^3$). In Chapter 9 the reader will find Exercise 9.36, 5, in which it is required to prove that the algebras $C^\infty(W_i)$, $i = 1, 2, 3$, are mutually nonisomorphic.

7. $W \subset \mathbb{R}^2$ is the closure of the graph of the function $y = \sin \frac{1}{x}$.

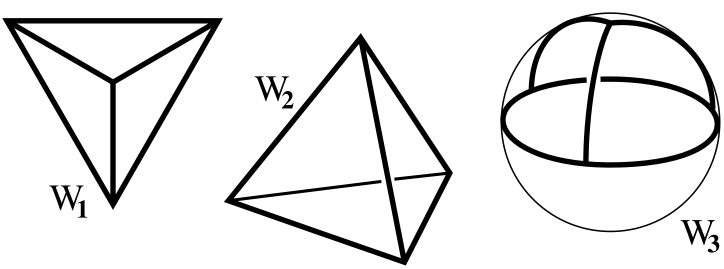


Figure 7.1. 1-skeleton of the tetrahedron.

Of course, there are various alternative descriptions of the algebras in question. For instance, perhaps the most direct and constructive way to represent smooth functions on the triangle (see Problem 3 above) is by means of triples (f_1, f_2, f_3), $f_i \in C^\infty([0, 1])$, such that $f_1(1) = f_2(0)$, $f_2(1) = f_3(0)$, $f_3(1) = f_1(0)$. Try to give a similar description of smooth functions on the cross and on the polyhedra mentioned in Problems 1 and 6, respectively.

7.15. Smooth sets sometimes appear implicitly in various mathematical problems. We illustrate this in the following exercises.

Exercises. 1. Show that the configuration space of a hinge mechanism can be viewed as a smooth set.

2. Determine which of the smooth sets corresponding to quadrilaterals and pentagons listed in Exercise 1 from Section 1.14 are not smooth

manifolds and describe their smooth function algebras (cf. Exercise 6.25).

3. Prove that the smooth set corresponding to a nonrigid nongeneric pentagon (see Exercise 6.25), e.g., $(2; 2, 1, 1, 2)$, is not a smooth manifold and describe its singular points (a pentagon is called rigid if its configuration space consists of one point, e.g., $(4; 1, 1, 1, 1)$).

Remark. The smooth set corresponding to the quadrilaterals $(5; 3, 3, 1)$ is "diffeomorphic" to a pair of tangent circles at a point, say z. Smooth functions on this set can be viewed as pairs (f_1, f_2), where $f_1, f_2 \in C^\infty(S^1)$, such that $f_1(z) = f_2(z)$ and $f_1'(z) = f_2'(z)$. Try to give similar descriptions for smooth functions on smooth sets appearing in Exercises 2 and 3.

7.16. The coincidence of the two definitions of a smooth map given in 6.1 and 6.14 is guaranteed by the following theorem.

Theorem. *Let M and N be manifolds with smooth atlases A and B and smooth function algebras $\mathcal{F}_M$ and $\mathcal{F}_N$, respectively. A map $\varphi\colon M \to N$ is smooth with regard to A and B* (Section 6.14) *if and only if $\varphi^*(\mathcal{F}_N) \subset \mathcal{F}_M$, where*

$$\varphi^*\colon \mathcal{F}_N \to \mathcal{F}_M, \quad f \mapsto f \circ \varphi.$$

The proof is given below, with Sections 7.17 and 7.18 corresponding to the "only if" and the "if" parts of the theorem, respectively.

7.17. *If $\varphi\colon M \to N$ is smooth, then $\varphi^*(\mathcal{F}_N) \subset \mathcal{F}_M$.*
◁ Suppose $f \in \mathcal{F}_N$ and $a \in M$. Choose a chart (V, y) in a neighborhood of $\varphi(a)$ and a chart (U, x) in a neighborhood of a, the charts (V, y) and (U, x) being compatible with B and A, respectively, and satisfying $\varphi(U) \subset V$. Then

$$\varphi^*(f) \circ x^{-1} = (f \circ \varphi) \circ x^{-1} = \left(f \circ y^{-1}\right) \circ \left(y \circ \varphi \circ x^{-1}\right)$$

is smooth as the composition of two smooth maps of Euclidean domains. Thus locally the map $\varphi^*(f)$ coincides with a map locally coinciding with an element of $\mathcal{F}_M$. Since $\mathcal{F}_M$ is complete (see 7.5), it follows that $\varphi^*(f) \in \mathcal{F}_M$. ▷

7.18. *If $\varphi^*(\mathcal{F}_N) \subset \mathcal{F}_M$, then $\varphi\colon M \to N$ is smooth.*
◁ Choose arbitrary charts $(U, x) \in A$ and $(V, y) \in B$ such that $\varphi(U) \subset V$. We must prove, according to Definition 6.13, that the local coordinates of the point $\varphi(a)$ are smooth functions of the local coordinates of $a \in U$. In other words, the functions $y^i \circ \varphi = \varphi^*\left(y^i\right)$ must be smooth. For every function y^i, let us choose a function $f_i \in \mathcal{F}_N$ such that $y^i = f_i|_V$. By the assumption $\varphi^*(\mathcal{F}_N) \subset \mathcal{F}_M$, the functions $\varphi^*(f_i) = f_i \circ \varphi$ are smooth, and thus the functions $y^i \circ \varphi|_U = \varphi^*(f_i)|_U$ are smooth as well. ▷

7.19. We have proved that two categories, the category of manifolds (as smooth atlases) with smooth maps in the sense of 6.13 and the category of manifolds (as smooth $\mathbb{R}$-algebras) with smooth maps in the sense of 6.1, are equivalent. From now on we shall not differentiate between the two categories. The notation $(M, \mathcal{F})$ or $(M, C^\infty(M))$ will be used to denote a manifold; the identifications $M = |\mathcal{F}|$ and $\mathcal{F} = C^\infty(M)$ will always be implied.

The reader acquainted with the notion of complex manifold has probably noticed already that in general, such a manifold does not coincide with the complex spectrum of its algebra of holomorphic functions. For example, as is well known from the elementary theory of functions of a complex variable, all holomorphic functions on the Riemann sphere (and on any compact complex manifold) reduce to constants. For this reason, it might seem that the "spectral approach" adopted in this book is less universal than the standard one, based on charts and atlases. Nevertheless, the observability principle forces us to understand complex manifolds as smooth ones, but equipped with an additional (complex) structure. In other words, complex manifolds within this approach are understood as solutions of certain differential equations, while complex charts appear as local solutions of these equations. This viewpoint, going back to Riemann, has many advantages, despite its apparent lack of simplicity. For example, it can be generalized to any commutative algebra by the methods of the "algebraic" differential calculus described below in Chapters 9 and 11.

8

Spectra and Ghosts

8.1. In the previous chapters we have tried to develop in detail the theory of the main notions of this book, i.e., of smooth $\mathbb{R}$-algebras and smooth manifolds. In this way we have determined the main class of geometric objects with which we shall be working.

Nevertheless, most of the definitions, constructions and results that the reader has met in this book are valid for algebras that are more general than smooth $\mathbb{R}$-algebras. In order to show how geometric intuition works in these more general situations, we shall need certain generalizations of the notions of point, smooth algebra, and manifold.

8.2. We can indicate a whole series of reasons for which it is desirable to generalize the notion of point.

(i) Although the "readings of measuring devices" (see Section 1.9) that we have used to motivate our constructions are usually real numbers, it is often necessary to consider measurements of a more general nature (complex numbers, matrices, residues modulo some positive integer, etc.). A striking example is the complex phase method used in the elementary theory of electricity.

(ii) If the solution of some problem (in physics or mathematics) reduces to the solution of algebraic equations with coefficients in a ring A (or in a field A that is not algebraically closed), then as a rule, it is useful to seek the solution in an extension $B \supset A$ of the ring, rather than in the ring A itself. The simplest example is the use of complex roots of a polynomial with real coefficients.

A less trivial example is given by the so-called Pauli operators, which arise in quantum mechanics of electrons as matrix solutions σ_1, σ_2, σ_3 of the system of equations

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = -1, \quad \sigma_2\sigma_3 - \sigma_3\sigma_2 = \sigma_1,$$
$$\sigma_1\sigma_2 - \sigma_2\sigma_1 = \sigma_3, \quad \sigma_1\sigma_3 - \sigma_3\sigma_1 = -\sigma_2.$$

(iii) The mathematician likes to avoid exceptions, strives for the aesthetic unification of any theory; this often leads him to invent a language in which the exceptions turn out to be part of the general rule. Thus parallel lines "intersect at infinity," imaginary points and points at infinity appear in many situations, in particular in connection with $\mathbb{R}$-algebras (see Example 1.18). This trick of giving intuitive geometric meaning to different algebraic situations is particularly fruitful in algebraic geometry, whose ideas will be used extensively here.

8.3. Motivating example. If $f \in \mathbb{R}[x]$ is an irreducible polynomial of degree 2, then there obviously exists no $\mathbb{R}$-point of the algebra $\mathbb{R}[x]$ that would be a root of this polynomial; i.e., there is no homomorphism $\alpha\colon \mathbb{R}[x] \to \mathbb{R}$ such that $\alpha(f) = 0$.

However, there exist exactly two homomorphisms into the field of complex numbers, $\alpha, \overline{\alpha}\colon \mathbb{R}[x] \to \mathbb{C}$, such that $\alpha(f) = \overline{\alpha}(f) = 0$. This will be proved below (see Theorem 8.6), but we suggest that the reader try to prove this now as an exercise. Homomorphisms such as α and $\overline{\alpha}$ should be viewed as complex points for the algebra $\mathbb{R}[x]$.

8.4. Suppose $\Bbbk$ is an arbitrary field, $K \supset \Bbbk$ a ring without zero divisors, and $\mathcal{F}$ a commutative $\Bbbk$-algebra with unit. In view of Example 8.3, it is natural to define a point of the algebra $\mathcal{F}$ "over the ring K" as an epimorphism of $\Bbbk$-algebras $a\colon \mathcal{F} \to K$. It is not less natural to consider two points

$$a_i\colon \mathcal{F} \to K_i \supset \Bbbk, \quad i = 1, 2,$$

identical (*equivalent*) if there exists a $\Bbbk$-algebra isomorphism $i\colon K_1 \to K_2$ such that $a_2 = i \circ a_1$. Now we can give the following definition:

Definition. An equivalence class of $\Bbbk$-algebra epimorphisms

$$a\colon \mathcal{F} \to K \supset \Bbbk$$

is called a *K-point* of the $\Bbbk$-algebra $\mathcal{F}$. The ring K is then referred to as the *domain* of the point a.

Further, we shall often speak of points of the algebra without specifying their domain. The reader should keep in mind that each point of the $\Bbbk$-algebra $\mathcal{F}$ has its own domain, and different points may have different (or identical) domains.

8.5. From the viewpoint of our physical interpretation (in terms of measurements), the isomorphism i appearing in Definition 8.4 can be construed as an equivalent change of the "system of observations," which, of course, should not influence the collection of points (states) determined by the algebra $\mathcal{F}$.

Definition 8.4 also possesses a purely algebraic motivation, which will appear in the next section.

8.6 Theorem. *Points of a commutative* $\Bbbk$*-algebra* $\mathcal{F}$ (*understood in the sense of* Definition 8.4) *correspond bijectively to prime ideals of the algebra* $\mathcal{F}$, *the correspondence being given by the map*

$$(a\colon \mathcal{F} \to K) \mapsto \operatorname{Ker} a \subset \mathcal{F}.$$

◀ The fact that the ideal $\operatorname{Ker} a$ is prime and depends only on the equivalence class of the homomorphism a and that different equivalence classes correspond to different ideals is obvious.

To prove surjectivity, take an arbitrary prime ideal $p \subset \mathcal{F}$. Then the ring $\mathcal{F}/p$ has no zero divisors, and the quotient map $q\colon \mathcal{F} \to \mathcal{F}/p$ is a point of the algebra $\mathcal{F}$ for which $\operatorname{Ker} q = p$. ▶

The set of prime ideals of a $\Bbbk$-algebra $\mathcal{F}$ is called the *prime spectrum* of $\mathcal{F}$ and is denoted by $\operatorname{Spec}\mathcal{F}$. According to the theorem, $\operatorname{Spec}\mathcal{F}$ may be viewed as constituting the set of all points of the $\Bbbk$-algebra $\mathcal{F}$. Obviously, when $\Bbbk = \mathbb{R}$, we have $\operatorname{Spec}\mathcal{F} \supset |\mathcal{F}|$, so that elements of $\operatorname{Spec}\mathcal{F}$ generalize the notion of $\mathbb{R}$-point; i.e., $\mathbb{R}$-points in the sense of Section 3.4 are "points over the field $\mathbb{R}$" in the sense of Section 8.4.

8.7. Examples (based on Theorem 8.6)**.** I. A smooth manifold has no points over the field of complex numbers $\mathbb{C}$.

◀ Suppose $a\colon C^\infty(M) \to \mathbb{C}$ is a $\mathbb{C}$-point and $a(f) = i \in \mathbb{C}$. So $a\left(1 + f^2\right) = 0$, and hence $1 + f^2 \in \ker a$. On the other hand, the element $1 + f^2$ is obviously invertible in $C^\infty(M)$ and as such does not belong to any proper ideal of $C^\infty(M)$. ▶

II. The set of all points of the polynomial algebra $\mathbb{R}[x]$ can be identified with the complex half-plane $\{z \mid \operatorname{Im} z \geqslant 0\}$ to which a point ω, corresponding to the ideal $\{0\} \subset \mathbb{R}[x]$, has been added.

◀ Since $\mathbb{R}[x]$ is a principal ideal domain, any prime ideal in it is of the form $\mathbb{R}[x]f$, where f is an irreducible polynomial. But then either $f \equiv 0$ or $f = a(x - b)$ or $f = a(x - c)(x - \overline{c})$, where a, $b \in \mathbb{R}$, while c and $\overline{c}$ are conjugate complex numbers. ▶

Note that we have proved the statement mentioned at the end of Section 8.3.

Exercises. 1. Let $\Bbbk$ be a field, and X a formal variable; describe $\operatorname{Spec}\Bbbk[X]$.

2. Show that $\mu_z^\infty \overset{\text{def}}{=} \bigcap_k \mu_z^k$, $z \in M$, is a prime ideal in $C^\infty(M)$.

8.8. We have learned to assign a set of points, $\operatorname{Spec}\mathcal{F}$, to every $\Bbbk$-algebra $\mathcal{F}$. It is only natural to try to supply this set with a topology. In a similar situation, working with $|\mathcal{F}|$, we introduced the topology induced from the topology in $\mathbb{R}$, but in the case of an arbitrary $\Bbbk$-algebra $\mathcal{F}$ we no longer have a fixed field $\mathbb{R}$ with a trustworthy topology. A new idea is needed to find a topology in $\operatorname{Spec} \mathcal{F}$.

Suppose C is a closed set in $\mathbb{R}^n$; then there exists a function $f \in C^\infty(\mathbb{R}^n)$ such that $f(r) = 0 \iff r \in C$. The same is true for a closed set B in a manifold M: There exists a function $f \in \mathcal{F} = C^\infty(M)$ such that $f(p) = 0 \iff p \in B$ (Section 4.17 (i)). This circumstance will be the basis for introducing a topology in $\operatorname{Spec} \mathcal{F}$.

First note that any element $f \in \mathcal{F}$ may be viewed as a function on $\operatorname{Spec} \mathcal{F}$. Namely, for any prime ideal $p \in \operatorname{Spec} \mathcal{F}$, we put

$$f(p) = f \bmod p.$$

For any subset $E \subset \mathcal{F}$ denote by $V(E) \subset \operatorname{Spec} \mathcal{F}$ the set of all prime ideals containing E. In other words, if $p \in V(E)$, then for any function $f \in E$ we have $f(p) = 0$. We can now define the *Zariski topology* in the prime spectrum $\operatorname{Spec} \mathcal{F}$ of an arbitrary $\Bbbk$-algebra $\mathcal{F}$ as the topology whose basis of closed sets is the collection $\{V(E) \mid E \subset \mathcal{F}\}$.

8.9. The definition of the Zariski topology in $\operatorname{Spec} \mathcal{F}$ can also be given in terms of the closure operation. For any $M \subset \operatorname{Spec} \mathcal{F}$ consider the ideal $I_M = \bigcap_{p \in M} p$ and define the closure $\overline{M}$ of M as

$$\overline{M} = \{p \in \operatorname{Spec} \mathcal{F} \mid p \supset I_M\}.$$

This definition has a clear algebraic meaning: If the element $f \in \mathcal{F}$ vanishes on M (i.e., $p \in M \Rightarrow f(p) = 0 \iff f \in p$), then it also vanishes on $\overline{M}$; conversely, if any f that vanishes on M is also zero on some point $p \in \operatorname{Spec} \mathcal{F}$, then $p \in \overline{M}$. (The fact that this construction gives the same topology as 8.8 follows directly from the two definitions.)

The reader has perhaps wondered what price we shall have to pay for the extreme generality of this construction. It turns out that the Zariski topology in $\operatorname{Spec} \mathcal{F}$ is non-Hausdorff. In particular, $\operatorname{Spec} \mathcal{F}$ contains nonclosed points, which will be considered in the next section.

8.10 Exercises. 1. Prove that a one-point set $\{p\} \subset \operatorname{Spec} \mathcal{F}$ is closed in the Zariski topology iff the ideal p is maximal.

2. Describe the Zariski topology of $\big|\mathbb{R}[X]\big|$. Compare the Zariski topology of $\big|\mathbb{R}[X]\big| = \mathbb{R}$ and the Zariski topology of $\big|C^\infty(\mathbb{R})\big| = \mathbb{R}$.

3. Describe the Zariski topology of $\operatorname{Spec} \mathbb{R}[X]$.

8.11 Proposition. *The closure of a point* $q \in \operatorname{Spec} \mathcal{F}$ *coincides with the set*

$$V(q) = \{p \in \operatorname{Spec} \mathcal{F} \mid q \subset p\}$$

(*in other words,* $\overline{\{q\}}$ *is the set of all common zeros of all the functions from the ideal* q).

◀ By Definition 8.8, the set $V(q)$ is closed. On the other hand, $E \subset \mathcal{F}$, and $q \in V(E) \Rightarrow q \supset E \Rightarrow V(q) \subset V(E)$. ▶

8.12. Corollaries and examples. I. If a $\mathbb{k}$-algebra has no zero divisors, then $\{0\} \in \operatorname{Spec}\mathcal{F}$ and $\overline{\{0\}} = \operatorname{Spec}\mathcal{F}$. For this reason, the ideal $\{0\}$ is said to be the *common point* of the set $\operatorname{Spec}\mathcal{F}$.

II. In particular, the point ω from Example 8.7, II, is such a common point.

III. Suppose $f \in \mathbb{k}[x_1, \ldots, x_n] = \mathcal{F}$ is an irreducible polynomial in n variables over the algebraically closed field $\mathbb{k}$. Consider the prime ideal $p = \mathbb{k}[x_1, \ldots, x_n]\cdot f$. The closure of the point p in the prime spectrum $\operatorname{Spec}\mathcal{F}$ of the polynomial algebra $\mathcal{F}$ contains all the maximal ideals corresponding to the points of the hypersurface

$$H_f = \{(r_1, \ldots, r_n) \in \mathbb{k}^n \mid f(r_1, \ldots, r_n) = 0\}.$$

The point p is therefore called the common point of this hypersurface. (To each point $(r_1, \ldots, r_n) \in \mathbb{k}^n$ there corresponds the maximal ideal of $\mathcal{F}$ generated by all monomials $x_1 - r_1, \ldots, x_n - r_n$.)

IV. By Proposition 8.11, the closure of the point p in Example III above contains all the prime ideals containing p. From the point of view of "maximal ideal geometry," each such ideal determines a surface of "lesser dimension" contained in the hypersurface H_f and is the common point of this lesser surface.

8.13 Theorem. *The prime spectrum* $\operatorname{Spec}\mathcal{F}$ *of any* $\mathbb{k}$*-algebra* $\mathcal{F}$ *is compact.*

Let us restate the theorem in terms of closed sets: *if* $\{M_\alpha\}_{\alpha\in A}$ *is a family of closed sets such that* $\bigcap_{\alpha\in A} M_\alpha = \varnothing$, *then we can choose a finite subfamily* $M_{\alpha_1}, \ldots, M_{\alpha_N}$ *with empty intersection.* We shall prove the theorem in this (equivalent) form.

◀ Without loss of generality we can assume that $M_\alpha = V(E_\alpha)$, where $E_\alpha \subset \mathcal{F}$ is some subset. Denote by $[E_\alpha] \subset \mathcal{F}$ the ideal generated by E_α and notice that

$$\varnothing = \bigcap_{\alpha\in A} M_\alpha = \bigcap_{\alpha\in A} V(E_\alpha) = V\left(\sum_{\alpha\in A}[E_\alpha]\right).$$

But this means that the ideal $\sum_{\alpha\in A}[E_\alpha]$ is not contained in any prime ideal and hence in any maximal ideal of the algebra $\mathcal{F}$. In other words,

$$\sum_{\alpha\in A}[E_\alpha] = \mathcal{F} \ni 1.$$

Therefore, we can find $\alpha_1, \dots, \alpha_N \in A$ and $f_i \in [E_{\alpha_i}]$, $i = 1, \dots, N$, such that $\sum_{i=1}^N f_i = 1$. But in this case

$$\bigcap_{i=1}^N M_{\alpha_i} = V\left(\sum_{i=1}^N [E_{\alpha_i}]\right) = V(\mathcal{F}) = \varnothing. \quad \blacktriangleright$$

8.14. Let us now investigate how prime spectra behave under homomorphisms of their $\Bbbk$-algebras. Suppose $\mathcal{F}_1$, $\mathcal{F}_2$ are $\Bbbk$-algebras and $\alpha\colon \mathcal{F}_2 \to \mathcal{F}_1$ is a $\Bbbk$-algebra homomorphism. We claim that *if $p \subset \mathcal{F}_1$ is a prime ideal, then so is $\alpha^{-1}(p)$.*

◀ If $q_1 \in \alpha^{-1}(p)$, then for any $q_2 \in \mathcal{F}_2$, we have

$$\alpha(q_1 q_2) \in \alpha(\alpha^{-1}(p) q_2) = p\alpha(q_2) \subset p.$$

Thus $q_1 q_2 \in \alpha^{-1}(p)$, and $\alpha^{-1}(p)$ is an ideal. To show that it is prime, let q_1, $q_2 \in \mathcal{F}$ and $q_1 q_2 \in \alpha^{-1}(p)$. Then $\alpha(q_1)\alpha(q_2) \in p$, and since p is prime, at least one of the elements $\alpha(q_1)$, $\alpha(q_2)$, say the first, belongs to p. Then we see that

$$q_1 \in \alpha^{-1}(\alpha(q_1)) \subset \alpha^{-1}(p);$$

i.e., the ideal $\alpha^{-1}(p)$ is prime. ▶

Now, to every $\Bbbk$-algebra homomorphism $\alpha\colon \mathcal{F}_2 \to \mathcal{F}_1$ we can assign the map of prime spectra

$$|\alpha|\colon \operatorname{Spec} \mathcal{F}_1 \to \operatorname{Spec} \mathcal{F}_2, \quad p \mapsto \alpha^{-1}(p).$$

8.15 Proposition. *For any $\Bbbk$-algebra homomorphism $\alpha\colon \mathcal{F}_2 \to \mathcal{F}_1$ the corresponding prime spectra map $|a|\colon \operatorname{Spec} \mathcal{F}_1 \to \operatorname{Spec} \mathcal{F}_2$ is continuous in the Zariski topology.*

◀ The proof is a straightforward verification of definitions; we leave it to the reader. ▶

8.16. Now, copying a similar definition for smooth manifolds (see Section 6.1), we can give the following definition:

Definition. A map $\beta\colon \operatorname{Spec} \mathcal{F}_1 \to \operatorname{Spec} \mathcal{F}_2$ is said to be *smooth* if there exists a $\Bbbk$-algebra homomorphism $\alpha\colon \mathcal{F}_2 \to \mathcal{F}_1$ such that $\beta = |\alpha|$.

Thus the reader has met with another example of a category: the category of $\Bbbk$-algebra prime spectra, in which the morphisms are smooth maps of spectra.

8.17. Suppose the ideal $p \in \operatorname{Spec} \mathcal{F}$ of the $\Bbbk$-algebra $\mathcal{F}$ satisfies $\mathcal{F}/p = \Bbbk$. Then it is easy to show that p is a maximal ideal. We also suggest that the reader work out the following exercises:

Exercises. 1. The ideal p is maximal iff $\mathcal{F}/p$ is a field.

2. If $\mathcal{F}$ is finitly generated and p is maximal, then $\mathcal{F}/p$ is a finite algebraic extension of $\Bbbk$.

8.18. Definition. The *maximal spectrum* $\operatorname{Spm}\mathcal{F}$ of the $\mathbb{k}$-algebra $\mathcal{F}$ is the set of maximal ideals of $\mathcal{F}$.

The previous section, as well as Sections 8.10–8.11, where we studied the closure of one-point sets in $\operatorname{Spec}\mathcal{F}$, suggests that instead of $\operatorname{Spec}\mathcal{F}$ we should have been studying $\operatorname{Spm}\mathcal{F}$. Indeed, we would thus have avoided such pathology as nonclosed points, and the set of domains of points would be more manageable. However, this is not quite reasonable because of the fact that *unlike prime ideals, maximal ideals may have inverse images that are no longer maximal* (see Example 8.19). Therefore, generally there is no map of maximal spectra that could naturally be assigned to a homomorphism of the corresponding algebras. In other words, the correspondence $A \mapsto \operatorname{Spm} A$ is not a functor from the category of K-algebras to the category of topological spaces.

8.19. Example. Suppose $\mathcal{F}_2 = \mathbb{k}[x_1, x_2]$ is the algebra of polynomials in two variables, $\mathcal{F}_1 = \mathbb{k}(x_1)$ is the field of rational functions, and $\alpha\colon \mathcal{F}_2 \to \mathcal{F}_1$ is the composition of the quotient epimorphism

$$\mathcal{F}_2 = \mathbb{k}[x_1, x_2] \to \mathbb{k}[x_1, x_2]/(x_2) = \mathbb{k}[x_1]$$

and the inclusion $\mathbb{k}[x_1] \hookrightarrow \mathbb{k}(x_1) = \mathcal{F}_1$.

The ideal $\{0\} \subset \mathbb{k}(x_1)$ is a maximal one; however, $\alpha^{-1}(\{0\}) = \mathcal{F}_2 \cdot x_2$ is a prime ideal, but not a maximal one. This establishes the statement in italics in the previous section.

8.20 Exercises. 1. Let M be a compact manifold. Prove that any maximal ideal in $C^\infty(M)$ is of the form μ_z, $z \in M$ (see Exercise 2 from Section 8.7).

2. Show that for any noncompact manifold M there exist maximal ideals in $C^\infty(M)$ different from the μ_z's.

3. Show that any such maximal ideal has an infinite number of generators.

8.21. In the remainder of this section we shall discuss certain aspects of prime spectra that distinguish them from smooth manifolds.

As we saw in Example 8.7, II, a simple spectrum may contain, besides "visible" points, certain points whose geometric interpretation is not obvious (e.g., the prime ideal ω in that example).

We shall begin with a few examples showing that "points" of that type may already appear in the maximal spectrum $\operatorname{Spm}\mathcal{F}$ of a $\mathbb{k}$-algebra.

8.22. Ghosts. The reader who has studied Example 8.7, I, in detail has undoubtedly noticed that the maximal ideals of the algebra of smooth functions on a compact manifold correspond bijectively to ordinary $\mathbb{R}$-points of this algebra (i.e., to ordinary points of the manifold). For noncompact manifolds this is no longer the case.

Indeed, suppose I_c is the ideal of all functions with compact support on a noncompact manifold. None of the maximal ideals containing I_c is the kernel of any $\mathbb{R}$-point (the proof is left to the reader as an exercise). Such maximal ideals correspond to "points" of noncompact manifolds that we call *ghosts*. In Sections 8.24–8.26 we shall see how such ghosts can actually materialize.

8.23. Before continuing, the following remark is called for. Suppose $\mathcal{F}$ is the $\mathbb{R}$-algebra of smooth functions on the compact manifold M. As was pointed out above, $M = \operatorname{Spm}\mathcal{F}$, but the reader should not think that $M = \operatorname{Spec}\mathcal{F}$. To that end, he should work out the following exercise.

Exercise. The function $f \in \mathcal{F}$ is called *flat* at the point $a \in M$ if it vanishes with all its derivatives at that point. The set of all functions that are flat at the given point $a \in M$ constitutes an ideal. Prove that this ideal is prime (i.e., is a point of $\operatorname{Spec}\mathcal{F}$ that corresponds to no point of M).

8.24. Example (compactifications of the line)**.** Let $\mathcal{F}$ be the $\mathbb{R}$-algebra of smooth functions on the line $\mathbb{R}$, and I_c, as above, its ideal of functions with compact support. One can try to describe the maximal ideals that contain I_c, but most of these ideals–ghosts have no reasonable constructive description.

Nevertheless, there are two very nice sets of functions that contain I_c, namely the sets $\overline{\mu}_{+\infty}$ and $\overline{\mu}_{-\infty}$:

$$\overline{\mu}_{\pm\infty} = \left\{ f \in \mathcal{F} \mid \lim_{r\to\pm\infty} f(r) = 0 \right\}.$$

These sets, unfortunately, are not ideals (in poetic language we can say that "were they ideals, they would be maximal ones"). This unpleasant circumstance can be overcome as follows. Put

$$\mathcal{F}_1 = \left\{ f \in \mathcal{F} \mid \forall k \geqslant 0 \ \lim_{r\to+\infty} \frac{d^k f}{dr^k} \quad \text{and} \quad \lim_{r\to-\infty} \frac{d^k f}{dr^k} \ \text{ exist} \right\}$$

and define $\mu_{\pm\infty}$ by putting $\mu_{\pm\infty} = \mathcal{F}_1 \cap \overline{\mu}_{\pm\infty}$. Then:

(i) $\mu_{+\infty}$ and $\mu_{-\infty}$ are maximal ideals of the algebra $\mathcal{F}_1$ containing the ideal I_c.

(ii) If $M = [0,1] \subset \mathbb{R}^1$, and the algebra $\mathcal{F}_M$ consists of the restrictions of all the functions from $\mathcal{F}$ to M, then the manifold with boundary M is diffeomorphic to $|\mathcal{F}_1|$.

(iii) Under this identification, the ideal I_c becomes the ideal of functions that vanish near the end points of the closed interval $[0,1]$, while the ideals $\mu_{-\infty}$ and $\mu_{+\infty}$ become the points 0 and 1, respectively.

(iv) This implies that all the maximal ideals of the algebra $\mathcal{F}_1$, except $\mu_{+\infty}$ and $\mu_{-\infty}$, correspond to ordinary points of the line $\mathbb{R}^1$; as for the ideals $\mu_{\pm\infty}$, they are "ghosts", which are adjoined to the line in order to make it compact.

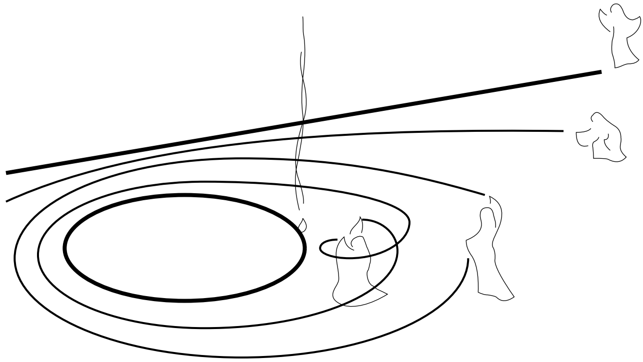


Figure 8.1. Gluing “infinite end points.”

(v) Prove that the algebra $\mathcal{F}_1$ is not isomorphic to the algebra $\mathcal{F}_M$. Try to find another subalgebra of $\mathcal{F}$, different from $\mathcal{F}_1$, isomorphic to $\mathcal{F}_M$.

8.25. Another example. Among the numerous other methods of compactifying the line, we consider only one more. Let

$$\mathcal{F}_2 = \left\{ f \in \mathcal{F}_1 \mid \forall k \geqslant 0 \lim_{r \to +\infty} \frac{d^k f}{dr^k} = \lim_{r \to -\infty} \frac{d^k f}{dr^k} \right\}$$

and put

$$\mu_\infty = \mu_{+\infty} \cap \mathcal{F}_2 \quad (= \mu_{-\infty} \cap \mathcal{F}_2).$$

It is easy to prove that the algebra $\mathcal{F}_2$ is isomorphic to $C^\infty(S^1)$, the algebra of smooth functions on the circle. This can be visualized by imagining that the “infinite end points” of the line are glued together by means of the “ghost” corresponding to the ideal μ_∞ (see Figure 8.1).

8.26. Further examples. I. Denote by $\mathcal{F}$ the algebra of complex-valued functions of a complex variable, defined, holomorphic, and bounded in the domain $|z| < 1$. Among the maximal ideals of the algebra $\mathcal{F}$, we know, of course, the ideals corresponding to points a, $|a| < 1$, namely

$$\mu_a = \{ f \in \mathcal{F} \mid f(a) = 0 \}.$$

It is also possible to define μ_a when $|a| = 1$ by putting

$$\mu_a = \{ f \in \mathcal{F} \mid \lim_{z \to a} f(z) = 0 \}.$$

Prove that the maximal spectrum of the algebra $\mathcal{F}$ consists of all the ideals μ_a, $|a| \leqslant 1$.

II. Find the maximal spectrum of the algebra $\mathcal{F}_2$ of complex-valued functions of two complex variables, analytic and bounded in the open polydisk

$$\{ (z_1, z_2) \mid |z_1| < 1,\ |z_2| < 1 \}.$$

9

The Differential Calculus as a Part of Commutative Algebra

9.1. The formal approach to observation procedures in classical physics, described in the previous chapters, leads to rather important conclusions. Let us first note that *the differential calculus is the natural language of classical physics*. On the other hand, *all information about some classical physical system is encoded in the corresponding algebra of observables*. From this it follows that the *differential calculus needed to describe physical problems is a part of commutative algebra*.

The basic aim of this chapter is to explain how to obtain a purely algebraic definition of differential operators using common facts known from the classical calculus. It is very important that the constructions obtained below can be used for arbitrary, not necessarily smooth, algebras. But perhaps the key point is somewhat different.

We shall see that the differential calculus is a formal consequence of arithmetical operations. This unexpected and beautiful fact plays an important role not only in mathematics itself. It also allows us to reconsider some paradigms reflecting the relationship of mathematics to the natural sciences and, above all, with physics and mechanics. By including observability in our considerations, we ensure that mathematics may be regarded as a branch of the natural sciences.

9.2. Let us start with the simplest notion of calculus, namely, that of the derivative, which is the formal mathematical counterpart of velocity. From elementary mechanics we know that velocity is a vector, i.e., a directed segment. This point of view is hardly satisfactory, since it is completely unclear what a *directed segment* is in the case of an abstract (*curved*) manifold. For this reason, the founders of differential geometry defined the tangent vector as a quantity described in a given coordinate system as an n-tuple of numbers; when passing from one coordinate system to another, this n-tuple is to be transformed in a prescribed manner. This approach is also unsatisfactory, because it describes vectors in local coordinates and does not explain what they are in essence.

In many modern textbooks on differential geometry one can find another definition of a tangent vector, which does not use local coordinates: A tangent vector is an equivalence class of smooth curves tangent to each other at a given point of the manifold under consideration. But try to find the sum of such classes or multiply a class by a number (i.e., try to introduce the structure of a linear space), and you will immediately see that this definition is rather inconvenient for work.

The principal reason why these definitions of a tangent vector are unsatisfactory is their descriptive nature: They say nothing about the functional role of this notion in the differential calculus. This role can be understood if algebras of observables are used.

Let A be an algebra of observables. Then, by definition, $M = |A|$ is the manifold of states for the corresponding system, and a particular state is an element $h \in |A|$. Therefore, a time evolution of the system state is described by a family h_t. Consequently, the velocity of evolution at a moment t_0 is

$$\Delta_h \stackrel{\text{def}}{=} \left.\frac{dh_t}{dt}\right|_{t=t_0} : A \to \mathbb{R}, \tag{9.1}$$

where, by definition,

$$\frac{dh_t}{dt} = \lim_{\Delta t \to 0} \frac{1}{\Delta t}(h_{t+\Delta t} - h_t),$$

whatever the meaning of the limit above may be. In other words, the motion that we conceive is in fact the change in time in the readings given by the measuring devices, while the velocity of motion is the velocity of these changes.

The translation of the above to geometrical language is accomplished by the correspondence $M \ni z \leftrightarrow h = h_z \in |A|$, where $A = C^\infty(M)$ and $h_z(f) = f(z)$ for any $f \in C^\infty(M)$. After this translation, the family $\{h_t\}$ becomes a curve $z(t)$ on the manifold M. This curve is such that $h_t = h_{z(t)}$, while the derivative

$$\frac{dh_t}{dt}$$

is a tangent vector to $z(t)$ and consequently to the manifold M. Formula (9.1) now acquires the form

$$\Delta_z = \left.\frac{dh_{z(t)}}{dt}\right|_{t=t_0} : A \to \mathbb{R}, \tag{9.2}$$

where $z = z(t_0)$, and obviously $\Delta_z \leftrightarrow \Delta_h$.

Thus, we arrive at the following interpretation:

velocity of changes of system states	$\longleftrightarrow$	tangent vector to the manifold M

where the system is described by the algebra A.

The words *tangent vector* were used intuitively above. Our aim now is to define it rigorously, solely in terms of the algebra of observables A. The above interpretation allows us to do this in a natural way. It remains only to understand the mathematical nature of the operator Δ_h (or Δ_z), informally defined by formula (9.1) (respectively by (9.2)): Not all maps from A to $\mathbb{R}$ (for example, h) may be appropriately called velocities of state change.

9.3. The algebra of observables is a combination of two structures: that of a vector space and that of a multiplicative structure. The interaction of the operator Δ_h (or Δ_z) with the first structure is obvious: It is $\mathbb{R}$-linear, i.e.,

$$\Delta_h\left(\sum_{j=1}^{k}\lambda_j f_j\right) = \sum_{j=1}^{k}\lambda_j\Delta_h(f_j), \quad \lambda_j \in \mathbb{R},\ f_j \in C^\infty(U). \tag{9.3}$$

Exercise. Check this.

To understand how Δ_h interacts with the multiplicative structure of the algebra $C^\infty(M)$, one needs to compute the action of this operator on the product of two observables. We have

$$\begin{aligned}\Delta_h(fg) &= \frac{dh_t(fg)}{dt}\Big|_{t=t_0} = \frac{d(h_t(f)h_t(g))}{dt}\Big|_{t=t_0}\\ &= \frac{dh_t(f)}{dt}\Big|_{t=t_0} h_{t_0}(g) + h_{t_0}(f)\frac{dh_t(g)}{dt}\Big|_{t=t_0}\\ &= \Delta_h(f)h(g) + h(f)\Delta_h(g);\end{aligned}$$

i.e., Δ_h satisfies the following *Leibniz rule*:

$$\Delta_h(fg) = \Delta_h(f)h(g) + h(f)\Delta_h(g). \tag{9.4}$$

In geometrical form, this rule can be written as

$$\Delta_z(fg) = \Delta_z(f)g(z) + f(z)\Delta_z(g). \tag{9.5}$$

Thus, the rules (9.3) and (9.4) completely govern the interrelations between the operator Δ_h and the basic structures in the algebra of observables. Therefore, we have a good reason for giving the following definition:

9.4. Definition. A map

$$\xi\colon C^\infty(M) \to \mathbb{R}$$

is said to be a *tangent vector to the manifold* M at a point $z \in M$ if it satisfies the two following conditions:

(1) $\mathbb{R}$-linearity:

$$\xi\left(\sum_{j=1}^{k}\lambda_j f_j\right)=\sum_{j=1}^{k}\lambda_j\xi(f_j),\quad \lambda_j\in\mathbb{R},\ f_j\in C^\infty(M).$$

(2) The local Leibniz rule (or the Leibniz rule at a point z):

$$\xi(fg)=f(z)\xi(g)+g(z)\xi(f),\quad f,g\in C^\infty(M).$$

Obviously, if we now define the sum of two tangent vectors and the multiplication of a tangent vector by a real number using the rules

$$\begin{aligned}(\xi+\xi')(f)&=\xi(f)+\xi'(f),\\ (\lambda\xi)(f)&=\lambda\xi(f),\quad \lambda\in\mathbb{R},\end{aligned}$$

then in both cases the result will be an $\mathbb{R}$-linear operator satisfying the Leibniz rule; i.e., we shall obtain a tangent vector again. In other words, the set T_zM of all tangent vectors at a point $z\in M$ possesses a natural structure of a vector space over $\mathbb{R}$. This space is called the *tangent space of the manifold M at z.*

Remark. The zero vector $0_z\in T_zM$ is just the zero map from $C^\infty(M)$ to $\mathbb{R}$ and as such coincides with $0_{z'}$ for any other point z'. But it is natural to distinguish between vectors 0_z and $0_{z'}$, $z\neq z'$, since they are tangent to M at two different points (are subject to different Leibniz rules). For a formally satisfactory explanation of this distinction see Section 9.52.

9.5. Let us now describe the operators $\xi\in T_zM$ in local coordinates. Let U be a domain in $\mathbb{R}^n$ and fix a local coordinate system $x_1,\dots,x_n$. Assume that

$$z=(z_1,\dots,z_n),\quad y=y(\Delta t)=(z_1+\alpha_1\Delta t,\dots,z_n+\alpha_n\Delta t),\quad \alpha_i\in\mathbb{R},$$

in this system. Then obviously $z=y(0)$ and

$$\Delta_z(f)=\lim_{\Delta t\to 0}\frac{f(y(\Delta t))-f(z)}{\Delta t}=\frac{df(y(t))}{dt}\Big|_{t=0}=\sum_{i=1}^{n}\alpha_i\frac{\partial f}{\partial x_i}\Big|_z.$$

This is nothing but the derivation of the function f in the direction $\alpha=(\alpha_1,\dots,\alpha_n)$. Hence, the operator Δ_z is described by an n-tuple of real numbers. This observation explains the hidden meaning of the classical descriptive definition of tangent vectors.

The arguments of Section 9.3 and 9.5 use the operator Δ_z, which was not defined rigorously, and so these arguments are not rigorous either. Nevertheless, they make it possible to define tangent vectors in terms of the algebra of observables (see Definition 9.4). Using this definition as a starting point, we can now compare our approach with the usual one.

9.6 Tangent vector theorem. *Let M be a smooth manifold, $z\in M$, and let $x_1,\dots,x_n$ be a local coordinate system in a neighborhood $U\ni z$. Then,* *in this coordinate system, any tangent vector $\xi \in T_zM$ can be represented in the form*

$$\xi = \sum_{i=1}^{n} \alpha_i \frac{\partial}{\partial x_i}\Big|_z, \qquad \alpha_i \in \mathbb{R}.$$

In other words, the notions of tangent vector and of differentiation in a given direction coincide.

slt The proof of this theorem consists of several steps. The first of them if proposed to the reader.

9.7 Lemma–Exercise. *Let $f = \text{const} \in \mathbb{R}$; then $\xi(f) = 0$.* $\triangleright$

9.8 Lemma. *Tangent vectors are local operators, i.e., if two functions $f, g \in \mathcal{F}$ coincide on an open set $U \ni z$, then for any tangent vector $\xi \in T_zM$ the equality $\xi(f) = \xi(g)$ holds.*

$\triangleleft$ To prove this statement, it suffices to check that if the equality $f|_U = 0$ holds in some neighborhood $U \ni z$, then $\xi(f) = 0$. Indeed, in this case, by Corollary 2.5, there exists a function $h \in C^\infty(M)$ such that $h(z) = 0$ and $h|_{M\setminus U} = 1$. Consequently, $f = hf$, and, by the Leibniz rule,

$$\xi(f) = \xi(hf) = f(z)\xi(h) + h(z)\xi(f) = 0. \qquad \triangleright$$

9.9 Lemma. *The spaces T_zU and T_zM are naturally isomorphic for any open neighborhood $U \ni z$.*

$\triangleleft$ The embedding $i\colon U \subset M$ of the open set U into the manifold M induces the map $d_zi\colon T_zU \to T_zM$ as follows: Let $\xi \in T_zU$ and $f \in C^\infty(M)$; set $d_zi(\xi)(f) = \xi(f\big|_U)$. (Check that the map $d_zi(\xi)$ is indeed a tangent vector.) Obviously, the map d_zi is $\mathbb{R}$-linear.

Let us construct the inverse map. To this end, note first that for any function $g \in C^\infty(U)$ one can find a function $f \in C^\infty(M)$ coinciding with g in some neighborhood in U of the point z. Indeed, consider a function $h \in C^\infty(U)$ vanishing outside some compact neighborhood $V_0 \subset U$ and equal to 1 in a neighborhood $V_1 \subset V_0$ of the point z (see Section 2.5). Then for $f \in C^\infty(M)$ we can take a function vanishing outside U and coinciding with gh in U. Now define an A-linear map $\pi_U\colon T_zM \to T_zU$ by setting $\pi_U(\eta)(g) = \eta(f)$. It follows from Lemma 9.8 that the value $\eta(f)$ does not depend on the choice of the function f; i.e., the homomorphism π_U is well defined. It is now easy to see that $d_zi \circ \pi_U = \text{id}$ and $\pi_U \circ d_zi = \text{id}$. $\triangleright$

9.10. From the lemma above it follows that we may confine ourselves to the case $M = U \subset \mathbb{R}^n$. Moreover, we may assume the domain U to be *star-shaped* with respect to the point z (i.e., such that $y \in U$ implies $[z, y] \subset U$ for the entire closed interval $[z, y]$). By Corollary 2.9, any smooth function

f in a star-shaped neighborhood of z can be represented in the form

$$f(x) = f(z) + \sum_{i=1}^{n}(x_i - z_i)\frac{\partial f}{\partial x_i}(z) + \sum_{i,j=1}^{n}(x_i - z_i)(x_j - z_j)g_{ij}(x).$$

Applying the tangent vector ξ to the last equality and using the Leibniz rule, we immediately see that for any derivation $\xi \in T_zM$,

$$\xi(f) = \sum_{i=1}^{n}\alpha_i\frac{\partial f}{\partial x_i}(z),$$

where $\alpha_i = \xi(x_i - z_i) = \xi(x_i)$, which concludes the proof of the theorem. ▶

9.11. From the tangent vector theorem it follows that T_zM is an n-dimensional vector space over $\mathbb{R}$. In fact, by this theorem the tangent vectors

$$\left.\frac{\partial}{\partial x_1}\right|_z, \quad \ldots, \quad \left.\frac{\partial}{\partial x_n}\right|_z$$

generate the space T_zM for any coordinate neighborhood (U, x) of the point z. Let $\xi = \sum_{j=1}^{n}\alpha_j\partial/\partial x_j$ be a linear combination of the vectors $\partial/\partial x_j$. Since obviously $\alpha_j = \xi(x_j)$, the vector ξ does not vanish if at least one of the coefficients α_j is not zero. Hence, the vectors $\partial/\partial x_j$ are linearly independent and form a basis of the tangent space T_zU. The isomorphism $d_zi\colon T_zU \to T_zM$ constructed during the proof of the theorem now shows that $\dim T_zM = n$. Below we shall identify the vectors $\partial/\partial x_j$ forming a basis of T_zU with the vectors $d_zi\,(\partial/\partial x_j)$ forming a basis of T_zM.

It follows from the above that the dimension of the tangent space T_zM is equal to the number of local coordinates in any chart containing z. In other words, it is equal to the dimension of the manifold M.

9.12. Let $y_1, \ldots, y_n$ be another local coordinate system in a neighborhood of the point z. Then in the corresponding basis of the tangent space $\partial/\partial y_1\big|_z, \ldots, \partial/\partial y_n\big|_z$ one has

$$\xi = \sum_{k=1}^{n}\beta_k\left.\frac{\partial}{\partial y_k}\right|_z.$$

Further, in view of the "chain rule,"

$$\xi = \sum_{i=1}^{n}\alpha_i\left.\frac{\partial}{\partial x_i}\right|_z = \sum_{i=1}^{n}\alpha_i\sum_{k=1}^{n}\left.\frac{\partial y_k}{\partial x_i}\frac{\partial}{\partial y_k}\right|_z = \sum_{k=1}^{n}\left(\sum_{i=1}^{n}\alpha_i\frac{\partial y_k}{\partial x_i}\right)\left.\frac{\partial}{\partial y_k}\right|_z,$$

i.e.,

$$\beta_k = \sum_{i=1}^{n}\alpha_i\frac{\partial y_k}{\partial x_i}, \qquad k = 1, \ldots, n.$$

The matrix that transforms the basis

$$\frac{\partial}{\partial x_1}\Big|_z, \quad \ldots, \quad \frac{\partial}{\partial x_n}\Big|_z,$$

corresponding to the local coordinates $x_1, \ldots, x_n$, to the basis

$$\frac{\partial}{\partial y_1}\Big|_z, \quad \ldots, \quad \frac{\partial}{\partial y_n}\Big|_z,$$

corresponding to the coordinates $y_1, \ldots, y_n$, is the *Jacobi matrix*

$$J_z = \begin{pmatrix} \dfrac{\partial y_1}{\partial x_1} & \cdots & \dfrac{\partial y_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial y_n}{\partial x_1} & \cdots & \dfrac{\partial y_n}{\partial x_n} \end{pmatrix}_z .$$

The subscript z indicates that the elements of the Jacobi matrix are computed at the point z. We see that the coordinate change rules obtained here for tangent vectors are in agreement with the approach accepted in the tensor calculus.

9.13. The differential of a smooth map. It is quite natural that any smooth map of manifolds generates a map of tangent vectors. (Try to check this yourself by looking at Figure 9.1 and continuing the informal arguments of Section 9.2.) A rigorous construction is as follows. Let $\varphi\colon M \to N$ be a smooth map and $\xi \in T_zM$. Then the map $\eta = \xi \circ \varphi^*\colon C^\infty(N) \to \mathbb{R}$ is a tangent vector to the manifold N at the point $\varphi(z)$.

In fact, its $\mathbb{R}$-linearity is obvious. In addition, for any $f, g \in C^\infty(N)$ one has

$$\begin{aligned} \eta(fg) &= \xi(\varphi^*(fg)) = \xi(\varphi^*(f)\varphi^*(g)) \\ &= \xi(\varphi^*(f))(\varphi^*(g)(z)) + (\varphi^*(f)(z))\xi(\varphi^*(g)) \\ &= \eta(f)g(\varphi(z)) + f(\varphi(z))\eta(g). \end{aligned}$$

Definition. The map

$$d_z\varphi\colon T_z(M) \to T_z(N), \quad \xi \mapsto \xi \circ \varphi^*, \quad \xi \in T_zM,$$

is called the *differential of the map* φ at the point $z \in M$.

Obviously, the differential $d_z\varphi$ is a linear map.

9.14 Exercise. Prove that if $\psi\colon N \to L$ is another smooth map, then one has

$$d_z(\psi \circ \varphi) = d_{\varphi(z)}\psi \circ d_z\varphi. \tag{9.6}$$

Prove also that if $N = L$ and $\psi = \mathrm{id}_N$, then $d_z\psi = \mathrm{id}_{T_zN}$.

Formula (9.6), applied to $\psi = \varphi^{-1}$, shows that

$$d_{\varphi(z)}\varphi^{-1} = (d_z\varphi)^{-1}.$$

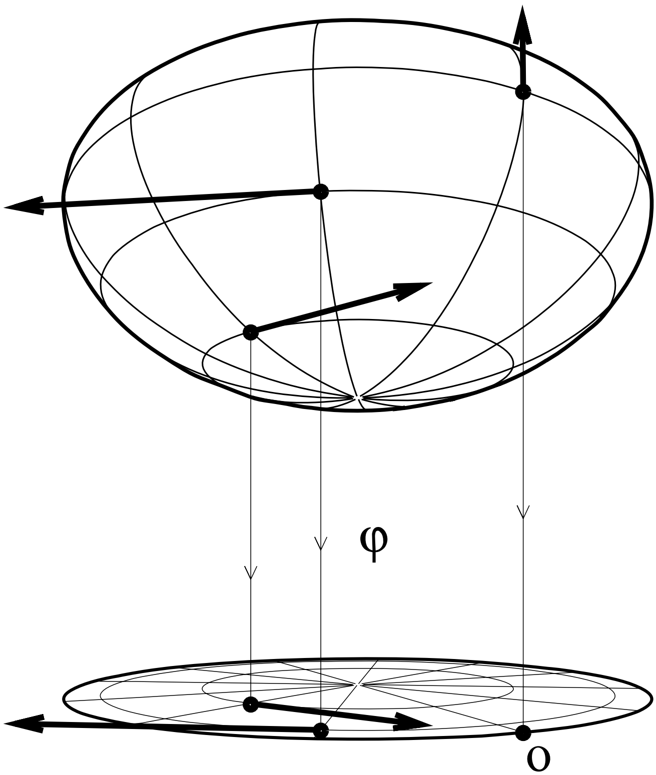


Figure 9.1. Mapping tangent vectors.

In particular, $d_z\varphi$ is an isomorphism if φ is a diffeomorphism.

We can now return to the discussion of Section 4.10 and prove the following result:

9.15 Proposition. *The algebras $C^\infty(M)$ and $C^\infty(N)$ are not isomorphic if $\dim M \neq \dim N$. In particular, the algebras $C^\infty(\mathbb{R}^n)$ and $C^\infty(\mathbb{R}^m)$ are not isomorphic if $m \neq n$.*

◀ Indeed, let $\Phi\colon C^\infty(N) \to C^\infty(M)$ be an isomorphism. Then $\varphi = |\Phi|\colon M \to N$ is a diffeomorphism. As was observed earlier, the differential

$$D_z\varphi\colon T_zM \to T_{\varphi(z)}N$$

is an isomorphism for all $z \in M$ in this case. Therefore,

$$\dim M = \dim T_zM = \dim T_{\varphi(z)}N = \dim N. \quad ▶$$

9.16. Let us now describe $d_z\varphi$ in coordinates. As in Section 6.15, choose local charts $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_m)$ in M and N containing the points z and $\varphi(z)$, respectively. Let $\varphi^*(y_i) = \varphi_i(x_1, \ldots, x_n)$ be the functions describing the map φ in coordinates. Then, for $g \in C^\infty(N)$, we have

$$\begin{aligned}
\left[d_z(\varphi)\left(\frac{\partial}{\partial x_i}\Big|_z\right)\right](g) &= \left[\frac{\partial}{\partial x_i}\Big|_z \circ \varphi^*\right](g) \\
&= \frac{\partial g(\varphi_1(x_1,\dots,x_n),\dots,\varphi_m(x_1,\dots,x_n))}{\partial x_i}\Big|_z \\
&= \sum_{j=1}^m \frac{\partial g(\varphi_1(x_1,\dots,x_n),\dots))}{\partial y_j}\Big|_z \frac{\partial \varphi_j(x_1,\dots,x_n)}{\partial x_i}\Big|_z \\
&= \sum_{j=1}^m \frac{\partial \varphi_j(x_1,\dots,x_n)}{\partial x_i}\Big|_z \left[\varphi * \left(\frac{\partial g}{\partial y_j}\right)(z)\right] \\
&= \sum_{j=1}^m \frac{\partial \varphi_j(x_1,\dots,x_n)}{\partial x_i}\Big|_z \frac{\partial g}{\partial y_j}\varphi(z) = \sum_{j=1}^m \frac{\partial \varphi_j}{\partial x_i}(z)\frac{\partial g}{\partial y_j}(\varphi(z)) \\
&= \left[\sum_{j=1}^m \frac{\partial \varphi_j}{\partial x_i}(z)\frac{\partial}{\partial y_j}\Big|_{\varphi(z)}\right](g).
\end{aligned}$$

In other words,

$$d_z(\varphi)\left(\frac{\partial}{\partial x_i}\Big|_z\right) = \sum_{j=1}^m \frac{\partial \varphi_j}{\partial x_i}(z)\frac{\partial}{\partial y_j}\Big|_{\varphi(z)}; \tag{9.7}$$

i.e., the matrix of the linear map $d_z\varphi$ in the bases

$$\left\{\frac{\partial}{\partial x_i}\Big|_z\right\} \subset T_z(M), \quad \left\{\frac{\partial}{\partial y_j}\Big|_{\varphi(z)}\right\} \subset T_{\varphi(z)}(N),$$

respectively, is the Jacobi matrix $\left\|\partial\varphi_i/\partial x_j\right\|_z$ of φ at the point z. The subscript z indicates here that all the derivatives in this matrix are taken at the point z. Thus, the coordinate representation for the differential $d_z\varphi$ is of the form

$$\begin{pmatrix}\beta_1\\ \vdots\\ \vdots\\ \beta_m\end{pmatrix} = \begin{pmatrix}\dfrac{\partial\varphi_1}{\partial x_1} & \dots & \dfrac{\partial\varphi_1}{\partial x_n}\\ \vdots & \ddots & \vdots\\ \dfrac{\partial\varphi_m}{\partial x_1} & \dots & \dfrac{\partial\varphi_m}{\partial x_n}\end{pmatrix}_z \cdot \begin{pmatrix}\alpha_1\\ \vdots\\ \vdots\\ \alpha_n\end{pmatrix}, \tag{9.8}$$

where $(\alpha_1,\dots,\alpha_n)$ and $(\beta_1,\dots,\beta_m)$ are the coordinates of the vector $v \in T_z(M)$ and of its image $d_z\varphi(v)$ in the bases

$$\left\{\frac{\partial}{\partial x_i}\Big|_z\right\}, \quad \left\{\frac{\partial}{\partial y_j}\Big|_{\varphi(z)}\right\},$$

respectively.

9.17. Tangent manifolds. As we know from elementary mechanics, any particular state of a mechanical system S is determined by its position (configuration) and instantaneous velocity. If $M = M_S$ is the configuration space (see Section 1.1, 5.12) of this system, then, as follows from the arguments of Section 9.2, the notion of a tangent vector to the manifold M_S is identical to the concept of the system state. More precisely, if we consider a tangent vector $\xi \in T_z M$, then z is the position of the system, while ξ is its instantaneous velocity. Thus, the set of all system states (pay attention to Remark 9.4) is

$$TM \stackrel{\text{def}}{=} \bigcup_{z \in M} T_z M.$$

This set can be equipped with a smooth manifold structure in a natural way. The object obtained is called the *tangent manifold* of the manifold M. Since the system evolution is uniquely determined by its initial state, differential equations describing possible evolutions should be equations on the manifold TM.

Besides mechanics, tangent manifolds naturally arise in various branches of mathematics, and first of all in differential geometry.

9.18. To introduce a smooth manifold structure on TM, we shall need the following simple facts:

I. Any smooth map $\Phi\colon M \to N$ generates the map of sets

$$T\Phi\colon TM \to TN,$$

taking a tangent vector $\xi \in T_z M$ to $d_z\Phi(\xi) \in T_{\Phi(z)}N$. By (9.6),

$$T(\Phi \circ \Psi) = T\Phi \circ T\Psi.$$

II. If $W \subset \mathbb{R}^n$ is an open domain of an arithmetical space, then TW is naturally identified with the domain $W \times \mathbb{R}^n \subset \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$. Namely, if $z \in W$, $z = (z_1, \dots, z_n)$ and $\xi \in T_z W$, then

$$\xi \Longleftrightarrow (z_1, \dots, z_n, \alpha_1, \dots, \alpha_n) \quad \text{if} \quad \xi = \sum_{i=1}^{n} \alpha_i \frac{\partial}{\partial x_i}.$$

III. A natural map of sets

$$\pi_T = \pi_{TM}\colon TM \to M, \quad T_z M \ni \xi \mapsto z \in M,$$

is defined, taking any tangent vector to its point of application.

IV. If $U \subset M$ is an open subset, then

$$\pi_T^{-1}(U) = \bigcup_{z \in U} T_z M = \bigcup_{z \in U} T_z U = TU.$$

9.19. Let us now note that any chart (U, x) of the manifold M, by the above, generates the map

$$Tx\colon TU \to TW \subset \mathbb{R}^{2n}, \quad \text{where} \quad W = x(U).$$

This map is obviously a bijection. Using 9.18, IV, we can identify TU with $\pi_T^{-1}(U)$ and obtain a $2n$-dimensional chart $\big(\pi_T^{-1}(U), Tx\big)$ in TM. In other words, coordinate functions $\{x_i, q_j\}$ associated with Tx are such that x_i for $(z,\xi) \in TU$ is the ith coordinate of the point z, while q_j is the jth component of the coordinate presentation for the vector ξ in the basis $\partial/\partial x_i|_z$. Charts of this kind are called *special*. If the charts (U,x) and (U',y) are compatible on M, the corresponding special charts $\big(\pi_T^{-1}(U), Tx\big)$ and $\big(\pi_T^{-1}(U'), Ty\big)$ are also compatible. In fact, the analytical form for the coordinate change

$$(Ty)\circ(Tx)^{-1}\colon T(W) \to T(W'), \quad W' = y\,(U')\,,$$

is (see (9.12))

$$y_i = y_i(x), \quad \beta_j = \sum_{k=1}^{n} \alpha_k \frac{\partial y_j}{\partial x_k}(x),$$

and consequently the map under consideration has the Jacobi matrix

$$\begin{pmatrix} J & * \\ 0 & J \end{pmatrix},$$

where J is the Jacobi matrix for the coordinate change $y \circ x^{-1}$, while the asterisk denotes an $(n\times n)$–matrix. Hence, $(Ty)\circ(Tx)^{-1}$ is a diffeomorphism of open sets in $\mathbb{R}^{2n}$.

Let $A = \{(U_k, x_k)\}$ be an atlas on M. Then, by the above, $TA = \{(\pi_T^{-1}(U_k), Tx_k)\}$ is a *special* atlas on TM. If two atlases A_1 and A_2 are compatible on M, then the corresponding special atlas TA_1 is compatible with TA_2 as well. If A is a countable and Hausdorff atlas, then TA enjoys the same properties. For all these reasons, the smooth manifold structure on the set TM determined by the atlas TA does not depend on the choice of the atlas A on M. The set TM equipped with the described smooth structure is called the *tangent manifold* of the manifold M. Let us also note that the map

$$\pi_T\colon TM \to M$$

described in Section 9.18, III, is smooth. It is called the *tangent fiber bundle* of the manifold M. We use here the words *fiber bundle* for the first time. The exact definition will be given in Section 10.10; Chapters 10 and 11 are completely devoted to the study of this notion. In the case under consideration, these words mean that the tangent spaces T_zM (*fibers* of the projection π_T) are identical (i.e., diffeomorphic to each other) and "fiber" the tangent manifold TM. Moreover, these fibers are mutually isomorphic vector spaces. Fiber bundles of this type are called *vector bundles* and studied in detail in Chapter 11.

Exercise. Prove that the map $d\Phi\colon TM \to TN$, corresponding to $\Phi\colon M \to N$, is smooth.

9.20. Sticking to the observability principle, it would be much more attractive to associate with any (smooth) algebra A an algebra TA such that $|TA| = T|A|$. This can really be done, but a discussion of the corresponding constructions is beyond the framework of this book. That is why we have to confine ourselves to the example of the *cotangent manifold* considered in the next sections.

9.21. Besides the description of a mechanical system in terms of position–velocity, there exists another, often more convenient, description in terms of position–momentum. The fundamental relation

$$p = mv,$$

which ties the velocity and the momentum of a mass point, shows that momenta are linear functionals on the space of velocities. In other words, the momentum of a system S in a position $z \in M = M_S$ is a linear functional on the tangent space T_zM; i.e., it is an element of the dual space

$$T_z^*M \stackrel{\text{def}}{=} \operatorname{Hom}_{\mathbb{R}}(T_zM, \mathbb{R}).$$

The space T_z^*M is called the *cotangent space* to M at the point z, and its elements are called *tangent covectors* to the manifold M at the point z. So, the momenta of a mechanical system S are tangent covectors to the configuration manifold $M = M_S$.

These and many other considerations lead us to the notion of the cotangent manifold. To give a formal definition of the latter, let us consider the set

$$T^*M \stackrel{\text{def}}{=} \bigcup_{z \in M} T_z^*M$$

together with the natural projection

$$\pi_T^* = \pi_{T^*M} \colon T^*M \to M, \quad T_z^*M \ni \theta \mapsto z \in M.$$

9.22. A natural smooth manifold structure on T^*M can be defined using the scheme applied already to TM. It is extremely helpful here to understand tangent covectors as differentials of functions on M.

Namely, let $U \ni z$ be a neighborhood of a point $z \in M$ and let $f \in C^\infty(U)$. Let us define a function d_zf on T_zM by setting

$$d_zf(\xi) \stackrel{\text{def}}{=} \xi(f), \quad \xi \in T_zM.$$

Exercise. Prove the following statements:

1. If $f = \text{const}$, then $d_zf = 0$.
2. $d_z(fg) = f(z)d_zg + g(z)d_zf$.

By the definition of a linear space structure on T_zM (see Section 9.4), the function d_zf is linear. Therefore, d_zf is a tangent covector at the point

z, called the *differential of the function* f at z. If (U, x) is a chart containing the point z, then

$$d_z x_i\Big(\frac{\partial}{\partial x_j}\Big|_z\Big) = \frac{\partial x_i}{\partial x_j}(z) = \delta^i_j,$$

where δ^i_j is the Kronecker symbol. This shows that $(d_z x_1, \dots, d_z x_n)$ is the basis of the space $T^*_z M$ dual to the basis

$$\frac{\partial}{\partial x_1}\Big|_z, \dots, \frac{\partial}{\partial x_n}\Big|_z$$

in $T_z M$. Therefore, any covector $\theta \in T^*_z M$ is uniquely represented in the form

$$\theta = \sum_{i=1}^n p_i d_z x_i, \quad p_i \in \mathbb{R}.$$

It is useful to note that

$$p_i = \theta\Big(\frac{\partial}{\partial x_i}\Big|_z\Big).$$

In particular, if $\theta = d_z f$, then

$$\theta\Big(\frac{\partial}{\partial x_i}\Big|_z\Big) = \frac{\partial f}{\partial x_i}(z) \quad \text{and hence} \quad d_z f = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(z) d_z x_i.$$

This formula justifies the adopted terminology and shows that any covector θ can be represented in the form $\theta = d_z f$. Thus, tangent covectors at a given point are exhausted by differentials of functions at this point.

9.23. Any smooth map $\Phi\colon M \to N$ generates the linear map

$$d_z\Phi^*\colon T^*_{\Phi(z)}N \to T^*_z M$$

dual to the linear map

$$d_z\Phi\colon T_z M \to T_{\Phi(z)}N.$$

If $\xi \in T_z M$ and $g \in C^\infty(N)$, then, by definition,

$$\begin{aligned} d_z\Phi^*(d_{\Phi(z)}g)(\xi) &= d_{\Phi(z)}g(d_z\Phi(\xi)) \\ &= \big(d_z\Phi(\xi)\big)(g) = \xi\big(\Phi^*(g)\big) = \big(d_z\Phi^*(g)\big)(\xi). \end{aligned}$$

This means that

$$d_z\Phi^*(d_{\Phi(z)}g) = d_z\Phi^*(g).$$

Note also that, in the notation of Section 9.16, the matrix of the map $d_z\Phi^*$ in the bases $\{d_z x_i\}$ and $\{d_{\Phi(z)} y_j\}$ in $T^*_z M$ and $T^*_{\Phi(z)}N$, respectively, is the transposed Jacobi matrix $J_z = \|\partial y_i/\partial x_j(z)\|$.

9.24. Construction of special charts on T^*M is accomplished in a way similar to that used above for TM. Instead of properties I–IV from Section 9.18, the following facts should be used:

I. Any diffeomorphism $\Phi\colon M \to N$ generates the bijection

$$\Phi_*\colon T^*M \to T^*N, \quad T_z^*M \ni \theta \mapsto (d_z\Phi^*)^{-1}(\theta).$$

II. If $W \subset \mathbb{R}^n$ is an open domain, then

$$T^*W = W \times \mathbb{R}^n \subset \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n},$$

and the identification $T^*W = W \times \mathbb{R}^n$ follows the rule

$$T^*W \ni \theta \Longleftrightarrow (z_1, \ldots, z_n, p_1, \ldots, p_n),$$

where $z = (z_1, \ldots, z_n)$, $\theta = \sum_{i=1}^n p_i d_z x_i$ and $(x_1, \ldots, x_n)$ are the standard coordinates in $\mathbb{R}^n$.

Now let (U, x) be a chart on M and $W = x(U) \subset \mathbb{R}^n$. By duality, the natural identification of T_zU with T_zM allows the identification of T_z^*U with T_z^*M. In turn, this leads to the identification of T^*U with $\pi_{T^*}^{-1}(U)$. Now using II, we obtain a *special chart* $(\pi_{T^*}^{-1}(U), T^*x)$ on T^*M. Here T^*x denotes the system of coordinate functions $\{x_i, p_j\}$, where x_i for $(z, \theta) \in T^*U$ is the ith coordinate of the point z, while p_j is the jth component of the decomposition of the vector θ in the basis dx_i.

If $A = \{(U_k, x_k)\}$ is an atlas on M, then $T^*A \stackrel{\text{def}}{=} \{(\pi_{T^*}^{-1}(U_k), T^*x_k)\}$ is an atlas (of dimension $2n$) on T^*M. If two atlases A_1 and A_2 are compatible on M, then so are the atlases T^*A_1 and T^*A_2. For this reason, the atlas T^*A determines a smooth manifold structure on T^*M independent of the choice of a particular atlas A.

The $2n$-dimensional manifold T^*M thus obtained is called the *cotangent manifold* of the manifold M. Note also that the map

$$\pi_{T^*}: T^*M \to M$$

is smooth. It is called the *cotangent bundle* of M.

9.25. Any function $f \in C^\infty(M)$ generates the smooth map

$$s_{df}\colon M \to T^*M, \quad s_{df}(z) = d_z(f).$$

This map is characterized by the fact that any point $z \in M$ is taken to a point in the fiber of the cotangent bundle $\pi_{T^*}^{-1}(z) = T_z^*M$ over z. Such maps are called *sections*. This notion will be discussed in more detail in subsequent chapters; see Section 10.12 and 11.7.

Exercise. Describe s_{df} in special local coordinates.

9.26. Any map $\Phi\colon M \to N$ generates a family $d_z\Phi^*: T_{\Phi(z)}^*N \to T_z^*M$ of maps taking cotangent spaces of points in M to those in N. Unfortunately, it does not allow one, in general, to construct a map of cotangent manifolds $T^*N \to T^*M$ that reduces to $d_z\Phi^*$ when restricted to $T_{\Phi(z)}^*M$.

Exercise. Show that such a map exists if and only if Φ is a bijection; it is smooth only if Φ is a diffeomorphism.

If $\dim M = \dim N = n$ and the map $\Phi\colon M \to N$ is *regular* at all points of the manifold, i.e., all differentials are isomorphisms, then one can define a smooth map $T^*M \to T^*N$ *covering* Φ and reducing to $(d_z\Phi^*)^{-1}$ when restricted to T_z^*M. The map Φ_* considered in Section 9.24 is its particular case.

9.27. It is remarkable that the cotangent space T_z^*M can be defined in a purely algebraic way, as it is done in algebraic geometry. Let μ_z be the ideal consisting of all functions vanishing at the point z:

$$\mu_z \stackrel{\text{def}}{=} \{f \in C^\infty(M) \mid f(z) = 0\}.$$

Proposition. *There exists a natural isomorphism between T_z^*M and the quotient μ_z/μ_z^2.*

◀ Consider the quotient algebra

$$J_z^1 M \stackrel{\text{def}}{=} C^\infty(M)/\mu_z^2$$

and the map

$$\bar{d}_z\colon J_z^1M \to T_z^*M, \quad \bar{d}_z([f]) = d_z f,$$

where $f \in C^\infty(M)$ and $[f] = f \bmod \mu_z^2$. Since $d_z f = 0$ for $z \in \mu_z^2$ (see the exercise from Section 9.22), this map is well defined. Obviously, it is $\mathbb{R}$-linear and surjective (see Section 9.22) because any covector θ can be presented as the differential of some function, $\theta = d_z f$.

The decomposition of the algebra $C^\infty(M)$ into the direct sum of linear spaces

$$C^\infty(M) = \mathbb{R} \oplus \mu_z, \quad f = f(z) + (f - f(z)),$$

gives the direct sum decomposition

$$J_z^1 M = \mathbb{R} \oplus \mu_z/\mu_z^2. \tag{9.9}$$

By the exercise from Section 9.22, the map d_z annihilates the first summand. On the other hand, Hadamard's lemma, lemma 2.8, shows that $f \in \mu_z$ and $d_z f = 0$ imply $f \in \mu_z^2$. Therefore, the restriction $\bar{d}_z$ to μ_z/μ_z^2 is an isomorphism. ▶

Corollary. $\dim J_z^1 M = n + 1$.

◀ Indeed, the above proposition allows us to rewrite equality (9.9) in the form

$$J_z^1 M = \mathbb{R} \oplus T_z^* M. \quad ▶$$

9.28. The quotient algebra $J^1_z M$ was useful in the proof of Proposition 9.27; as we shall see later, this algebra is one of the most important constructions of the differential calculus. It is called the *algebra of first-order jets* (or of 1-*jets*) at the point $z \in M$ for the algebra of smooth functions $C^\infty(M)$.

The union

$$J^1 M = \bigcup_{z \in M} J^1_z M$$

can be endowed with a natural smooth manifold structure in the same way as was done above for T^*M.

Exercise. Check the corresponding constructions in detail. Describe the special coordinates in $J^1 M$.

The manifold $J^1 M$ is called the *manifold of first-order jets* for the manifold M.

Similar to tangent and cotangent manifolds, $J^1 M$ is fibered over M by means of the natural map

$$\pi_{J^1} \colon J^1 M \to M, \quad \pi_{J^1}([f]^1_z) = z,$$

where $[f]^1_z$ denotes the image of the function f under the quotient map $C^\infty(M) \to J^1_z M$. Similar to π_T and π_{T^*} (see Sections 9.19 and 9.24, respectively), the map π_{J^1} is also a vector bundle over M. Its fibers are of dimension $(n+1)$. For any function $f \in C^\infty(M)$, one can consider the smooth map

$$s_{j_1 f} \colon M \to J^1 M, \quad s_{j_1 f}(z) = [f]^1_z,$$

which is a section of the bundle π_{J^1}.

The map

$$\pi_{J^1,T^*} \colon J^1 M \to T^*M, \quad \pi_{J^1,T^*}([f]^1_z) = d_z f,$$

which relates the manifold of 1-jets in the natural way to the cotangent manifold, is a one-dimensional vector bundle over T^*M.

A remarkable feature of the manifold $J^1 M$ is that it allows us to construct an exhaustive theory of first-order partial differential equations in one unknown. In this theory, differential equations are interpreted as submanifolds in $J^1 M$.

The tangent vector theorem allowed us to make the first step in understanding the differential calculus as a part of commutative algebra. The next step is to define tangent vectors to the spectrum of an arbitrary commutative algebra.

Let A be an arbitrary unital commutative K-algebra. Denote by $|A|$ its K-spectrum, i.e., the set of all (unital) K-homomorphisms from A to K.

9.29. Definition. A map $\xi \colon A \to K$ is called a *tangent vector*, or a *derivation* at a point $h \in |A|$, if it

(i) is K-linear, i.e.,

$$\xi\left(\sum_{j=1}^{k}\lambda_j f_j\right)=\sum_{j=1}^{k}\lambda_j\xi(f_j),\qquad \lambda_j\in K,\quad f_j\in A;$$

(ii) satisfies the Leibniz rule at h, i.e.,

$$\xi(fg)=f(h)\xi(f)+g(h)\xi(f),\qquad f,g\in A.$$

This definition, in the case $K=\mathbb{R}$ and $A=C^\infty(M)$, coincides with the definition of tangent vectors to the manifold M $(=|A|)$ at a point $z\in M$. To understand this fact, it suffices to recall the identification $M=|C^\infty(M)|$ and to treat z as the homomorphism $h_z\colon f\mapsto f(z)$, $f\in C^\infty(M)$. The set of all tangent vectors at a given point is naturally endowed with a K-module structure (or that of a vector space over K when K is a field):

1. $(\xi_1+\xi_2)(a)\overset{\text{def}}{=}\xi_1(a)+\xi_2(a)$, $a\in A$;
2. $(k\xi)(a)\overset{\text{def}}{=}k\xi(a)$, $k\in K, a\in A$.

Let us denote this K-module by T_hA. If $K=\mathbb{R}$ and $A=C^\infty(M)$, then, under the above identification of points $z\in M$ with K-homomorphisms h_z, the space T_zM will coincide with $T_{h_z}A$.

Remark. In algebraic geometry, one considers various spectra of algebras, maximal, primitive, etc. Treating the symbol h in the previous definition in an adequate way, the reader will easily define tangent vectors for points of all these spectra.

9.30. Cotangent spaces of commutative algebra spectra. Proposition 9.27, revealing a purely algebraic nature of cotangent bundles, shows how to define the cotangent space of the spectrum $|A|$ for an arbitrary commutative K-algebra A at some point $h\in|A|$. Namely, set

$$T_h^*A\overset{\text{def}}{=}\mu_h/\mu_h^2,\tag{9.10}$$

where μ_h is the kernel of a K-algebra homomorphism $h\colon A\to K$. By definition, T_h^*A is a K-module. Its role is illustrated by the following proposition:

Proposition. *For any K-algebra A, the natural surjection of K-modules*

$$\nu_h\colon \operatorname{Hom}_K(T_h^*A,K)\to T_hA\tag{9.11}$$

is defined. If K is a field, then ν_h is an isomorphism.

◀ Let us first note that any K-linear map $\varphi\colon T_h^*A\to K$ determines the tangent vector

$$\xi_\varphi\in T_hA,\quad \xi_\varphi(a)=\varphi([a-h(a)\cdot 1_A]),$$

where $[b] = b \bmod \mu_h^2$ (check it). The correspondence $\varphi \mapsto \xi_\varphi$ in an obvious way determines the K-module homomorphism

$$\nu_h \colon \operatorname{Hom}_K(T_h^* A, K) \to T_h A.$$

The map ν_h is a surjection. In fact, let $\xi \in T_h A$. Consider the K-linear map

$$\varphi_\xi \colon T_h^* A \to K, \qquad \varphi_\xi([a]) = \xi(a), \quad a \in \mu_h.$$

The Leibniz rule implies $\xi(\mu_h^2) \subset \mu_h$, and thus the map φ_ξ is well defined. Obviously, $\nu_h(\varphi_\xi) = \xi$.

If K is a field, then ν_h is also an injection. In fact, let now $a, b \in \mu_h$ and $[a] \neq [b]$. Since K is a field, one can always find a linear function φ defined on the vector K-space $T_h^* A$ and satisfying $\varphi([a]) \neq \varphi([b])$, i.e., $\xi_\varphi(a) \neq \xi_\varphi(b)$. ▶

9.31. To find an algebraic counterpart for the concept of the differential of a smooth map, let us note that to any (unital) K-algebra homomorphism $F \colon A_1 \to A_2$ there corresponds a map of K-spectra, namely

$$|F| \colon |A_2| \to |A_1|, \quad |A_2| \ni h \mapsto h \circ F \in |A_1|.$$

If, in addition, $\xi \in T_h(A_1)$, then the map

$$d_h|F|(\xi) \stackrel{\text{def}}{=} \xi \circ F \colon A_1 \to K$$

is a tangent vector to the space $|A_1|$ at the point $|F|(h) = h \circ F$. Thus we obtain the K-linear map

$$d_h|F| \colon T_h(A_1) \to T_{h \circ F}(A_2)$$

(prove this fact). If $F = \varphi^*$, where $\varphi \colon M_2 \to M_1$ is a smooth map and $A_i = C^\infty(M_i)$, $i = 1, 2$, then the differentials $d_h\varphi$ and $d_h|F|$ coincide.

9.32 Exercises. 1. Prove that $T_{\mathrm{id}_K} K = 0$, where $\mathrm{id}_K \colon K \to K$, is the only point of the K-spectrum for K.

2. Let $i \colon K \to A$, $k \mapsto k \cdot 1_A$, be the canonical embedding and $\xi \in T_h A$. Prove that $\xi\big|_{\operatorname{Im} i} = 0$; i.e., any derivation at a given point takes constants to zero.

3. Let $F \colon A_1 \to A_2$ be a K-algebra epimorphism. Prove that the map $d_h|F|$ is a monomorphism for any point $h \in |A_2|$.

4. Let $C^0(M)$ be the algebra of all continuous functions on M. Prove that $T_z(C^0(M)) = 0$ for any point $z \in M$.

9.33. The advantages of the algebraic approach to the differential calculus can already be shown at this point, though so far we have succeeded only in giving the definition of tangent vectors. For example, we can define tangent spaces to manifolds with singularities and even more—to arbitrary smooth sets—and obtain the simplest invariants of singular points. Some examples will be given below. The following statement, whose proof is literally carried over from Section 9.9, will be quite useful in analyzng these examples. Below we use the notation of Sections 3.23–3.25.

Proposition. *Suppose $\mathcal{F}$ is an arbitrary geometrical $\mathbb{R}$-algebra and $U \subset |\mathcal{F}|$ is an open subset. Then the restriction homomorphism*

$$\rho_U : \mathcal{F} \to \mathcal{F}\big|_U$$

induces the isomorphism

$$d_h(\rho_U)\colon T_h(\mathcal{F}\big|_U) \to T_{\rho_U \circ h}(\mathcal{F}), \quad h \in |\mathcal{F}\big|_U|. \quad \blacktriangleright$$

Of course, a similar statement is valid for arbitrary K-algebras. Try to prove this fact yourself.

9.34 Exercises. 1. Let $W = \{(x, y) \in \mathbb{R}^2 \mid y^2 = x^3\}$ be the semicubical parabola. Show that $T_z W$ is two-dimensional for $z = (0, 0)$ and one-dimensional otherwise.
An obvious consequence of this fact is that the algebra

$$C^\infty(W) = C^\infty\left(\mathbb{R}^2\right) / \left(y^2 - x^3\right) C^\infty\left(\mathbb{R}^2\right)$$

is not smooth (cf. 4.16).

2. Give an example of a smooth set whose tangent spaces are all 1-dimensional except for one point in which it is 3-dimensional.

9.35. Example. Suppose $\mathbf{K}$ is the coordinate cross on the plane (see Section 7.14, 1), $\mathbf{K} = \{(x, y) \subset \mathbb{R}^2 \mid xy = 0\}$, and $\mathcal{F} = C^\infty(\mathbf{K}) = C^\infty\left(\mathbb{R}^2\right)\big|_{\mathbf{K}}$ is the algebra of smooth functions on $\mathbf{K}$. Let us describe $T_z C^\infty(\mathbf{K})$ for all points $z \in \mathbf{K}$. Elements of the algebra $C^\infty(\mathbf{K})$ may be understood as pairs $(f(x), g(y))$ of smooth functions on the line satisfying $f(0) = g(0)$. In other words,

$$C^\infty(\mathbf{K}) = \{(f(x), g(y)) \mid f(0) = g(0)\}.$$

Note that for any nonsingular point on the cross, i.e., for a point of the form $(x, 0)$, with $x \neq 0$, or $(0, y)$, with $y \neq 0$, the tangent space is one-dimensional. Let us consider, say, $z = (x, 0)$, $x \neq 0$, and $U = \{(x', 0) \mid xx' > 0\}$. Then U is open in $\mathbf{K} = |\mathcal{F}|$, and consequently, by Proposition 9.33, we have $T_z\mathcal{F} = T_z\mathcal{F}\big|_U$. It remains to note that $\mathcal{F}\big|_U = C^\infty(\mathbb{R}^1_+) = C^\infty(\mathbb{R}^1)$. For a basis vector in the space $T_{(x,0)}$ one can take the operator

$$\frac{d}{dx}\bigg|_{(x,0)}, \qquad \frac{d}{dx}\bigg|_{(x,0)}(f(x), g(y)) = \frac{df}{dx}(x),$$

while for tangent spaces of the form $T_{(0,y)}$ the operator

$$\frac{d}{dy}\bigg|_{(0,y)}, \qquad \frac{d}{dy}\bigg|_{(0,y)}(f(x), g(y)) = \frac{dg}{dy}(y)$$

can be taken.

Now consider the point $(0,0)$. Obviously, the operators

$$\left.\frac{d}{dx}\right|_{(0,0)} \quad \text{and} \quad \left.\frac{d}{dy}\right|_{(0,0)}$$

will be tangent vectors at this point. They are linearly independent, and hence the space $T_{(0,0)}$ is at least two-dimensional. Since the natural restriction map $\tau\colon C^\infty(\mathbb{R}^2) \to C^\infty(\mathbf{K})$ is an epimorphism, then by Problem 3 of Section 9.32, the map

$$d_{(0,0)}\tau\colon T_{(0,0)}(C^\infty(\mathbf{K})) \to T_{(0,0)}(C^\infty(\mathbb{R}^2)) = \mathbb{R}^2$$

has trivial kernel. Therefore, the tangent space $T_{(0,0)}(C^\infty(\mathbf{K}))$ is isomorphic to $\mathbb{R}^2$.

Thus, the property of the point $(0,0) \in \mathbf{K}$ to be singular manifests itself, in particular, in the fact that the dimension of the tangent space at this point is greater than for “normal” ones.

Let us stress that the standard coordinate approach does not allow one to define tangent vectors at the point $(0,0)$. But if one tried to understand a tangent vector as an equivalence class of curves, then there would be no linear space structure in the set of such tangent vectors to $\mathbf{K}$ at this point.

9.36 Exercise. Let $W \subset \mathbb{R}^n$ be a smooth set. Recall that by definition, $C^\infty(W) = \{f\big|_W \mid f \in C^\infty(\mathbb{R}^n)\}$. Describe T_zW for all points $z \in W$ in the following cases (see Exercise 7.14):

1. $W \subset \mathbb{R}^2$ is given by the equation $y = \sqrt{|x|}$.
2. W is the triangle in $\mathbb{R}^2$: $W = W_1 \cup W_2 \cup W_3$, where

$$\begin{aligned} W_1 &= \{(x,y) \mid 0 \leqslant y \leqslant 1,\ x = 0\},\\ W_2 &= \{(x,y) \mid 0 \leqslant x \leqslant 1,\ y = 0\},\\ W_3 &= \{(x,y) \mid x + y = 1,\ x, y \geqslant 0\}. \end{aligned}$$

3. W is the triangle from the previous problem together with the interior domain: $W = \{(x,y) \mid x + y \leqslant 1,\ x, y \geqslant 0\}$.
4. W is the cone $x^2 + y^2 = z^2$ in $\mathbb{R}^3$.
5. $W = W_i$, $i = 1, 2, 3$, is one of the one-dimensional homeomorphic polyhedra shown in Figure 7.1. Explain why the algebras $C^\infty(W_i)$, $i = 1, 2, 3$, are pairwise nonisomorphic.
6. $W \subset \mathbb{R}^2$ is the closure of the graph of the function $y = \sin 1/x$.

9.37 Exercise. Let $\dim T_zW = 0$, where W is a smooth set. Prove that z is an isolated point of W. This is no longer true for arbitrary algebras. Give an example of an algebra $\mathcal{F}$ such that $|\mathcal{F}| \simeq \mathbb{R}$ but $\dim T_z\mathcal{F} = 0$ for all $z \in |\mathcal{F}|$.

Can you construct another algebra $\mathcal{F}$ with $|\mathcal{F}| \simeq \mathbb{R}$ and $\dim T_z\mathcal{F} = 2$ for some $z \in |\mathcal{F}|$? And the same for all $z \in |\mathcal{F}|$?

9.38. More complicated objects of the differential calculus, which can be constructed using tangent vectors, are vector fields. Vivid geometrical images of vector fields are provided by numerous fields of forces in mechanics and physics, velocity fields of continuous media, etc. A "field" of arrows on a meteorological map may be considered as the velocity field of moving air masses.

Let us try to formalize this notion in the same spirit as was done for tangent vectors. The first step in this direction is obvious: A vector field on a manifold M is a family of tangent vectors $\{X_z\}_{z\in M}$, where $X_z \in T_zM$. In terms of the algebra of observables $C^\infty(M)$, this means that we deal with the family of operators

$$X_z\colon C^\infty(M) \to \mathbb{R}, \quad z \in M.$$

In particular, such a family of operators assigns to each function $f \in C^\infty(M)$ the set of numbers $\{X_z(f),\ z \in M\}$, which a physicist would call a *scalar field*, while a mathematician would just call it a function on M. Denoting this function by $X(f)$, we obtain by definition

$$X(f)(z) \overset{\text{def}}{=} X_z(f), \quad z \in M.$$

In this notation it becomes clear that the words *vector field* must be understood as a sort of operation on the algebra $C^\infty(M)$:

$$X\colon C^\infty(M) \to \,?\,,$$

where the question mark means some set of functions on M. A natural way to formalize the idea of smoothness of a vector field X is to set $? = C^\infty(M)$: $X(f) \in C^\infty(M)$ for any function $f \in C^\infty(M)$. Thus, a smooth vector field X on M is an operator acting on $C^\infty(M)$:

$$X\colon C^\infty(M) \to C^\infty(M).$$

By the $\mathbb{R}$-linearity of the maps X_z, of which the operator X "consists," this operator is also $\mathbb{R}$-linear. Moreover, the Leibniz rule for a tangent vector X_z at a point z implies

$$\begin{aligned} X(fg)(z) &= X_z(fg) = X_z(f)g(z) + f(z)X_z(g) \\ &= \big(X(f)(z)\big)\,g(z) + f(z)\big(X(g)(z)\big) = [X(f)g + fX(g)](z); \end{aligned}$$

i.e., the operator X satisfies the Leibniz rule

$$X(fg) = X(f)g + fX(g), \quad f, g \in C^\infty(M). \tag{9.12}$$

The above said motivates the following definition:

Definition. An $\mathbb{R}$-linear operator $X\colon C^\infty(M) \to C^\infty(M)$ satisfying the Leibniz rule (9.12) is called a *smooth vector field* on the manifold M.

Everywhere below the word "smooth" is omitted, since we shall deal with smooth vector fields only.

9.39. The above definition of a vector field was formulated in terms of the base algebra $C^\infty(M)$ and completely satisfies the principle of observability. Moreover, we can now a posteriori justify the use of the words *vector field* in this definition: We can associate with any vector field X the family of tangent vectors $\{X_z \in T_zM\}_{z\in M}$. Namely, setting

$$X_z(f) = X(f)(z), \quad z \in M, \tag{9.13}$$

we easily see that the maps $X_z\colon C^\infty(M) \to \mathbb{R}$ thus defined are $\mathbb{R}$-linear and satisfy the Leibniz rule at any point z, i.e., they are tangent vectors at z.

Exercise. Prove this fact.

9.40 Proposition. (Locality of vector fields.) *Let X be a vector field on M. If functions $f, g \in C^\infty(M)$ coincide on an open set $U \subset M$, then the functions $X(f), X(g)$ also coincide on U.*

◀ In fact, by Lemma 9.8 one has $X_z(f) = X_z(g)$ for all $z \in U$. Hence, $X(f)(z) = X(g)(z)$ for all $z \in U$. ▶

The interpretation of a vector field as a family of tangent vectors allows one to consider the section of the tangent bundle

$$s_X\colon M \to TM, \quad z \mapsto X_z \in T_zM \subset TM,$$

related to this field.

Exercise. Prove that s_X is a smooth section, and vice versa, any (smooth) section of the tangent bundle is of the form s_X. Therefore, vector fields on M can be understood as sections of the tangent bundle.

9.41. Equality (9.13) shows that the family of tangent vectors $\{X_z\}_{z\in M}$ generated by X determines this vector field uniquely. Using this fact, we can easily understand how vector fields are described in terms of local coordinates. In fact, if (U, x) is a chart on M and $z \in U$, then X_z, as a tangent vector at the point $z \in U$, is presented in the form

$$X_z = \sum_{i=1}^n \alpha_i(z) \frac{\partial}{\partial x_i}\Big|_z. \tag{9.14}$$

The notation $\alpha_i(z)$ underlines the fact that the coordinates of the vector X_z depend on a point $z \in U$; i.e., they are functions on U. By (9.13) and (9.14), we have

$$X(f)(z) = X_z(f) = \sum_{i=1}^n \alpha_i(z)\frac{\partial f}{\partial x_i}(z) = \left(\Big(\sum_{i=1}^n \alpha_i \frac{\partial}{\partial x_i}\Big)(f)\right)(z),$$

and consequently

$$X(f) = \sum_{i=1}^n \alpha_i \frac{\partial}{\partial x_i}(f).$$

Therefore,

$$X = \sum_{i=1}^{n} \alpha_i \frac{\partial}{\partial x_i}.$$

Note that all functions α_i belong to $C^\infty(U)$. In fact, let $z \in U$. Consider a function $\tilde{x}_i \in C^\infty(M)$ coinciding with x_i in a neighborhood V of the point z. By definition, $X(\tilde{x}_i) \in C^\infty(M)$. Further, $X(\tilde{x}_i)\big|_V = \alpha_i$, since $\tilde{x}_i\big|_V = x_i\big|_V$, and consequently $\alpha_i \in C^\infty(V)$. Since $z \in U$ is an arbitrary point, we obtain that α_i is a smooth function on U.

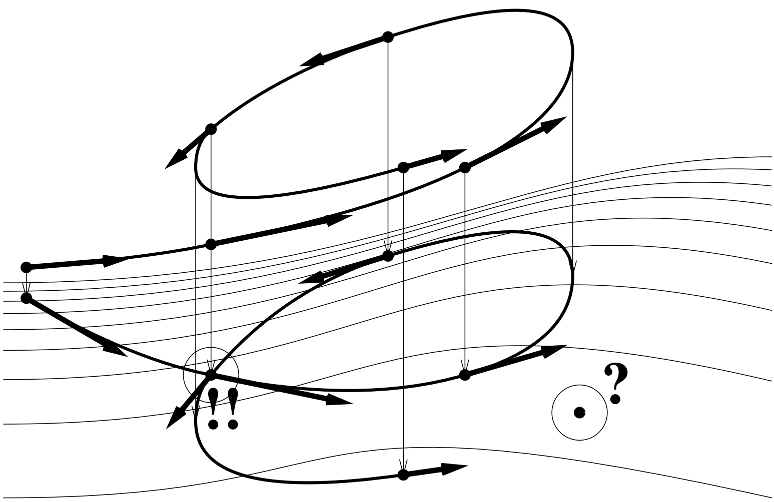


Figure 9.2. Trying to map a vector field.

9.42. Transformation of vector fields. Let X be a vector field on M, and $\varphi\colon M \to N$ a smooth map. The differential $d_z\varphi$ takes any vector X_z to the tangent vector $Y_{\varphi(z)} = d_z\varphi(X_z) \in T_{\varphi(z)}N$. In general, the family of tangent vectors $\{Y_{\varphi(z)}\}_{z\in M}$ does not constitute a vector field on N. In fact, if $u \in N \setminus \varphi(M)$, then the vector Y_u is undefined, while a point $u \in \varphi(M)$ may have several inverse images, and thus the vector Y_u may be defined ambiguously (see Figure 9.2).

So, as a rule, there are no maps of vector fields corresponding to maps of manifolds. But diffeomorphisms are exceptions from the general rule, and if φ is a diffeomorphism, then its action on a vector field X can be defined by the formula

$$Y = (\varphi^{-1})^* \circ X \circ \varphi^*.$$

Exercise. 1. Prove that Y is really a vector field.

2. Prove that $Y_{\varphi(z)} = d_z\varphi(X_z)$.

Below (see Section 9.47) it will be shown that the image of a vector field may be defined in a reasonable way, provided that the notion of the vector field can be adequately generalized.

9.43. The definition of a vector field given above is a particular case of the general algebraic notion of derivation, which is as follows. Let A be a commutative K-algebra.

Definition. A K-linear map $\Delta\colon A \to A$ is called a *derivation* of the algebra A if it satisfies the Leibniz rule

$$\Delta(ab) = a\Delta(b) + b\Delta(a) \quad \forall\, a, b \in A.$$

Let us denote the set of all derivations of A by $D(A)$. Let $\Delta, \nabla \in D(A)$ and $a \in A$. Then obviously, $\Delta + \nabla \in D(A)$ and $a\Delta \in D(A)$. These operations endow $D(A)$ with a natural A-module structure.

Any derivation of the K-algebra A can be understood as a vector field on $|A|$. To see this, it suffices to carry over formula (9.13) to the algebraic setting. Let $\Delta \in D(A)$ and $h \in |A|$. Put

$$\Delta_h = h \circ \Delta\colon A \to K.$$

Then, obviously, the operator Δ_h, being the composition of two K-linear operators, is also K-linear, and

$$\begin{aligned}\Delta_h(ab) &= h(\Delta(ab)) = h(\Delta(a)b + a\Delta(b)) \\ &= h(\Delta(a))h(b) + h(a)h(\Delta(b)) = \Delta_h(a)h(b) + h(a)\Delta_h(b).\end{aligned}$$

Thus $\Delta_h \in T_h(A)$. In what follows, for brevity we shall write $D(M)$ instead of $D(C^\infty(M))$.

Let us note that the definition of a tangent vector Δ_h written in the form

$$\Delta_h(f) = h(\Delta(f)), \quad f \in A,$$

is identical to (9.13) if $K = \mathbb{R}$, $A = C^\infty(M)$, while $h = h_z \in M = |A|$, is, as usual, understood as a homomorphism taking f to $f(z)$. If the algebra A is geometric, then the system $\{\Delta_h\}_{h\in|A|}$ of vectors tangent to $|A|$ determines the "vector field" Δ uniquely.

9.44. Just as in the case of tangent vectors (see Sections 9.33–9.36), one can construct the theory of vector fields for geometrical objects of a much more general nature than smooth manifolds. For example, using Section 9.33, it is possible to obtain a theory of vector fields on arbitrary closed subsets of smooth manifolds, just as is done for the manifolds themselves. In the examples below we use the notation introduced for smooth manifolds. The locality of vector fields (Proposition 9.40) is valid in this more general situation and is proved in the same way.

9.45. Example. Let us describe vector fields on the cross **K** using the notation of Section 9.35, where we studied tangent vectors. By A_x and A_y we denote the algebras of smooth functions on the line with fixed coordinate functions x and y, respectively. The natural embeddings

$$i_x\colon A_x \to C^\infty(\mathbf{K}), \qquad f(x) \mapsto (f(x), f(0)),$$
$$i_y\colon A_y \to C^\infty(\mathbf{K}), \qquad g(y) \mapsto (g(0), g(y)),$$

are defined together with the projections

$$\pi_x\colon C^\infty(\mathbf{K}) \to A_x, \qquad (f(x), g(y)) \mapsto f(x),$$
$$\pi_y\colon C^\infty(\mathbf{K}) \to A_y, \qquad (f(x), g(y)) \mapsto g(y).$$

Obviously, $\pi_x \circ i_x = \mathrm{id}$ and $\pi_y \circ i_y = \mathrm{id}$. Therefore, if $\Delta \in D(\mathbf{K})$, then $\Delta^x = \pi_x \circ \Delta \circ i_x \in D(A_x)$ and $\Delta^y = \pi_y \circ \Delta \circ i_y \in D(A_y)$.

Let us show that

$$\Delta(f(x), g(y)) = (\Delta^x(f), \Delta^y(g)) \tag{9.15}$$

and the fields Δ^x, Δ^y vanish at the point 0.

◀ Indeed, let

$$\varphi = i_x(f) = (f(x), c) \in C^\infty(\mathbf{K}),\ \ c = f(0).$$

Consider the point $z = (0, y) \in \mathbf{K},\ y \neq 0$. Then a sufficiently small neighborhood U of the point z is an interval and $\varphi|_U \equiv c$. By the locality of tangent vectors, one has $\Delta_z(\varphi) = 0$. In other words, $\Delta(\varphi)(z) = 0$ for all points of the y-axis except for the point $(0, 0)$. By continuity, $\Delta(\varphi)$ is zero identically on the whole axis. Therefore, $i_x(\pi_x(\Delta(\varphi))) = \Delta(\varphi)$, or $i_x(\Delta^x(f)) = \Delta(i_x(f))$. The last equality means that

$$i_x \circ \Delta^x = \Delta \circ i_x.$$

In addition, $\Delta^x(f)(0) = 0$, since $\Delta(\varphi)(0, 0) = 0$. Consequently, the vector field $\Delta^x \in D(A_x)$ vanishes at the point $(0, 0)$. In a similar way,

$$i_y \circ \Delta^y = \Delta \circ i_y,$$

and the vector field $\Delta^y \in D(A_y)$ also vanishes at $(0, 0)$.

Note now that

$$(f(x), g(y)) = i_x(f) + i_y(g) - (c, c), \quad c = f(0) = g(0),$$

and $\Delta\big((c, c)\big) = 0$, since functions of the form (c, c) are constants in the algebra $C^\infty(\mathbf{K})$. From this we eventually obtain that

$$\begin{aligned}\Delta\big((f(x), g(y))\big) &= \Delta(i_x(f)) + \Delta(i_y(g))\\ &= i_x(\Delta^x(f)) + i_y(\Delta^y(g)) = \big(\Delta^x(f), \Delta^y(g)\big),\end{aligned}$$

where the last equality is a consequence of

$$\Delta^x(f)(0) = \Delta^y(g)(0) = 0. \quad ▶$$

Obviously, the inverse statement is also valid: Any pair of vector fields $\Delta^x \in D(A_x)$, $\Delta^y \in D(A_y)$ determines a vector field Δ on $\mathbf{K}$ by formula (9.15), provided that these fields vanish at the point $(0, 0)$.

Exercise. Describe the A-modules of vector fields for the following algebras:

1. For all algebras from Exercise 9.36.
2. For the $\mathbb{R}$-algebra $C^m(\mathbb{R}^1)$, $m \geqslant 1$, of m-times differentiable functions on the line. (Hint: Start from the case $m = 0$.)
3. For the algebra $A = K[X]/X^{l+1}K[X]$ of truncated polynomials, where $K = \mathbb{R}$ or $\mathbb{Z}/m\mathbb{Z}$.
4. For a Boolean algebra, i.e., a commutative algebra over the field $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$ whose elements satisfy the relation $a^2 = a$.

9.46. Vector fields on submanifolds. In the above considerations we formalized two geometric images: a manifold at one point of which an arrow "grows" and a manifold on which arrows "grow" at all points. Clearly, an intermediate situation also exists: Arrows may grow at points of some submanifold (or, more generally, of a closed subset). Examples of this kind are velocity fields on a moving thread or on an oscillating membrane. Arguments similar to those that have led us to the definition of vector fields on a manifold M lead to the desired formalization in this case also.

Let $N \subset M$ be a submanifold of a manifold M.

Definition. An $\mathbb{R}$-linear map

$$X\colon C^\infty(M) \to C^\infty(N)$$

is said to be a *tangent* (to M) *vector field along* N if

$$X(fg) = X(f)g\big|_N + f\big|_N X(g). \tag{9.16}$$

The sum of two vector fields along N is obviously a vector field along N. One can also define multiplication of such fields by functions from $C^\infty(M)$:

$$(fX)(g) \stackrel{\text{def}}{=} f\big|_N X(g), \quad f, g \in C^\infty(M).$$

The set $D(M, N)$ of all tangent fields along N on the manifold M is a $C^\infty(M)$-module with respect to these operations. If $z \in N$, then the following analogue of formula (9.13),

$$X_z(f) = X(f)(z), \tag{9.17}$$

determines a tangent vector to the manifold M at a point z, corresponding to the vector field X along the submanifold N. Note that (9.17) makes no sense when $z \notin N$, and the definition above really introduces a field of vectors along the submanifold N.

9.47. Vector fields along maps. Relation (9.16) may be also rewritten in the form

$$X(fg) = X(f)i^*(g) + i^*(f)X(g),$$

where $i\colon N \hookrightarrow M$ denotes the inclusion map. After this, it becomes clear that it still makes sense, provided that i is an arbitrary map of N to M.

Definition. Let $\varphi\colon N \to M$ be a smooth map of manifolds. An $\mathbb{R}$-linear map

$$X\colon C^\infty(M) \to C^\infty(N)$$

is said to be a *tangent* (to M) *vector field along the map* φ if

$$X(fg) = X(f)\varphi^*(g) + \varphi^*(f)X(g) \quad \forall f, g \in C^\infty(M). \tag{9.18}$$

The set $D_\varphi(M)$ of all vector fields along a given map φ is a $C^\infty(M)$-module if the multiplication of a field $X \in D_\varphi(M)$ by $f \in C^\infty(M)$ is defined by the rule $(fX)(g) \overset{\text{def}}{=} \varphi^*(f)X(g)$. The $C^\infty(M)$-module $D_\varphi(M)$ also becomes a $C^\infty(N)$-module if the multiplication of its elements by elements of the algebra $C^\infty(N)$ is defined by

$$(fX)(g) \overset{\text{def}}{=} fX(g), \quad f \in C^\infty(N),\ g \in C^\infty(M),\ X \in D_\varphi(M).$$

Exercise. Check that formula (9.17), in the context under consideration, allows one to relate any point $z \in M$ with a tangent vector to M at the point $\varphi(z)$.

Any vector field $X \in D_\varphi(M)$ may be understood as an *infinitesimal deformation* of the map φ. In fact, since $X_z \in T_{\varphi(z)}M$, this vector can be naturally understood as an *infinitesimal shift* of the image of z under the map φ.

Vector fields along maps are also often called *relative* vector fields.

Example. Let $\varphi\colon N \to M$ be an arbitrary smooth map, $X \in D(N)$, and $Y \in D(M)$. Then $X \circ \varphi^*$ and $\varphi^* \circ Y$ are vector fields along the map φ. It is appropriate to interpret the relative field $X \circ \varphi^*$ as the *image of the field* X *under the map* φ (cf. Section 9.42).

9.48. An important example of a relative vector field is the *universal vector field* on M. It is constructed in the following way.

Consider the tangent bundle $\pi_T\colon TM \to M$ (see Section 9.17). Let ξ be a tangent vector to M, also understood as a point of the manifold TM. The universal vector field Z on M is defined as the following vector field along the map π_T:

$$Z(f)(\xi) = \xi(f), \quad f \in C^\infty(M).$$

9.49 Exercises. 1. Show that $Z(f) \in C^\infty(TM)$. (Hint: Use special local coordinates on TM.)

2. Check that Z is indeed a vector field along π_T.

3. Let $X \in D(M)$. Prove that

$$X = s_X^* \circ Z, \tag{9.19}$$

where $s_X\colon M \to TM$, $z \mapsto X_z \in T_zM$, is the section of the tangent bundle corresponding to the vector field X. Formula (9.19) explains why the field Z is universal: Any vector field M can be obtained from Z by using the appropriate section.

9.50. The only difference between the two definitions of vector fields discussed above is the interpretation of the Leibniz rule, i.e., the rule for differentiation of products. Let us rewrite it without specifying the range of the map X:

$$X(fg) = fX(g) + gX(f). \tag{9.20}$$

This formula makes sense when the product of the objects $X(g), X(f)$ and the functions f, g is defined, or in the other words, when the range of X is a module over its domain. For this reason, the following definition exhausts everything discussed above in relation to tangent vectors and vector fields.

Definition. Let A be a commutative K-algebra and let P be an arbitrary A-module. A K-linear map $\Delta\colon A \to P$ is called a *derivation* of the algebra A with values in P if it satisfies the Leibniz rule (9.20), i.e.,

$$\Delta(fg) = f\Delta(g) + g\Delta(f) \quad \forall\, f, g \in A.$$

The set $D(P)$ of all derivations of the algebra A with values in P carries a natural A-module structure.

9.51. Let $X \in D(P)$, and $h\colon P \to Q$ an A-module homomorphism. Then $h \circ X \in D(Q)$. (Check this.)

Moreover, the map

$$D(h)\colon D(P) \to D(Q), \quad D(P) \ni X \mapsto h \circ X \in D(Q),$$

is obviously an A-module homomorphism, and

$$D(\mathrm{id}_P) = \mathrm{id}_{D(P)},$$
$$D(h_1 \circ h_2) = D(h_1) \circ D(h_2).$$

This means that the correspondence $P \mapsto D(P)$ is a functor in the category of A-modules and their homomorphisms. This functor is one of the basic ones of the differential calculus. Some others will be discussed below. A complete and systematic description of the algebra of these functors together with specific features of its realization for concrete commutative algebras is the object of differential calculus in its modern meaning. Therefore, it may be asserted that the construction of the differential calculus, started by Newton and Leibniz, is not finished yet; it must be accomplished in the future.

9.52. Let us show how to specify Definition 9.50 in order to obtain the definitions of tangent vectors and various vector fields considered above. We shall also describe the procedures that assign to a vector field a tangent vector at a fixed point. In all our considerations here, we assume that $K = \mathbb{R}$.

I. The tangent vector to the manifold M at a point z:

$$A = C^\infty(M), \quad P = A/\mu_z = \mathbb{R},$$

where μ_z is an ideal of the point z.

II. A vector field on the manifold M:

$$A = P = C^\infty(M).$$

The tangent vector $X_z \in T_zM$ is assigned to a vector field $X \in D(M)$ and a point $z \in M$ in the following way:

$$D(A) \ni X \mapsto X_z = h \circ X \in D(A/\mu_z),$$

where $h\colon A \to A/\mu_z$ is the natural projection. In other words, $X_z = D(h)(X)$.

III. A vector field along a submanifold $N \subset M$:

$$A = C^\infty(M), P = A/\mu_N, \quad \text{where } \mu_N = \{f \in C^\infty(M) \mid f\big|_N = 0\}.$$

Let us consider the natural isomorphism $C^\infty(N) = A/\mu_N$. If we are given $X \in D(M, N) = D(P)$ and $z \in N$, then $\mu_z \supset \mu_N$, and the natural projection

$$h\colon P = A/\mu_N \to A/\mu_z$$

is defined. One also has $X_z = h \circ X = D(h)(X)$.

IV. A vector field along a map $\varphi\colon N \to M$:

$$A = C^\infty(M), \quad P = C^\infty(N), \quad \varphi = |F|,$$

where $F\colon A \to P$ is an $\mathbb{R}$-algebra homomorphism. Note also that an A-module structure in the algebra $P = C^\infty(N)$ is defined by the rule

$$(f, g) \mapsto F(f)g = \varphi^*(f)g, \quad f \in C^\infty(M),\ g \in C^\infty(N).$$

Let $X \in D_\varphi(M)$. If $z \in N$ and $h\colon P \to Q = P/\mu_z$ is the natural projection, then $X_z = h \circ X$.

By the way, at this point let us answer a question that naturally arises: What is a continuous vector field on M? It is an element of the $C^\infty(M)$-module $D(C^0(M))$, where $C^0(M)$ is the algebra of continuous functions on M equipped with a natural $C^\infty(M)$-module structure. Vector fields of the C^m class are defined in a similar way.

9.53. To conclude our discussion of geometric and algebraic problems related to the notion of vector field, let us note that the module $D(M)$ of vector fields (or more generally, derivations of the algebra A) carries another important algebraic structure. Namely, $D(A)$ is a *Lie algebra* due to the following $\mathbb{R}$-linear skew-symmetric operation satisfying the Jacobi identity.

Proposition. *The commutator* $[\Delta,\nabla]\stackrel{\text{def}}{=}\Delta\circ\nabla-\nabla\circ\Delta$ *of two derivations* Δ, $\nabla\in D(A)$ *is again a derivation.*

◀ Indeed,

$$\begin{aligned}[\Delta,\nabla](fg) &= (\Delta\circ\nabla-\nabla\circ\Delta)(fg)\\ &= \Delta(f\nabla(g)+g\nabla(f))-\nabla(f\Delta(g)+g\Delta(f))\\ &= \Delta(f)\nabla(g)+f\Delta(\nabla(g))+\Delta(g)\nabla(f)+g\Delta(\nabla(f))\\ &\quad -\nabla(f)\Delta(g)-f\nabla(\Delta(g))-\nabla(g)\Delta(f)-g\nabla(\Delta(f))\\ &= f\Delta(\nabla(g))-f\nabla(\Delta(g))+g\Delta(\nabla(f))-g\nabla(\Delta(f))\\ &= f[\Delta,\nabla](g)+g[\Delta,\nabla](f). \quad ▶\end{aligned}$$

Since the commutator is obviously skew-symmetric, it remains only to check the Jacobi identity.

9.54 Proposition. *Let* $\nabla,\Delta',\Delta''\in D(A)$. *Then*

$$[\nabla,[\Delta',\Delta'']]=[[\nabla,\Delta'],\Delta'']+[\Delta',[\nabla,\Delta'']].$$

◀ Indeed,

$$\begin{aligned}[[\nabla,\Delta'],\Delta'']+[\Delta',[\nabla,\Delta'']] &= [[\nabla,\Delta'],\Delta'']-[[\nabla,\Delta''],\Delta']\\ &= [\nabla\circ\Delta'-\Delta'\circ\nabla,\Delta'']-[\nabla\circ\Delta''-\Delta''\circ\nabla,\Delta']\\ &= \nabla\circ\Delta'\circ\Delta''-\Delta'\circ\nabla\circ\Delta''-\Delta''\circ\nabla\circ\Delta'+\Delta''\circ\Delta'\circ\nabla\\ &\quad -\nabla\circ\Delta''\circ\Delta'+\Delta''\circ\nabla\circ\Delta'+\Delta'\circ\nabla\circ\Delta''-\Delta'\circ\Delta''\circ\nabla\\ &\quad -\nabla\circ\Delta'\circ\Delta''-\Delta''\circ\Delta'\circ\nabla-\nabla\circ\Delta''\circ\Delta'+\Delta'\circ\Delta''\circ\nabla\\ &= \nabla\circ(\Delta'\circ\Delta''-\Delta''\circ\Delta')-(\Delta'\circ\Delta''-\Delta''\circ\Delta')\circ\nabla\\ &= \nabla\circ[\Delta',\Delta'']-[\Delta',\Delta'']\circ\nabla=[\nabla,[\Delta',\Delta'']]. \quad ▶\end{aligned}$$

9.55. The local coordinate description of vector fields given in Section 9.41 shows that they are (scalar) first-order differential operators. On the other hand, a first-order differential operator Δ of general form on the manifold M can be locally written as

$$\Delta=\sum_{i=1}^{n}\alpha_i\frac{\partial}{\partial x_i}+\beta,\quad \alpha_i,\beta\in C^\infty(U).$$

Let us note that its free term β has an invariant meaning: $\beta=\Delta(1)$. Therefore, we can assert that Δ is a first-order differential operator if and only if $\Delta-\Delta(1)$ is a derivation. This gives us a coordinate-free definition of (scalar) linear first-order differential operators. But being insufficiently "clever," it does not allow us to guess a similar definition for operators of higher orders. Let us trim this definition: Note that the Leibniz rule for the derivation $\Delta-\Delta(1)$ is equivalent to the following equality:

$$\Delta(fg)-f\Delta(g)=g\Delta(f)-fg\Delta(1),\quad\text{or}\quad [\Delta,f](g)=g[\Delta,f](1),$$

where $[\Delta, f] = \Delta \circ f - f \circ \Delta$ is the commutator of the operator Δ with the multiplication by f. Now let $g = hs$. Then, taking into account the equality $s[\Delta, f](1) = [\Delta, f](s)$, we obtain

$$[\Delta, f](hs) = h[\Delta, f](s).$$

The last equality can be rewritten in the form $[[\Delta, f], h](s) = 0$. Since s was arbitrary, this means that we have proved the following result:

9.56 Proposition. *An $\mathbb{R}$-linear map*

$$\Delta\colon C^\infty(M) \to C^\infty(M)$$

is a first-order differential operator if and only if

$$[[\Delta, f], g] = 0 \quad \forall f,\ g \in C^\infty(M). \tag{9.21}$$

Let us note that (9.21) is equivalent to the fact that the commutator $[\Delta, f]$ is a $C^\infty(M)$-homomorphism for any $f \in A$.

Looking at the last equality, the reader has probably understood already how to define a differential operator of any order over an arbitrary commutative algebra A. Before stating this definition, let us observe that the expression $[[\Delta, f], g]$ is not manifestly symmetric with respect to f and g, while in fact f and g enter this expression symmetrically:

$$[[\Delta, f], g] = \Delta \circ fg + fg\Delta - g\Delta \circ f - f\Delta \circ g = [[\Delta, g], f].$$

Therefore, we shall change our notation and for any element $f \in A$ introduce the map

$$\delta_f\colon \operatorname{Hom}_K(A, A) \to \operatorname{Hom}_K(A, A), \quad \delta_f(\Delta) \overset{\text{def}}{=} [\Delta, f].$$

By the above, the operators δ_f and δ_g commute, and condition (9.21) acquires the form

$$(\delta_g \circ \delta_f)(\Delta) = 0 \quad \forall f,\ g \in A.$$

We can now give the following fundamental definition:

9.57. Definition. Let A be a K-algebra. Then a K-homomorphism $\Delta\colon A \to A$ is called a *linear differential operator of order* $\leqslant l$ with values in A if for any $f_0, \dots, f_l \in A$ we have the identity

$$(\delta_{f_0} \circ \dots \circ \delta_{f_l})(\Delta) = 0. \tag{9.22}$$

Let us denote the set of all differential operators of order $\leqslant l$ acting from A to A by $\operatorname{Diff}_l A$. Like to $D(A)$, this set is stable with respect to summation and multiplication by elements of the algebra A. Therefore, it is naturally endowed with an A-module structure. Moreover, another A-module structure can be introduced in this module, by defining the action of an element $f \in A$ on an operator Δ as the composition $\Delta \circ f$. This structure is called the *right* one, while the action of an element $f \in A$ on an operator Δ is sometimes denoted by $f^+\Delta$ instead of $\Delta \circ f$. The set $\operatorname{Diff}_l A$

endowed with a module structure with respect to the right multiplication will be denoted by $\mathrm{Diff}_l^+ A$. The two multiplicative structures in $\mathrm{Diff}_l A$ commute and thus determine a bimodule structure, denoted by $\mathrm{Diff}_l^{(+)} A$.

9.58 Exercises. 1. Prove the last statement. Namely, check that the set $\mathrm{Diff}_l A$ is stable with respect to the right multiplication and that the left and right multiplications commute in $\mathrm{Diff}_l A$.

2. Check whether the set $D(A)$ is stable with respect to the right multiplication.

To deduce some natural and useful properties of differential operators, we shall need the following notation. Let $\varkappa^n = (1, 2, \dots, n)$ be the ordered set of integers, and $\varkappa = (i_1, \dots, i_l)$, $l \leqslant n$, an ordered subset of it. Let us set by definition $|\varkappa| = l$, $a_\varkappa = a_{i_1} \cdots a_{i_l}$ and $\delta_{a_\varkappa} = \delta_{a_{i_1}} \circ \dots \circ \delta_{a_{i_l}}$. The ordered complement of $\varkappa$ in $\varkappa^n$ will be denoted by $\overline{\varkappa}$.

Exercise. Let A be a K-algebra, and Δ, ∇ K-linear maps from A to A. Then

$$\delta_{a_{\varkappa^n}}(\Delta \circ \nabla) = \sum_{|\varkappa| \leqslant n} \delta_{a_\varkappa}(\Delta) \circ \delta_{a_{\overline{\varkappa}}}(\nabla), \quad a_i \in A, \tag{9.23}$$

$$\delta_{a_{\varkappa^n}}(\Delta)(b) = \sum_{|\varkappa| \leqslant n} (-1)^{|\varkappa|} a_\varkappa \Delta(a_{\overline{\varkappa}} b), \quad a_i, b \in A. \tag{9.24}$$

For the case $\Delta \in \mathrm{Diff}_m A$, $m < n$, the left-hand side of the last equality vanishes by Section 9.57, and the equality can be rewritten in the following form:

$$\Delta(a_{\varkappa^n} b) = - \sum_{0 < |\varkappa| \leqslant n} (-1)^{|\varkappa|} a_\varkappa \Delta(a_{\overline{\varkappa}} b). \tag{9.25}$$

These formulas allow one to readily prove the following two important statements:

9.59 Proposition. *Let ∇ and Δ be differential operators of orders $\leqslant l$ and $\leqslant m$, respectively. Then their composition $\Delta \circ \nabla$ is a differential operator of order $\leqslant l + m$.*

◀ Indeed, let us set $n = m + l + 1$ in formula (9.23). Then each monomial on the right-hand side of the equality thus obtained will vanish by the definition of differential operators: Either $|\varkappa| \geqslant m + 1$ and therefore $\delta_{a_\varkappa}(\Delta) = 0$, or $|\overline{\varkappa}| \geqslant l + 1$ and, respectively, $\delta_{a_{\overline{\varkappa}}}(\nabla) = 0$. ▶

9.60 Proposition. *Let $I \subset A$ be an arbitrary ideal, $a \in I^k$, $\Delta \in \mathrm{Diff}_n A$, and $n < k$. Then $\Delta(a) \in I^{k-n}$.*

◀ To prove the proposition, it suffices to confine oneself to the case $a = a_1 \cdots a_k$, $a_i \in I$. Let $k = n + 1$. Consider equality (9.25) with $b = 1$. Then every summand on the right-hand side will contain at least one element

$a_i \in I$ and consequently will belong to I itself. The passage from $k = n+r$ to $k = n+r+1$ is accomplished as follows. Let us use formula (9.25) again. Each of the summands on the right-hand side is of the form

$$a_{i_1} \cdots a_{i_m} \Delta(a_{j_1} \cdots a_{j_{k-m}}). \tag{9.26}$$

Note that $a_{i_1} \cdots a_{i_m} \in I^m$, $a_{j_1} \cdots a_{j_{k-m}} \in I^{k-m}$. If $m \geqslant k-n$, then the monomial 9.26 obviously belongs to I^{k-n}. Otherwise, if $k-m > n$, we see that $\Delta(a_{j_1} \cdots a_{j_{k-m}}) \in I^{k-m-n}$ by the induction hypothesis, and the monomial (9.26) as a whole belongs to I^{k-n}. ▶

Now let us prove that for algebras of smooth functions Definition 9.57 *coincides with the usual definition of linear differential operator.* The desired result will be a consequence of the previous proposition, its corollary, and Theorem 9.62.

9.61 Corollary. *If functions f and g coincide in some neighborhood $U \ni z$, then for any differential operator Δ the equality $\Delta(f)(z) = \Delta(g)(z)$ is valid. In other words, differential operators are local.*

◀ Indeed, let Δ be an operator of order $\leqslant l$. Since $f-g \in \mu_z^{l+1}$ for any l, then by Proposition 9.60 one has $\Delta(f-g) \in \mu_z$. ▶

This corollary allows one to obtain, for any differential operator $\Delta \in \operatorname{Diff}_l C^\infty(M)$, the well-defined restriction $\Delta\big|_U \colon C^\infty(U) \to C^\infty(U)$ to any open domain $U \subset M$ by setting

$$\Delta\big|_U(f)(z) = \Delta(g)(z), \quad f \in C^\infty(U),\ g \in C^\infty(M),\ z \in U,$$

where g is an arbitrary function coinciding with f in some neighborhood of the point z. This definition implies $\Delta\big|_U(f\big|_U) = \Delta(f)\big|_U$ for $f \in C^\infty(M)$. Obviously, any operator is uniquely determined by its restrictions on charts of an arbitrary atlas.

It remains to prove that the following result is valid:

9.62 Theorem. *Let $\Delta \in \operatorname{Diff}_l C^\infty(M)$, and $x_1, \ldots, x_n$ local coordinates in a neighborhood $U \subset M$. Then the operator $\Delta\big|_U$ can be presented in the form*

$$\Delta\big|_U = \sum_{|\sigma|=0}^{l} \alpha_\sigma \frac{\partial^{|\sigma|}}{\partial x^\sigma}, \qquad \alpha_\sigma \in C^\infty(U).$$

◀ Let $z \in U$ and $f \in C^\infty(M)$. Consider an arbitrary star-shaped neighborhood $U_z \subset U$ of the point z and, using Section 2.9, present the function f in this neighborhood in the form

$$f = \sum_{|\sigma|=0}^{l} \frac{\partial^{|\sigma|} f}{\partial x^\sigma}(z) \left(\frac{(x-z)^\sigma}{\sigma!} \right) + h(x),$$

where $h(x) \in \mu_z^{l+1}$ and $(x-z)^\sigma \stackrel{\text{def}}{=} (x_1 - a_1)^{\sigma_1} \cdots (x_n - a_n)^{\sigma_n}$. Therefore,

$$\Delta(f)(z) = \Delta\big|_U (f\big|_U)(z) = \sum_{|\sigma|=0}^{l} \frac{\partial^{|\sigma|} f}{\partial x^\sigma}(z) \alpha_\sigma(z)$$

with

$$\alpha_\sigma(x) \stackrel{\text{def}}{=} \Delta\big|_U \left(\frac{(x-z)^\sigma}{\sigma!} \right).$$

It remains to note that functions $\alpha_\sigma(x)$ are smooth by construction. ▶

To understand how the algebraic Definition 9.57 of differential operators works for the case in which the algebra A is not the smooth function algebra on a smooth manifold, let us do the following exercise:

9.63 Exercises. 1. Describe the modules of differential operators for the algebra $C^\infty(\mathbf{K})$. (See Examples 9.35 and 9.45.)

2. Do the same for the algebra of truncated polynomials

$$A = K[X]/X^n K[X], \quad \text{where} \quad K = \mathbb{R},\ K = \mathbb{Z}_m.$$

(See Exercise 3 from Section 9.45.)

3. In the classical situation $A = C^\infty(\mathbb{R})$, any differential operator of order > 1 may be represented as the sum of compositions of first-order operators. May one assert the same thing for the algebras from the previous exercises?

9.64. Jets of order l at a point. Let us formulate an important consequence of Proposition 9.60. Recall that μ_z^{l+1} denotes the $(l+1)$st power of the ideal μ_z consisting of all functions on M vanishing at the point z.

Corollary. *Let $\Delta \in \operatorname{Diff}_l C^\infty(M)$, $f, g \in C^\infty(M)$, and $z \in M$. Then $\Delta(f)(z) = \Delta(g)(z)$ if $f = g \mod \mu_z^{l+1}$.*

◀ Indeed, in this case $f - g \in \mu_z^{l+1}$, and consequently, $\Delta(f-g) \in \mu_z$, i.e., $\Delta(f-g)(z) = 0$. ▶

It will be useful to consider this fact after introducing the vector *space of lth order jets*, or *l-jets*, of (smooth) functions on M at some point z (cf. Section 9.27):

$$J_z^l M \stackrel{\text{def}}{=} C^\infty(M)/\mu_z^{l+1}.$$

The image of the function f under the natural projection

$$C^\infty(M) \to C^\infty(M)/\mu_z^{l+1} = J_z^l M$$

is called its *jet of order l* (or *l-jet*) at the point z and is denoted by $[f]_z^l$. In these terms, the condition $f = g \mod \mu_z^{l+1}$ means that $[f]_z^l = [g]_z^l$, while the previous corollary states that $\Delta(f)(z) = \Delta(g)(z)$ if $[f]_z^l = [g]_z^l$. In other words, the map

$$h_{\Delta,z}\colon J^l_z M \to \mathbb{R}, \quad [f]^l_z \mapsto \Delta(f)(z),$$

is well defined. It is obviously $\mathbb{R}$-linear. Its importance is explained by the fact that it completely determines the operator Δ at the point z.

Exercise. The map $h_{\Delta,z}$ is a linear function on the space $J^l_z M$. Find a basis of the space $J^l_z M$ in which the components of this function are the numbers $\alpha_\sigma(z)$ appearing in Theorem 9.62.

9.65. The manifold of jets. The family $\{h_{\Delta,z}\}_{z\in M}$ of linear functionals uniquely determines the operator Δ, since

$$\Delta(f)(z) = h_{\Delta,z}([f]^l_z). \tag{9.27}$$

Therefore, it makes sense to construct a new object combining the separate maps $h_{\Delta,z}$ into a single whole. To do this, one first needs to join their domains, in the same way as was done in Section 9.28 for $l = 1$:

$$J^l M = \bigcup_{z\in M} J^l_z M.$$

The set $J^l M$ is equipped with a smooth manifold structure by a procedure similar to that used for TM and T^*M. The details of this construction will be described in Section 10.11. This smooth manifold is called the *manifold of jets of order l* (or *of l-jets*) of the manifold M.

The map

$$\pi_{J^l} = \pi_{J^l M}\colon J^l M \to M, \quad J^l M \supset J^l_z M \ni \theta \mapsto z \in M,$$

fibers the manifold $J^l M$ over M. By this definition, $\pi^{-1}_{J^l}(z) = J^l_z M$. Moreover, to any function $f \in C^\infty(M)$ we can assign the section

$$s_{j_l(f)}\colon M \to J^l M, \quad z \mapsto [f]^l_z \in J^l_z M \subset J^l M,$$

of this bundle. This section is called the *l-jet of f*.

Any operator $\Delta \in \operatorname{Diff}_l C^\infty(M)$ determines the map

$$h_\Delta\colon J^l M \to M \times \mathbb{R}, \quad J^l_z M \ni \theta \mapsto (z, h_{\Delta,z}(\theta)).$$

Let $\pi_{\mathbb{R}}\colon M \times \mathbb{R} \to \mathbb{R}$ be the canonical projection. Then, by (9.27), $\pi_{\mathbb{R}}(h_\Delta([f]^l_z)) = \Delta(f)(z)$, and consequently,

$$\Delta(f) = \pi_{\mathbb{R}} \circ h_\Delta \circ s_{j_l(f)}, \tag{9.28}$$

where $\Delta(f)$ is understood as a smooth map from M to $\mathbb{R}$. This relation shows that all the information on the operator Δ is encoded in the map of smooth manifolds h_Δ. It will be shown in Section 11.47 that $s_{j_l}: f \mapsto s_{j_l(f)}$ is a differential operator of order l, whose range of values is the set of sections of the bundle π_{J^l}. It is natural to call this operator the *universal* differential operator of order l, since all particular operators are obtained by composing this operator with maps from $J^l M$ to $M \times \mathbb{R}$. The specifics of maps of the form h_Δ acting from $J^l M$ to $M \times \mathbb{R}$ can be described in the following way.

Consider the projection

$$\pi \colon M \times \mathbb{R} \to M, \quad (z, \lambda) \mapsto z.$$

It is a trivial bundle over M with fiber $\mathbb{R}$ (see Section 11.2). Then, as is easily seen, maps of the form h_Δ are morphisms of the vector bundle π_{J^l} to the vector bundle π (see Section 11.4), and in particular, they take the fiber $\pi_{J^l}^{-1}(z) = J^l_z M$ to the fiber $\pi^{-1}(z) = \mathbb{R}$. This map of fibers obviously coincides with $h_{\Delta,z}$.

All these facts reveal the fundamental role of vector bundles in the differential calculus. For this and many other reasons (some of them will appear in our subsequent exposition), the theory of vector bundles is a necessary part of the differential calculus over smooth manifolds. This theory will be considered in Chapter 11.

Due to the universality of the operator s_{j_l} expressed by formula (9.28), the manifolds $J^l M$ and their natural generalizations constitute an important part of the foundations of the modern theory of partial differential equations. The universal property of this operator is also revealed by the fact that the module of sections of the bundle

$$\pi_{J^l} \colon J^l M \to M$$

(see Section 11.7) is the representing object for the functor Diff_l of the differential calculus in the category of geometrical $C^\infty(M)$-modules (see Section 11.55).

9.66. It was shown above for the case in which $A = C^\infty(M)$ is the algebra of smooth functions on the manifold M that Definition 9.57 is equivalent to the usual definition of a linear differential operator acting on functions and taking its values in functions on M (i.e., to the definition of a scalar differential operator). In fact, the more general case, that of matrix differential operators, can also be described in purely algebraic terms. Let us recall that such an operator Δ is usually defined as a matrix composed of differential operators,

$$\Delta = \begin{pmatrix} \Delta_{1,1} & \dots & \Delta_{1,m} \\ \vdots & \ddots & \vdots \\ \Delta_{k,1} & \dots & \Delta_{k,m} \end{pmatrix},$$

where $\Delta_{i,j}$ are differential operators of order $\leqslant l$, while the action of this operator on a vector function $\bar{f} = (f_1, \dots, f_n)$ is defined in the following natural way:

$$\begin{pmatrix} \Delta_{1,1} & \dots & \Delta_{1,m} \\ \vdots & \ddots & \vdots \\ \Delta_{k,1} & \dots & \Delta_{k,m} \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} = \begin{pmatrix} \Delta_{1,1}(f_1) + \dots + \Delta_{1,m}(f_m) \\ \vdots \\ \Delta_{k,1}(f_1) + \dots + \Delta_{k,m}(f_m) \end{pmatrix}.$$

In Chapter 11 it will be shown that vector functions of the above type can naturally be considered as sections of an m-dimensional vector bundle over M and that the category of all vector bundles over M is equivalent to the category of projective modules over the algebra $C^\infty(M)$. This fact, together with the observations of Section 9.50, leads one to believe that differential operators of general nature should be maps connecting modules over some base algebra A. It is remarkable that to define a general differential operator it suffices simply to repeat the scalar Definition 9.57. The only thing that matters here is that for any K-linear map of A-modules $\Delta\colon P \to Q$ and $a \in A$ one can define the commutator

$$\delta_a(\Delta) \stackrel{\text{def}}{=} [\Delta, a]\colon P \to Q,$$

where the element $a \in A$ is understood as the operator of multiplication by a applied to elements of the corresponding A-module. In other words,

$$\delta_a(\Delta)(p) = \Delta(ap) - a\Delta(p), \quad p \in P.$$

So, one can hope that the following purely algebraic definition reduces to the usual notion of a (matrix) differential operator in the "standard" situation.

9.67. Definition. Let A be an arbitrary commutative K-algebra, and let P and Q be A-modules. A K-homomorphism $\Delta\colon P \to Q$ is called a *linear differential operator of order* $\leqslant l$ acting from P to Q if for any $a_0, \dots, a_l \in A$ one has

$$(\delta_{a_0} \circ \cdots \circ \delta_{a_l})(\Delta) = 0. \tag{9.29}$$

The fact that under an adequate specialization ($K = \mathbb{R}$, $A = C^\infty(M)$ with projective A-modules P and Q) the definition given above coincides with the usual one will be proved in Chapter 11, after we have established relations between vector bundles and projective modules.

Let us denote the set of all differential operators of order $\leqslant l$ acting from P to Q by $\mathrm{Diff}_l(P, Q)$. This set is stable with respect to summation and the ordinary (left) multiplication by elements of the algebra A:

$$(a\Delta)(p) \stackrel{\text{def}}{=} a \cdot \Delta(p), \quad a \in A,\ p \in P.$$

Therefore, it possesses a natural left A-module structure. One can also introduce another A-module structure, defining the action of an element $a \in A$ on the operator Δ as the composition $\Delta \circ a$. This structure is called *right*, and the action of $a \in A$ to Δ is denoted by $a^+\Delta$ instead of $\Delta \circ a$. The set $\mathrm{Diff}_l(P, Q)$, as a module with respect to the right multiplication, will be denoted by $\mathrm{Diff}_l^+(P, Q)$. Two multiplicative structures in $\mathrm{Diff}_l(P, Q)$ commute and thus determine the structure of a bimodule, denoted by $\mathrm{Diff}_l^{(+)}(P, Q)$. For the sake of brevity, we use the notation $\mathrm{Diff}_l^{(+)} Q$ for $\mathrm{Diff}_l^{(+)}(A, Q)$.

If $h\colon P \to Q$ is an A-module homomorphism, then the correspondence $\Delta \mapsto h \circ \Delta$, $\Delta \in \operatorname{Diff}_l P$, determines a homomorphism of the A-module $\operatorname{Diff}_l P$ to the A-module $\operatorname{Diff}_l Q$. Therefore, the correspondence $P \mapsto \operatorname{Diff}_l P$ is a functor on the category of A-modules. Let us denote this functor by Diff_l. We obtain another example of an *absolute* functor of differential calculus (see Section 9.51). Such functors are defined for all commutative unital algebras. If we choose an A-module P, we obtain an example of a *relative* functor of differential calculus, $\operatorname{Diff}_l(P, \cdot)\colon Q \mapsto \operatorname{Diff}_l(P, Q)$.

Formulas (9.23)–(9.25) and Proposition 9.59 are proved in the general situation exactly in the same way as for scalar operators. As to Proposition 9.60, its analogue in the general case is the following

Proposition. *Let $I \subset A$ be an ideal, P, Q A-modules, $p \in I^k P$, $\Delta \in \operatorname{Diff}_n(P, Q)$, and $n < k$. Then $\Delta(p) \in I^{k-n} Q$.*

The proof is the same as in the scalar case.

Exercises. 1. Check that

$$\operatorname{Diff}_0(P, Q) = \operatorname{Diff}_0^+(P, Q) = \operatorname{Hom}_K(P, Q).$$

2. Consider the maps i^+ and i_+ of A-modules that are the identities on the underlying sets:

$$i^+\colon \operatorname{Diff}_l(P, Q) \to \operatorname{Diff}_l^+(P, Q), \quad i^+(f) = f,$$
$$i_+\colon \operatorname{Diff}_l^+(P, Q) \to \operatorname{Diff}_l(P, Q), \quad i_+(f) = f.$$

Prove that these maps are differential operators of order $\leqslant l$.

9.68. Let us note that any differential operator of order $\leqslant l$ is an operator of order $\leqslant m$ as well if $l \leqslant m$. Therefore, one has a natural bimodule embedding $\operatorname{Diff}_l^{(+)}(P, Q) \subset \operatorname{Diff}_m^{(+)}(P, Q)$. Let us denote the direct limit of the sequence of embeddings

$$\operatorname{Diff}_0^{(+)}(P, Q) \subset \cdots \subset \operatorname{Diff}_l^{(+)}(P, Q) \subset \operatorname{Diff}_{l+1}^{(+)}(P, Q) \subset \cdots$$

by $\operatorname{Diff}^{(+)}(P, Q)$.

As we saw above, the composition of two differential operators, if it is defined, is again a differential operator. Therefore, the bimodule $\operatorname{Diff}^{(+)}(P, P)$ becomes a (noncommutative) A-algebra with respect to this operation. Moreover, $\operatorname{Diff}^{(+)}(P, Q)$ can be regarded as a right $\operatorname{Diff}^{(+)}(P, P)$- and left $\operatorname{Diff}^{(+)}(Q, Q)$-module.

9.69. We have now everything needed to answer the question asked in Section 9.20: *What is the algebra whose spectrum is the cotangent manifold T^*M?* Let us start with necessary algebraic definitions.

Let A be a K-algebra. The above-mentioned embedding of A-modules $\operatorname{Diff}_{k-1} A \subset \operatorname{Diff}_k A$ (see Section 9.68) allows one to define the quotient module

$$\mathcal{S}_k(A) \stackrel{\text{def}}{=} \operatorname{Diff}_k A / \operatorname{Diff}_{k-1} A,$$

which is called the *module of symbols of order k* (or the module of *k-symbols*). The coset of an operator $\Delta \in \mathrm{Diff}_k A$ modulo $\mathrm{Diff}_{k-1} A$ will be denoted by $\mathrm{smbl}_k \Delta$ and called the *symbol* of Δ. Let us define the *algebra of symbols* for the algebra A by setting

$$\mathcal{S}_*(A) = \bigoplus_{n=0}^{\infty} \mathcal{S}_n(A).$$

The operation of multiplication in $\mathcal{S}_*(A)$ is induced by the composition of differential operators. To be more precise, for two elements

$$\mathrm{smbl}_l \Delta \in \mathcal{S}_l(A), \quad \mathrm{smbl}_k \nabla \in \mathcal{S}_k(A)$$

let us set by definition

$$\mathrm{smbl}_l \Delta \cdot \mathrm{smbl}_k \nabla \stackrel{\text{def}}{=} \mathrm{smbl}_{k+l}(\Delta \circ \nabla) \in \mathcal{S}_{l+k}(A).$$

This operation is well defined, since the result does not depend on the choice of representatives in the cosets $\mathrm{smbl}_l \Delta$ and $\mathrm{smbl}_k \nabla$. Indeed, if, say, $\mathrm{smbl}_l \Delta = \mathrm{smbl}_l \Delta'$, then $\Delta - \Delta' \in \mathrm{Diff}_{l-1} A$ and consequently $(\Delta - \Delta') \circ \nabla \in \mathrm{Diff}_{l+k-1} A$.

Proposition. *$\mathcal{S}_*(A)$ is a commutative algebra.*

◀ We must check that

$$\Delta \circ \nabla - \nabla \circ \Delta = [\Delta, \nabla] \in \mathrm{Diff}_{l+k-1} A$$

if $\Delta \in \mathrm{Diff}_l A$ and $\nabla \in \mathrm{Diff}_k A$. Let us use induction on $l + k$. For $l + k = 0$, i.e., for $l = k = 0$, the statement is obvious, since scalar differential operators of order zero are the operators of multiplication by elements of the algebra A, and this algebra is commutative. The induction step from $l + k < n$ to $l + k = n$ is based on the following formula, which is a rather particular case of (9.23):

$$\begin{aligned}\delta_a(\Delta \circ \nabla - \nabla \circ \Delta) &= \delta_a(\Delta) \circ \nabla + \Delta \circ \delta_a(\nabla) - \delta_a(\nabla) \circ \Delta - \nabla \circ \delta_a(\Delta) \\ &= [\delta_a(\Delta), \nabla] + [\Delta, \delta_a(\nabla)].\end{aligned}$$

The orders of the operators $\delta_a(\Delta)$ and $\delta_a(\nabla)$ are $l-1$ and $k-1$, respectively. By the induction hypothesis, the last expression is an operator of order $\leqslant k+l-2$. Hence, the order of the operator $[\Delta, \nabla]$ does not exceed $l+k-1$. ▶

Let us note that $\mathcal{S}_0(A) = A$ is a subalgebra of the algebra $\mathcal{S}_*(A)$, and the operations of left (right) multiplication of differential operators by elements of the algebra A reduce to the left (right) multiplication by elements of this subalgebra. By the commutativity of the algebra $\mathcal{S}_*(A)$, these multiplication operations coincide.

9.70. Now let $\mathrm{smbl}_l \Delta \in \mathcal{S}_l(A)$ and $\mathrm{smbl}_k \nabla \in \mathcal{S}_k(A)$. Then, by the last proposition, $[\Delta, \nabla] \in \mathrm{Diff}_{l+k-1} A$. One can assign to the pair

$(\operatorname{smbl}_l \Delta, \operatorname{smbl}_k \nabla)$ the element

$$\{\operatorname{smbl}_l \Delta, \operatorname{smbl}_k \nabla\} \stackrel{\text{def}}{=} \operatorname{smbl}_{k+l-1}[\Delta, \nabla] \in \mathcal{S}_{k+l-1}(A),$$

which is well defined, i.e., does not depend on the choice of representatives in the cosets $\operatorname{smbl}_l \Delta$ and $\operatorname{smbl}_k \nabla$ (this is proved exactly in the same way as we proved that the multiplication in $\mathcal{S}_*(A)$ is well defined). The operation $\{\,\cdot\,,\,\cdot\,\}$ is K-linear and skew-symmetric. It satisfies the Jacobi identity, since the commutator of linear differential operators satisfies this identity. Thus, $\mathcal{S}_*(A)$ is a *Lie algebra* with respect to this operation. If $\mathfrak{s}_1, \mathfrak{s}_2 \in \mathcal{S}_1(A)$, then $\{\mathfrak{s}_1, \mathfrak{s}_2\} \in \mathcal{S}_1(A)$ as well. In other words, $\mathcal{S}_1(A) \subset \mathcal{S}_*(A)$ is a *Lie subalgebra* of the Lie algebra of symbols $S_*(A)$.

Exercises. 1. Let $\mathfrak{s} = \operatorname{smbl}_1 \Delta$. Prove that the correspondence

$$\mathfrak{s} \leftrightarrow \Delta - \Delta(1) \in D(A)$$

is well defined and establishes an isomorphism between the Lie algebras $\mathcal{S}_1(A)$ and $D(A)$.

2. Let us fix an arbitrary element $\mathfrak{s} \in \mathcal{S}_*(A)$. Show that the map

$$\{\mathfrak{s}, \,\cdot\,\} \colon \mathcal{S}_*(A) \to \mathcal{S}_*(A), \quad \mathfrak{s}_1 \mapsto \{\mathfrak{s}, \mathfrak{s}_1\},$$

is a derivation of the algebra $\mathcal{S}_*(A)$.

9.71. Assume that the ring K is an algebra over the field of rational numbers $\mathbb{Q}$. Then any element $a \in A$ determines a K-algebra homomorphism

$$\Xi_a \colon \mathcal{S}_*(A) \to A, \quad \operatorname{smbl}_k(\Delta) \mapsto \frac{[\delta_a^k(\Delta)]}{k!}(1).$$

Let us check this fact. Note first that $\delta_a^k(\nabla) = 0$ if $\nabla \in \operatorname{Diff}_{k-1} A$. Therefore, the map Ξ_a is well defined. Its K-linearity is obvious. Further, $\Xi_a\big|_{\mathcal{S}_0(A)} \colon \mathcal{S}_0(A) = A \to A$ is the identity map, and hence $\Xi_a(1_{\mathcal{S}_*(A)}) = 1_A$ (unitarity). Finally, if $\Delta \in \operatorname{Diff}_k A$, $\nabla \in \operatorname{Diff}_l A$, then from (9.23) it follows that

$$\delta_a^{k+l}(\Delta \circ \nabla) = \binom{k+l}{k} \delta_a^k(\Delta) \circ \delta_a^l(\nabla). \tag{9.30}$$

Since $\delta_a^k(\Delta) \in \operatorname{Diff}_0 A = A$ and $\delta_a^l(\nabla) \in \operatorname{Diff}_0 A = A$ are operators of multiplication by the elements $\left[\delta_a^k(\Delta)\right](1)$ and $\left[\delta_a^l(\nabla)\right](1)$ of the algebra A, the multiplicativity of the map Ξ_a is a direct consequence of (9.30).

Proposition. *Let $I \subset A$ be an ideal, and $\Pi \colon A \to A/I$ a natural projection. Then $\Pi \circ \Xi_a = 0$ if $a \in I^2$.*

◀ From formula (9.24) it follows that

$$\left[\delta_a^k(\Delta)\right](1) = \sum_{i=0}^{k} \binom{k}{i} a^i \Delta\left(a^{k-i}\right).$$

Therefore, if $a \in I^2$, then

$$\left[\delta_a^k(\Delta)\right](1) = \Delta\left(a^k\right) \mod I^2.$$

If, in addition, $\Delta \in \operatorname{Diff}_k A$, then, by Proposition 9.60, $\Delta\left(a^k\right) \in I^k$, $k \geqslant 1$, and consequently $\Pi(\Xi_a(\operatorname{smbl}_k \Delta)) = 0$. For $k = 0$ the assertion is obvious. ▶

9.72. We can now describe the K-spectrum $|\mathcal{S}_*(A)|$ of the algebra $\mathcal{S}_*(A)$. Let $h \in |A|$. Then the composition

$$\gamma_{h,a} \stackrel{\text{def}}{=} h \circ \Xi_a \colon \mathcal{S}_*(A) \to K$$

is a K-algebra homomorphism and thus is a point of the spectrum $|\mathcal{S}_*(A)|$.

Corollary. *Let* $\mu_h = \ker h$ *and*

$$a - h(a) \cdot 1_A = b - h(b) \cdot 1_A \mod \mu_h^2.$$

Then $\gamma_{h,a} = \gamma_{h,b}$.

◀ Note first that $\delta_{\lambda \cdot 1_A} = 0$ for $\lambda \in K$. Therefore, $\Xi_a = \Xi_{a-h(a)\cdot 1_A}$, $\Xi_b = \Xi_{b-h(b)\cdot 1_A}$, and we may confine ourselves to the case $h(a) = h(b) = 0$, which is equivalent to $a, b \in \mu_h$. Further, by our assumptions, we have $a' = a - b \in \mu_h^2$. Since

$$\delta_b^k(\Delta) = \delta_{a+a'}^k(\Delta) = \sum_{s=0}^{k} \binom{k}{s} \delta_{a'}^s(\delta_a^{k-s}(\Delta)),$$

by setting $\Delta_s = \frac{k!}{(k-s)!}\delta_a^{k-s}(\Delta)$ and assuming that $\Delta \in \operatorname{Diff}_k A$ we find that

$$\begin{aligned}
\Xi_b(\operatorname{smbl}_k \Delta) =& \frac{1}{k!}\left[\delta_{a+a'}^k(\Delta)\right](1) \\
=& \frac{1}{k!}\left[\delta_a^k(\Delta)\right](1) + \sum_{s=1}^{k} \frac{1}{s!}\left[\delta_{a'}^s(\Delta_s)\right](1) \\
=& \Xi_a(\operatorname{smbl}_k \Delta) + \sum_{s=1}^{k} \Xi_{a'}(\operatorname{smbl}_s(\Delta_s)).
\end{aligned}$$

Therefore, from Proposition 9.71 it follows that $h \circ \Xi_b = h \circ \Xi_a$. ▶

Let us recall that by definition (see Section 9.30) the cotangent space to the spectrum $|A|$ at a point $h \in |A|$ is the quotient module $T_h^*(A) \stackrel{\text{def}}{=} \mu_h/\mu_h^2$. Corollary 9.72 makes it possible to construct the map

$$i_h \colon T_h^*(A) \to |\mathcal{S}_*(A)|,$$

by setting $i_h([a]) = \gamma_{h,a}$, $a \in \mu_h$. In other words, to any “cotangent vector” to the “manifold” $|A|$ there corresponds a point of the “manifold $|\mathcal{S}_*(A)|$. Let us study this correspondence in more detail.

9.73 Proposition. *Let K be a field and assume that any tangent vector $\xi \in T_h A$ can be continued to a "vector field" $X \in D(A)$; i.e., for any $\xi \in T_h A$ there exists a derivation $X \in D(A)$ such that $\xi = h \circ X$. Then the map i_h is injective.*

◀ By (9.11), to any K-linear map $\varphi\colon T_h^* A \to K$ there corresponds a tangent vector $\xi_\varphi = \nu_h(\varphi) \in T_h A$. Let now $a, b \in \mu_h$ and $[a] \neq [b]$, where $[g] = g \bmod \mu_h^2$. Since K is a field, by Section 9.30, ν_h is an isomorphism and consequently $\xi_\varphi(a) \neq \xi_\varphi(b)$. Let us continue the tangent vector ξ_φ to a vector field $X \in D(A)$. Then the above inequality can be interpreted as $h(X(a)) \neq h(X(b))$. Now identifying $D(A)$ with $\mathcal{S}_1(A)$ (see Exercise 1 from Section 9.70), we see that $X(a) = \Xi_a(X)$, $X(b) = \Xi_b(X)$, and thus the last inequality can be rewritten in the form

$$(h \circ \Xi_a)(X) \neq (h \circ \Xi_b)(X).$$

Hence we have $\gamma_{h,a} \neq \gamma_{h,b}$, which is equivalent to the desired inequality $i_h([a]) \neq i_h([b])$. ▶

Obviously, the assumptions of the proposition proved above hold for the algebra $A = C^\infty(M)$. Therefore, setting $i_z = i_{h_z}$ for a point $z \in M$, we obtain the following:

9.74 Corollary. *The map $i_z\colon T_z^* M \to |\mathcal{S}_*(C^\infty(M))|$ is injective.* ▶

Combining the maps i_z for all points $z \in M$, we obtain the embedding

$$i\colon T^*M \to |\mathcal{S}_*(C^\infty(M))|, \quad i\big|_{T_z M} = i_z. \tag{9.31}$$

9.75. Let us discuss some other facts useful in our subsequent study of the K-spectrum of the algebra $\mathcal{S}_*(A)$.

Let $\widetilde{h} \in |\mathcal{S}_*(A)|$. Let us identify A with $\mathcal{S}_0(A)$. Then obviously $h \overset{\text{def}}{=} \widetilde{h}\big|_A \in |A|$.

Exercise. Show that $\widetilde{h} \in \operatorname{Im} i_h$ implies $\widetilde{h}\big|_A = h$. Check that the projection $\pi_{T^*}: T^*M \to M$ is the geometrical analogue of the map $\widetilde{h} \mapsto h$ in the case $A = C^\infty(M)$. (In other words, if $\widetilde{h} = h_\theta$, $\theta \in T^*M$, then $h = h_z$, where $z = \pi_{T^*}(\theta)$.)

Note now that if $a \in A$ and $X \in D(A) = \mathcal{S}_1(A)$, then we have $\widetilde{h}(aX) = h(a)\widetilde{h}(X)$. In particular, $\widetilde{h}(aX) = 0$ if $a \in \mu_h$. Therefore, the map of K-modules

$$\widetilde{\widetilde{h}}\colon D(A)/\mu_h D(A) \to K, \quad \widetilde{\widetilde{h}}\,(X \bmod \mu_h D(A)) = \widetilde{h}(X),$$

is well defined. On the other hand, we have the natural map

$$\tau_h\colon D(A)/\mu_h D(A) \to T_h A, \quad X \bmod \mu_h D(A) \mapsto h \circ X.$$

Lemma. *If $A = C^\infty(M)$, then τ_h is an isomorphism of vector spaces over $\mathbb{R}$.*

◀ From the spectrum theorem, Theorem 7.7, it follows that $h = h_z$ for some point $z \in M$, and consequently $h \circ X = X_z$ (see Section 9.52, II). Since any tangent vector $\xi \in T_h A = T_z M$ can obviously be continued to a vector field on M, τ_h is a surjective map. The injectivity of τ_h means that the equality $X_z = 0$ implies $X \in \mu_h D(A) = \mu_z D(M)$. The latter is easily proved using the following fact: If

$$X = \sum_{i=1}^{n} \alpha_i(x) \frac{\partial}{\partial x_i}$$

in a local coordinate system, then $\alpha_i(z) = 0$, i.e., the coefficients α_i belong to μ_z "locally." ▶

The following fact may be used to complete rigorously the proof of the above lemma.

Exercise. Let $X \in D(M)$ be such that $X_z = 0$, $z \in M$. Prove that $X = \sum_{i=1}^{n} f_i X_i + Y$, $f_i \in C^\infty(M)$, $Y, X_i \in D(M)$ where f_i (respectively, X_i) coincides with α_i (respectively, $\partial/\partial x_i$), $i = 1, \ldots, n$, in a neighborhood of z, while Y vanishes in this neighborhood.

We can now completely describe the $\mathbb{R}$-spectrum of the algebra of symbols $\mathcal{S}_*(C^\infty(M))$.

9.76 Theorem. *The map $i \colon T^*M \to |\mathcal{S}_*(C^\infty(M))|$ is an isomorphism; i.e., T^*M is the $\mathbb{R}$-spectrum of the algebra $\mathcal{S}_*(C^\infty(M))$.*

◀ The injectivity of the map i was proved in Corollary 9.74. Let us prove its surjectivity. Suppose that, in the notation of Section 9.75, $A = C^\infty(M)$, $\widetilde{h} \in |\mathcal{S}_*(A)|$, $h = \widetilde{h}\big|_A$, and $z \in M$ is a point such that $h = h_z$. Then $T_h A = T_z M$, and by Lemma 9.75 the map $\widetilde{\widetilde{h}}$ can be understood as an $\mathbb{R}$-linear map from $T_z M$ to $\mathbb{R}$, i.e., as a covector $d_z f \in T^*M$ (see Section 9.22). By the definition of $\widetilde{\widetilde{h}}$, we see that

$$\begin{aligned} \gamma_{h_z, f}(X) &= (h_z \circ \Xi_f)(X) = (h_z \circ X)(f) \\ &= X_z(f) = d_z f(X_z) = \widetilde{\widetilde{h}}(X_z) = \widetilde{h}(X). \end{aligned}$$

Thus, $\widetilde{h}\big|_{\mathcal{S}_1(A)} = \gamma_{h_z, f}\big|_{\mathcal{S}_1(A)}$. The following lemma, whose assumptions hold for the algebra $C^\infty(M)$ due to "partition of unity" (see Lemma 4.18), shows that $\widetilde{h}$ lies in the image of the map i. ▶

9.77 Lemma. *Let a K-algebra A be such that for all natural numbers l any differential operator of order $\leqslant l$ is representable as the sum of monomials of the form $X_1 \circ \cdots \circ X_s$, where $X_i \in D(A)$, $s \leqslant l$. In this case, if $\widetilde{h}_1, \widetilde{h}_2 \in |\mathcal{S}_*(A)|$, $\widetilde{h}_1\big|_A = \widetilde{h}_2\big|_A$ and $\widetilde{h}_1\big|_{\mathcal{S}_1(A)} = \widetilde{h}_2\big|_{\mathcal{S}_1(A)}$, then $\widetilde{h}_1 = \widetilde{h}_2$.*

◀ Passing to symbols of differential operators, we see that the algebra $\mathcal{S}_*(A)$ is generated by its submodule $\mathcal{S}_1(A) = D(A)$; i.e., any symbol

$s = \operatorname{smbl}_k(\delta)$ is represented as the sum of monomials $s_1 \cdots s_k$, where $s_i = \operatorname{smbl}_1 X_j$, $X_j \in D(A)$. Since $\widetilde{h}_i$ is an algebra homomorphism, $h_i(s_1 \cdots s_k) = h_i(s_1) \cdots h_i(s_k)$ and the required equality $\widetilde{h}_1(s) = \widetilde{h}_2(s)$ follows from the fact that by our assumptions, $\widetilde{h}_1(s) = \widetilde{h}_2(s)$ for all $s \in D(A) = \mathcal{S}_1(A)$. ▶

9.78 Exercise. Describe the $\mathbb{R}$-spectrum of the algebra of symbols $\mathcal{S}_*(C^\infty(\mathbf{K}))$ on the cross. Use a reasonable modification of the constructions that allowed us to describe the spectrum of the algebra $\mathcal{S}_*(C^\infty(M))$. By Proposition 9.71, this spectrum can be naturally interpreted as the cotangent space for the cross.

9.79. The algebra of symbols in coordinates. Theorem 9.76 allows one to understand elements of the algebra $\mathcal{S}_*(C^\infty(M))$ as functions on T^*M. Let us describe this interpretation in special coordinates (see Section 9.24).

Let (U, x) be a local chart on M. Then, by Section 9.24, (T^*U, T^*x) is a local chart on T^*M. The localization of differential operators defined on M to the domain U naturally generates the corresponding localization of the algebra $\mathcal{S}_*(C^\infty(M))$. This localization clearly coincides with $\mathcal{S}_*(C^\infty(U))$. Therefore, we can restrict ourselves to the interpretation of its elements as functions on T^*U. We shall use the notation of Section 9.76.

Let $\theta = d_z f \in T^*U$, $f \in C^\infty(U)$, and $\Delta \in \operatorname{Diff}_k C^\infty(M)$. Denote by $\mathfrak{s} = \mathfrak{s}_\Delta$ the function on T^*M corresponding to the symbol $\operatorname{smbl}_k \Delta$. Then, by definition,

$$\mathfrak{s}(\theta) = \gamma_{h_z, f}(\operatorname{smbl}_k(\Delta)).$$

Let us identify, as above, a vector field $X \in D(U)$ with its symbol. It was noted in Section 9.76 that $\gamma_{h_z, f}(X) = X_z(f)$. Therefore,

$$\mathfrak{s}(\theta) = X_z(f).$$

In particular, if $X = \partial/\partial x_i$, then $\mathfrak{s}(\theta) = \partial f/\partial x_i(z)$.

Recall that by the definition of special coordinates (x, p) in T^*U, the coordinate $p_i(\theta)$ is the ith component of the covector θ in the basis $\{d_z x_i\}$ in T_z^*M. Since $\theta = d_z f$ in our case, $p_i(\theta) = \partial f/\partial x_i(z)$ and consequently,

$$\mathfrak{s}_{\partial/\partial x_i} = p_i.$$

Note further that the equality

$$\operatorname{smbl}_{k+l}(\Delta \circ \nabla) = \operatorname{smbl}_k(\Delta) \cdot \operatorname{smbl}_l(\nabla),$$

for $\Delta \in \operatorname{Diff}_k C^\infty(M)$, $\nabla \in \operatorname{Diff}_l C^\infty(M)$, implies $\mathfrak{s}_{(\Delta\circ\nabla)} = \mathfrak{s}_\Delta \cdot \mathfrak{s}_\nabla$. Therefore, if $\Delta = \sum_{|\sigma| \leqslant h} a_\sigma \partial^{|\sigma|}/\partial x^\sigma$, then

$$\mathfrak{s}_\Delta = \sum_{|\sigma|=k} a_\sigma p^\sigma, \text{ where } p^\sigma = p_1^{i_1} \cdots p_n^{i_n} \text{ if } \sigma = (i_1, \ldots, i_n).$$

Thus the algebra $\mathcal{S}_*(C^\infty(U))$ is isomorphic to the algebra of polynomials in the variables $p_1, \ldots, p_n$ with coefficients in the algebra $C^\infty(U)$. From this we immediately obtain the following:

9.80 Proposition. *The algebra $\mathcal{S}_*(C^\infty(M))$ is isomorphic to the subalgebra of the algebra $C^\infty(T^*M)$ consisting of functions whose restrictions to the fibers T_z^*M of the cotangent bundle are polynomials. The algebra $C^\infty(T^*M)$ is isomorphic to the smooth closure of the algebra $\mathcal{S}_*(C^\infty(M))$.*

Exercise. 1. Prove that the restriction of $s_{df}^*\colon C^\infty(T^*M) \to C^\infty(M)$ to the subalgebra $\mathcal{S}_*(C^\infty(M))$ coincides with the map

$$\Xi_f\colon C^\infty(T^*M) \to C^\infty(M)$$

(see Section 9.71).

2. Let A be a K-algebra, $h \in |A|$, and $\mathcal{S}_+(A) = \sum_{i>0} \mathcal{S}_i(A)$. Then the map

$$\bar{h}\colon \mathcal{S}_*(A) \to K, \quad \bar{h}\big|_A = h, \quad \bar{h}\big|_{\mathcal{S}_+(A)} = 0,$$

is a K-algebra homomorphism, i.e., $\bar{h} \in |\mathcal{S}_*(A)|$. Show that for $A = C^\infty(M)$ the map $|A| \to |\mathcal{S}_*(A)|$, $h \mapsto \bar{h}$, coincides with s_{df} for $f \equiv 0$ (the canonical embedding of M into T^*M).

3. Find an analogue of the map s_{df} for arbitrary K-algebras.

4. Describe the algebra of symbols $\mathcal{S}_*(C^\infty(\mathbf{K}))$ and realize it as the algebra of functions on the spectrum $|\mathcal{S}_*(C^\infty(\mathbf{K}))|$.

9.81. Hamiltonian formalism in T^*M and $|\mathcal{S}_*(A)|$. Now consider the case $A = C^\infty(M)$; let us describe, in special coordinates, the bracket $\{\cdot,\cdot\}$ introduced in Section 9.70. By the skew-symmetry of this bracket and because of the relation

$$\{\mathfrak{s}, \mathfrak{s}_1\mathfrak{s}_2\} = \{\mathfrak{s}, \mathfrak{s}_1\}\mathfrak{s}_2 + \mathfrak{s}_1\{\mathfrak{s}, \mathfrak{s}_2\} \tag{9.32}$$

(see the exercise from Section 9.70), it suffices to compute this bracket for the coordinate functions. Using the notation of the previous section, we have by definition

$$\{\mathfrak{s}_\Delta, \mathfrak{s}_\nabla\} = \mathfrak{s}_{[\Delta,\nabla]}. \tag{9.33}$$

Since $[f,g] = [g,f] = 0$ for $f, g \in C^\infty(U)$,

$$\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0, \quad \left[\frac{\partial}{\partial x_i}, f\right] = \frac{\partial f}{\partial x_i},$$

we have by (9.33)

$$\{f(x), g(x)\} = 0, \quad \{p_i, p_j\} = 0, \quad \{p_i, f(x)\} = \frac{\partial f(x)}{\partial x_i}. \tag{9.34}$$

Further, applying (9.32) to $F = f(x)p^\sigma$, $G = g(x)p^\tau$, we have

$$\begin{aligned}\{F, G\} &= \{f(x)p^\sigma, g(x)p^\tau\} \\ &= \{f(x), g(x)\}p^\sigma p^\tau + \{f(x), p^\tau\}g(x)p^\sigma \\ &\quad + \{p^\sigma, g(x)\}f(x)p^\tau + \{p^\sigma, p^\tau\}f(x)g(x) \\ &= \{p^\sigma, g(x)\}f(x)p^\tau - \{p^\tau, f(x)\}g(x)p^\sigma.\end{aligned}$$

By the last equality in (9.34),

$$\{p^\sigma, g(x)\} = \sum_{i=1}^n \frac{\partial p^\sigma}{\partial p_i}\frac{\partial g}{\partial x_i}, \quad \{p^\tau, f(x)\} = \sum_{i=1}^n \frac{\partial p^\tau}{\partial p_i}\frac{\partial f}{\partial x_i}.$$

As a result of these computations, we finally obtain the formula:

$$\{F, G\} = \sum_{i=1}^n \left(\frac{\partial F}{\partial p_i}\frac{\partial G}{\partial x_i} - \frac{\partial G}{\partial p_i}\frac{\partial F}{\partial x_i}\right), \tag{9.35}$$

which is the standard Poisson bracket on T^*M. Moreover, this formula shows that the derivation $X_F \stackrel{\text{def}}{=} \{F, \cdot\}$ of the algebra of symbols (which is, geometrically, a vector field on its $\mathbb{R}$-spectrum, i.e., on T^*M) is of the form

$$X_F = \sum_{i=1}^n \left(\frac{\partial F}{\partial p_i}\frac{\partial}{\partial x_i} - \frac{\partial F}{\partial x_i}\frac{\partial}{\partial p_i}\right). \tag{9.36}$$

Thus X_F is the Hamiltonian vector field on T^*M with the Hamiltonian F. This fact justifies calling the bracket $\{\cdot, \cdot\}$ on the algebra of symbols $\mathcal{S}_*(A)$ of an arbitrary K-algebra A the *Poisson bracket*, while derivations $\{\mathfrak{s}, \cdot\}$, $\mathfrak{s} \in \mathcal{S}_*(A)$, are naturally called *Hamiltonian vector fields* on $|\mathcal{S}_*(A)|$. This is another evidence in favor of treating differential calculus as a part of commutative algebra (and this treatment is a consequence of the observability principle). The reader can now enjoy constructing Hamiltonian mechanics on smooth sets or over arithmetic fields.

Exercise. 1. Let $F = F(x, p) \in C^\infty(T^*M)$. Check that to find functions satisfying the condition $s_{df}^*(F) = 0$, $f \in C^\infty(M)$, is equivalent to solving the Hamilton–Jacobi equations. Find analogue of these equations for an arbitrary K-algebra A.

2. Describe Hamiltonian mechanics on the cross $\mathbf{K}$.

9.82. Thus we see that the differential calculus is a natural consequence of the classical observability principle and is developed simply and naturally if one keeps this fact in mind. The commutativity of the algebra of observables is a formalization of the fundamental idea of classical physics: the independence of observations. A more sophisticated realization of this idea by means of commutative graded algebras (traditionally called *superalgebras*) does not involve overcoming any additional difficulties. All definitions and constructions of the differential calculus are carried over to this case and to any other case in which commutativity can be treated in a reasonable way.

As is known, in quantum physics one has to reject the principle of independence of observations. Nevertheless, it would not be right to try to quantize the differential calculus in order to describe quantum phenomena by a simple change of commutative algebras to noncommutative ones. The reader will see this by trying systematically to carry over the constructions of this chapter to noncommutative algebras. The failure of such attempts becomes really catastrophic when one tries to repeat subtler and deeper constructions in the noncommutative situation. These and many other reasons show that it is hardly possible to obtain a mathematically adequate quantum principle of observability by using the language of noncommutative algebra (or noncommutative geometry).

There are serious reasons to believe now that this aim can be reached in a natural way by using the language of the *secondary differential calculus*, which is a sort of synthesis of the usual (= primary) differential calculus with homological algebra. In any case, there is no doubt that this calculus is the natural language for the geometry of nonlinear partial differential equations.

10
Smooth Bundles

10.1. Inner structure of the point. The concept of observability developed in this book assumes that a point is an elementary object that can be individualized with the help of a given set of instruments, i.e., a given algebra of observables A. We know, however, that points of the physical manifold where we live may have *inner parameters*, such as temperature, color, and humidity. To give a precise mathematical meaning to this phrase, we need the notion of fibering, which is the main protagonist of the present chapter.

A priori, there are two possibilities: The concept of inner structure can be either *relative* or *absolute*. The inner structure is relative if it can be expressed within the classical framework by simply adding new instruments. The mathematical meaning of this construction is that the algebra of observables A is extended to a bigger algebra B by means of the inclusion $i\colon A \hookrightarrow B$. The inner structure of a point $z \in |A|$ is then described by its inverse image $|i|^{-1}(z) \subset |B|$ under the map of $\mathbb{R}$-spectra $|i|: |B| \to |A|$ (see Section 3.19).

Example. The 3-dimensional world $\mathbb{R}^3$ can be made colored if to the set of instruments measuring a point's coordinates, we add one more instrument measuring the color, i.e., the frequency of electromagnetic waves. In algebraic language, this means that we pass from the algebra $A = C^\infty\left(\mathbb{R}^3\right)$ to the algebra $B = A \otimes_{\mathbb{R}} C$, where C is the algebra of smooth functions on the "manifold of colors," which is identified naturally with $\mathbb{R}^1_+$. The inclusion $i\colon A \hookrightarrow B$ is defined by the rule

$$A \ni a \mapsto a \otimes 1_C \in A \otimes_{\mathbb{R}} C.$$

Returning to the general case, note that $|i|^{-1}(z) = |B_z|$, where $B_z = B/(\mu_z \cdot B)$. The inner structure of a point $z \in |A|$ is thus observable by means of the algebra B_z. The assumption that all points $z \in |A|$ have the same inner structure means that all algebras of additional observables B_z are the same, i.e., isomorphic to each other. If this condition is fulfilled in a certain regular manner (see Section 10.9), then the map $|i|\colon |B| \to |A|$ is referred to as a *locally trivial smooth bundle*. In the above example this condition holds, and all algebras B_z are isomorphic to C.

At first glance, relative inner structures do not add anything new to the classical scheme of observability, because any such structure can be reduced to a standard one through an appropriate extension of the algebra A. This approach, however, is not convenient if the manifold $M = |A|$ is considered as a display that shows the points with different inner structures. For example, this is the case for real physical space. Moreover, the problem ceases to be a question of mere convenience if the inner structures of the points displayed on M are *absolute* in the sense that they cannot be described by the above classical approach. In particular, this is true of quantum phenomena that have survived many unsuccessful attempts of explanation based on so-called "latent parameters."

Unless otherwise specified, all algebras in this chapter are assumed to be smooth.

10.2. Fibering as an algebra extension. Before going on to bundles, we shall introduce the more general notion of a fibering. In algebraic terms it can be defined as follows.

Definition. A *smooth fibering* is an injective homomorphism of smooth algebras $i\colon A \hookrightarrow B$. The manifold $|A|$ is called the *base* of the fibering i, the manifold $|B|$ is its *total space*, while the map $|i|\colon |B| \to |A|$ is referred to as the *projection* of the fibering.

10.3. Examples. I. **Product fibering.** Let A and C be smooth algebras. Set $B = \overline{A \otimes_{\mathbb{R}} C}$ (we recall that the bar stands for the smooth envelope of an algebra; see Section 3.36) and define the inclusion $i\colon A \hookrightarrow B$ by the rule $a \mapsto a \otimes 1$. A concrete example of this construction is the fibering of the torus over the circle, defined as the extension $i\colon A \hookrightarrow B$, where

$$B = \{g \in C^\infty(\mathbb{R}^2) \mid g(x+1, y) = g(x, y+1) = g(x, y)\}$$

is the algebra of twice periodic functions in two variables and A is the subalgebra of B consisting of all functions that do not depend on y. Indeed, if

$$A = \{f \in C^\infty(\mathbb{R}) \mid f(x+1) = f(x)\}, \tag{10.1}$$

$$C = \{f \in C^\infty(\mathbb{R}) \mid f(y+1) = f(y)\}, \tag{10.2}$$

then it is readily verified that $\overline{A \otimes C} \cong B$.

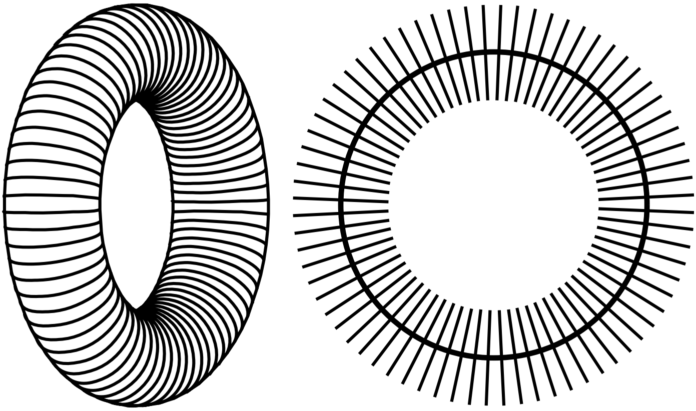


Figure 10.1. Product fibering.

II. **The Klein bottle fibered over the circle** (see Figure 10.2). This is the inclusion of the algebra of smooth periodic functions on the real line

$$A = \{f \in C^\infty(\mathbb{R}) \mid f(x+1) = f(x)\}$$

into the algebra

$$B = \{g \in C^\infty(\mathbb{R}^2) \mid g(x+1, y) = -g(x, y+1) = g(x, y)\}$$

by the rule $f \mapsto g\colon g(x, y) = f(x)$.

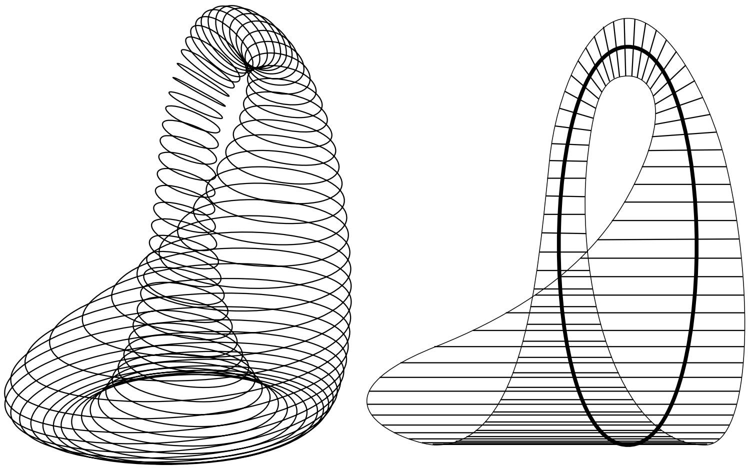


Figure 10.2. Non-trivial fibering.

III. **The two-sheeted covering of the circle.** This is the map of the algebra

$$A = \{f \in C^\infty(\mathbb{R}) \mid f(x+1) = f(x)\}$$

into itself defined by the formula $f \mapsto g\colon g(x) = f(2x)$.

IV. **One-sheeted fibering of the line over the circle.** Let $B = C^\infty(\mathbb{R})$ and let $A \subset B$ consist of all functions $f \in B$ for which the function $x \mapsto f(1/x)$ and all of its derivatives have finite limits as $x \to 0$. One can see that $A \cong C^\infty(S^1)$, and under a proper choice of this isomorphism, the inclusion $A \hookrightarrow B$ corresponds to the map

$$\mathbb{R} \to S^1 = \big\{(x,y) \mid x^2 + y^2 = 1\big\}: \quad t \mapsto \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right).$$

The image of this map is the entire circle with one point removed.

V. In Section 9.18 we described the map

$$\pi_T\colon TM \to M, \quad (z,\xi) \mapsto z,$$

from the tangent space TM of the manifold M onto the manifold itself. The corresponding homomorphism of smooth algebras $C^\infty(M) \to C^\infty(TM)$ is injective; therefore, π_T can be regarded as the projection of the fibering.

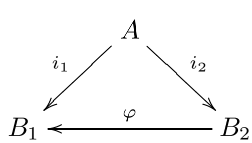
The most important class of fiberings consists of *bundles* (see Sections 10.9 and 10.10), defined as locally trivial fiberings. In the previous list, examples I–III, V possess this property, while example IV does not. To give a precise definition of local triviality, we need the notion of fiber and the procedure of localization.

10.4. Fiber of a fibering. Geometrically, the fiber of a given fibering $i\colon A \to B$ over the point $z \in |A|$ is the inverse image of z under the projection $|i|\colon |B| \to |A|$.

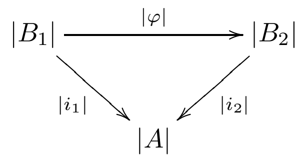
In examples I–II the fiber over any point is a circle; in example III it is two points; in example IV, depending on the choice of the point z, the fiber is either empty or consists of one point. Finally, in example V, the fiber T_zM is isomorphic to the linear space $\mathbb{R}^n$. Note that in examples I and II both the base space and the fiber of both fiberings are the same, whereas the total spaces are different.

An algebraic definition of the *fiber over a point* $a \in |A|$ can be given as follows: It is the quotient algebra of B over the ideal generated by the set $i(\mu_a)$, where $\mu_a \subset A$ is the maximal ideal of the point a.

10.5. The category of fiberings. By definition, a morphism of a fibering $i_1: A \to B_1$ into a fibering $i_2\colon A \to B_2$ is an algebra homomorphism $\varphi\colon B_2 \to B_1$ making commutative the diagram



An equivalent definition in terms of spectra (see Sections 3.4 and 8.6) reads that the diagram



commutes. This means that the map $|\varphi|$ takes the fibers of one fibering into the fibers of another: $|\varphi|(|i_1|^{-1}(a)) \subset |i_2|^{-1}(a)$, or, equivalently, that $\varphi(i_2(\mu_a) \cdot B_2) \subset i_1(\mu_a) \cdot B_1$ for any point $a \in |A|$.

The totality of all fiberings over a smooth algebra $A = C^\infty(M)$ together with all morphisms between them constitutes the *category of fiberings over* M.

If the homomorphism φ is an isomorphism, or, which is the same thing, the map $|\varphi|$ is a diffeomorphism, then the fiberings i_1 and i_2 are said to be *equivalent*. In the case where the homomorphism φ is surjective, which corresponds to a proper embedding of manifolds $|B_1| \to |B_2|$, the fibering i_1 is called a *subfibering* of the fibering $|i_2|$.

The simplest example of a fibering with base M and fiber F is provided by the direct product $M \times F$ with the natural projection on the first factor, or, in algebraic terms, the natural inclusion of the algebra $A = C^\infty(M)$ into the smooth envelope of the tensor product $\overline{A \otimes C}$, where $C = C^\infty(F)$.

A fibering equivalent (in the category of fiberings) to a fibering of this kind is referred to as a *trivial fibering*. A *bundle* is a fibering that is *locally* isomorphic to a trivial fibering. This phrase will become an exact definition after we have explained the meaning of the word "locally."

10.6. Localization. The aim of this section is to describe an algebraic construction that allows one to define the restriction of a smooth algebra to an open set (see Section 3.23) in purely algebraic terms.

Let A be a commutative ring with unit and let $S \subset A$ be a *multiplicative set*, i.e., a subset of A, containing 1, not containing 0, and closed with respect to multiplication. In the set of all pairs (a, s), where $a \in A$, $s \in S$, we introduce the equivalence relation

$$(a_1, s_1) \sim (a_2, s_2) \overset{\text{def}}{\iff} \exists s \in S \colon s(a_1 s_2 - a_2 s_1) = 0.$$

The equivalence class of a pair (a, s) is denoted by $\frac{a}{s}$ (or a/s) and called a *formal fraction*; we denote the set of all such classes by $S^{-1}A$. The sum and product of formal fractions are defined by the ordinary formulas

$$\frac{a_1}{s_1} \cdot \frac{a_2}{s_2} = \frac{a_1 a_2}{s_1 s_2}, \qquad \frac{a_1}{s_1} + \frac{a_2}{s_2} = \frac{a_1 s_2 + a_2 s_1}{s_1 s_2}.$$

The resulting ring $S^{-1}A$ is referred to as the *localization of the ring A over the multiplicative system S*. We leave the straightforward checks to the reader.

There is a *canonical homomorphism* $\iota\colon A \to S^{-1}A$ defined by $\iota(a) = a/1$. In general, ι is neither injective nor surjective.

Suppose now that P is a module over A. In the same way as above, in the set of pairs (p, s) with $p \in P$, $s \in S$, we can introduce the equivalence relation

$$(p_1, s_1) \sim (p_2, s_2) \overset{\text{def}}{\Longleftrightarrow} \exists s \in S\colon\ s(s_2 p_1 - s_1 p_2) = 0.$$

A *formal fraction* $\frac{p}{s}$ (or p/s) is the equivalence class of the pair (p, s). The set of all such classes, denoted by $S^{-1}P$, is referred to as the *localization of P over S*.

Exercises. 1. Introduce the addition of two elements of $S^{-1}P$ and the multiplication of an element of $S^{-1}P$ by an element of $S^{-1}A$ as follows:

$$\frac{p_1}{s_1} + \frac{p_2}{s_2} = \frac{s_2 p_1 + s_1 p_2}{s_1 s_2}, \qquad \frac{a_1}{s_1} \cdot \frac{p_2}{s_2} = \frac{a_1 p_2}{s_1 s_2}.$$

Verify that these operations are well defined and turn the set $S^{-1}P$ into an $(S^{-1}A)$-module.

2. Let $\varphi\colon P \to Q$ be a homomorphism of A-modules. Prove that the map $S^{-1}(\varphi)\colon S^{-1}P \to S^{-1}Q$, given by the formula

$$S^{-1}(\varphi)\left(\frac{p}{s}\right) \overset{\text{def}}{=} \frac{\varphi(p)}{s}, \quad p \in P,\ s \in S,$$

is well defined and represents a homomorphism of $(S^{-1}A)$-modules.

Summarizing, we can say that for a given multiplicative set $S \subset A$ we have defined a functor from the category of A-modules into the category of $(S^{-1}A)$-modules.

Examples. I. If A has no zero divisors and $S = A \setminus \{0\}$, then $S^{-1}A$ is the quotient field of A.

II. Let $A = \mathbb{Z}$ and let S be the set of all nonnegative powers of 10. Then $S^{-1}A$ consists of all rational numbers that have a finite decimal representation.

III. If M is a smooth manifold, $A = C^\infty(M)$, $x \in M$, and $S = A \setminus \mu_x$, then $S^{-1}A$ is the *ring of germs of smooth functions* on M at the point x (readers who are familiar with the notion of germ may prove this fact as an exercise; others may take it as the definition and try to understand its geometrical meaning).

10.7 Proposition. *Let U be an open subset of the manifold M,*

$$A = C^\infty(M), \quad \textit{and} \quad S = \{f \in A \mid f(x) \neq 0\ \forall x \in U\}.$$

Then $S^{-1}A \cong C^\infty(U)$, and the canonical homomorphism $\iota\colon C^\infty(M) \to C^\infty(U)$ coincides with the restriction $\iota(f) = f\big|_U$.

◀ To prove this fact, consider the map $\alpha\colon S^{-1}A \to C^\infty(U)$,

$$\alpha\left(\frac{f}{s}\right)(x) = \frac{f(x)}{s(x)}, \quad f \in A,\ s \in S,\ x \in U,$$

which converts a formal fraction into the ordinary quotient of two functions. This map is well defined, because the functions $s \in S$ do not vanish anywhere in U. Suppose that $\alpha(f_1/s_1) = \alpha(f_2/s_2)$. Then the function $f_1s_2 - f_2s_1$ is identically zero on U. By Lemma II of Section 4.17, there is a function $s \in S$ that is identically zero outside of U. The product of these two functions is identically zero on all of M, i.e., $s(f_1s_2 - f_2s_1) = 0$ as an element of the algebra A. By definition, the two formal fractions f_1/s_1 and f_2/s_2 are equal. This proves that α is injective. The fact that it is surjective follows from the next lemma. ▶

Exercise. Give an example of a geometric algebra A that is not smooth and of an open subset $U \subset |A|$ such that the algebras $S^{-1}A$ and $A|_U$ are not isomorphic.

10.8 Lemma. *Suppose that U is an open subset of the manifold M and $f \in C^\infty(U)$. Then there exists a function $g \in C^\infty(M)$, having no zeros on U, such that the product fg can be smoothly extended on all of the manifold M and therefore $f = \alpha(fg/g)$.*

◀ A rigorous proof of this fact can be obtained by the techniques of partition of unity described in Chapter 2. We leave the details to the reader. ▶

We are now in a position to give a precise algebraic definition of a bundle.

Recall that a homomorphism of K-algebras $\varphi\colon A \to B$ gives rise to the operation of the *change of rings*: Any B-module R can be regarded as an A-module with multiplication $a \cdot r \stackrel{\text{def}}{=} \varphi(a)r,\ a \in A,\ r \in R$. In particular, the algebra B itself can be regarded as an A-module. Therefore, for a given multiplicative set $S \subset A$ the $(S^{-1}A)$-module $S^{-1}B$ is defined.

10.9. Definition. Let A, B, and F be smooth algebras. An injective homomorphism $i\colon A \to B$ is called a *bundle $|B|$ over $|A|$ with fiber $|F|$* if every point $z \in |A|$ has an open neighborhood $U_z \subset |A|$ over which the localization of i is equivalent to the product fibering

$$S_z^{-1}A \to \overline{S_z^{-1}A \otimes F},$$

where $S_z \subset A$ is the multiplicative system of U_z, i.e., the set of all elements of A whose values at the points of U_z are nonzero. More exactly, there must exist an algebra isomorphism $p\colon S_z^{-1}B \to \overline{S_z^{-1}A \otimes F}$ that makes commutative the triangle

$$\begin{array}{ccc} S_z^{-1}A & \xrightarrow{S_z^{-1}(i)} & S_z^{-1}B \\ & {\scriptstyle j}\searrow \quad \swarrow{\scriptstyle p} & \\ & \overline{S_z^{-1}A \otimes F} & \end{array}$$

where j is the map taking every element a to $a \otimes 1$. (Recall once again that the bar over the notation of an algebra means that we take its smooth envelope.)

A bundle π is *trivial* if and only if the previous condition (*local triviality axiom*) is fulfilled for $U_z = M$.

The geometric definition of a bundle, which follows, is simpler. The equivalence of the two definitions is quite obvious.

10.10. Definition. Let E and M be smooth manifolds. A smooth map $\pi\colon E \to M$ is said to be a *fiber bundle*, or *bundle* for short, if for a certain manifold F the following condition holds: Any point $x \in M$ has a neighborhood $U \subset M$ for which there exists a *trivializing* diffeomorphism $\varphi\colon \pi^{-1}(U) \to U \times F$ that closes the commutative diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\varphi} & U \times F \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle p} & \\ & U & \end{array}$$

where p is the projection of the product on the first factor.

Under these conditions M, F, and E are referred to as the *base*, the *fiber*, and the *total space* of the bundle π, respectively. The whole thing is conventionally written as $E \xrightarrow{F} M$. The total space of the bundle π is usually denoted by E_π.

The fiber of bundle π *over a point* $x \in M$ is the set $\pi_x = \pi^{-1}(x)$. The fiber over any point, equipped with the structure of the submanifold of E, is diffeomorphic to the fixed "outer" fiber F.

10.11. Some more examples. In the examples that follow, we give only the geometric construction of the bundle, leaving it to the reader to describe the corresponding extension of smooth algebras.

I. The bundle of the line over the circle (cf. Section 6.2). Representing the circle S^1 as the set of complex numbers of modulus 1, we define the map $\mathbb{R}^1 \to S^1$ by $t \mapsto e^{it}$. Any interval containing a given point $x \in S^1$ can be taken as the neighborhood U in the condition of local triviality.

The fiber in this example is the zero-dimensional manifold $\mathbb{Z}$. Bundles with zero-dimensional fibers are called *coverings*.

II. Another example of a covering is provided by the map $S^n \to \mathbb{R}P^n$ that assigns to a point $x \in S^n \subset \mathbb{R}^n$ the line in $\mathbb{R}^{n+1}$ passing through that point and the origin (see the definitions in Section 5.10). This bundle is trivial over the complement to any hyperplane $\mathbb{R}P^n \setminus \mathbb{R}P^{n-1}$. Its fiber consists of two points.

III. The open Möbius band is a *line bundle* over the circle: $M \xrightarrow{\mathbb{R}} S^1$. If M is viewed as the band $[0,1] \times \mathbb{R} \subset \mathbb{R}^2$ with the points $(0, y)$ and $(1, -y)$ identified for any $y \in \mathbb{R}$, and the circle S^1 is viewed as the segment $[0,1]$ with identified endpoints 0 and 1, then the projection $\pi\colon M \to S^1$ is simply $\pi(x, y) = x$. A visual representation of this bundle is given in Figure 6.2, where the Möbius band embedded into $\mathbb{R}^3$ squeezes to its middle line; the fibers are the segments perpendicular to the middle line.

Let us check local triviality in this example. If $a \in S^1$ is an inner point of the segment $[0,1]$, then for the neighborhood U we can take the interval $]0,1[$. If $a = 0$, then we put $U = \{x \in S^1 \mid x \neq \frac{1}{2}\}$ and define the diffeomorphism $\varphi\colon \pi^{-1}(U) \to U \times \mathbb{R}$ as follows:

$$\varphi(x, y) = \begin{cases} (x, y), & \text{if } x < \frac{1}{2}, \\ (x, -y), & \text{if } x > \frac{1}{2}. \end{cases}$$

IV. The bundle of unit tangent vectors to the sphere $\pi\colon T_1S^2 \to S^2$. The space of this bundle

$$T_1S^2 = \{(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid |x| = 1, |y| = 1, x \perp y\}$$

is a submanifold of $\mathbb{R}^6$. Making the orthogonal group act on a fixed unit vector to the sphere, we obtain a diffeomorphism $T_1S^2 \cong \mathrm{SO}(3)$. The total space of the bundle under study thus provides another (fifth) realization of the manifold considered in the beginning of the book in Examples 1.1–1.4. The fiber is the circle S^1.

To prove local triviality, we shall show that this bundle is trivial over any open hemisphere S^2. Indeed, if U is a hemisphere and S^1 its boundary, then we can identify the points of U and S^1 with the vectors drawn from the center of the sphere and set $\varphi(x, z) = x \times z$ (cross product of vectors) for $x \in U$ and $z \in S^1$. The two vectors $x \in U$ and $z \in S^1$ are never collinear; hence the map $\varphi\colon U \times S^1 \to \pi^{-1}(U)$ is a diffeomorphism.

V. The composition of the maps $S^3 \to \mathbb{R}P^3$ and $\mathbb{R}P^3 \to S^2$ defined in Examples II and IV above is a bundle of S^3 over S^2 with fiber S^1, called the *Hopf fibration*, compare with Section 6.17, II. We leave it to the reader to check its local triviality.

VI. *Tautological bundle over a Grassmannian.* Suppose $G_{n,k}$ is the Grassmann manifold (see Example IV in Section 5.10) whose points are k-dimensional linear subspaces of the n-dimensional space $\mathbb{R}^n$; let $E_{n,k}$ be the set of all pairs (x, L) such that $x \in L \in G_{n,k}$; $E_{n,k}$ is viewed as a submanifold in $\mathbb{R}^n \times G_{n,k}$. The correspondence $(x, L) \mapsto L$ defines a fibering

$$\Theta = \Theta_{n,k} : E_{n,k} \to G_{n,k},$$

called the *tautological bundle.*

Exercise. Show that for $k = 1$ the tautological bundle $E_{n,1} \to G_{n,1} = \mathbb{R}P^{n-1}$ can be described as the projection

$$\pi\colon \mathbb{R}P^n \setminus \{L_0\} \to \mathbb{R}P^{n-1},$$

where L_0 is the $(n+1)$th coordinate axis in $\mathbb{R}^{n+1}$ (= "vertical" line), and π assigns to each "slanted" line its projection to $\mathbb{R}^n \cong \{x_{n+1} = 0\}$. In particular, if $k = 1$ and $n = 2$, we obtain the fibering of the Möbius band over the circle from Example III.

Let us prove that the tautological fibering is a bundle, i.e., possesses the property of local triviality. We shall use the covering of the manifold $G_{n,k}$ by the family of open sets

$$U_I, \quad \text{where } I = \{i_1, \dots, i_k\},\ 1 \leqslant i_1 < \dots < i_k \leqslant n.$$

By definition, the neighborhood U_I consists of all k-planes in $\mathbb{R}^n$ that do not degenerate under the projection on $\mathbb{R}^k_I$ along $\mathbb{R}^{n-k}_{\overline{I}}$, where $\overline{I} = \{1, \dots, n\} \setminus I$ and the symbol $\mathbb{R}^m_J$, $J = \{j_1, \dots, j_m\}$, stands for the m-plane in $\mathbb{R}^n$ spanned by the basic vectors numbered $j_1, \dots, j_m$. If $x \in L$ and $L \in U_I$, then to the pair $(x, L) \in E_{n,k}$ we assign the pair $(\overline{x}, L)$, where $\overline{x} \in \mathbb{R}^k_I \cong \mathbb{R}^k$ is the projection of x along $\mathbb{R}^{n-k}_{\overline{I}}$ onto $\mathbb{R}^k_I$. This assignment is a trivializing diffeomorphism for the tautological fibering over the set U_I. This is why $\Theta_{n,k}$ is in fact a bundle.

All bundles listed above are nontrivial. For example, the Möbius band is nonorientable and therefore not diffeomorphic to the cylinder $S^1 \times \mathbb{R}$. The nontriviality of the bundle of unit tangent vectors (Example IV) can be proved by using the basic facts about fundamental groups. In fact, the manifolds $\mathbb{R}P^3$ and $S^2 \times S^1$ are not diffeomorphic, because their fundamental groups are different: $\pi_1(\mathbb{R}P^3) = \mathbb{Z}_2$, $\pi_1(S^2 \times S^1) = \mathbb{Z}$. The same argument shows the nontriviality of the Hopf fibration (Example V).

VII. *Tangent bundle* $\pi_T\colon TM \to M$ (see Section 9.19). If (U, x) is a chart on M, then the corresponding trivializing diffeomorphism $U \times \mathbb{R}^n \to \pi_T^{-1}(U)$ is the composition of natural identifications

$$U \times \mathbb{R}^n \longleftrightarrow TU \quad \text{and} \quad TU \longleftrightarrow \pi_T^{-1}(U),$$

described in Section 9.18 II and IV:

$$(z, q) \mapsto \sum_{i=1}^{n} q_i \frac{\partial}{\partial x_i}\Big|_z \in T_z M \subset TU,$$

where $q = (q_1, \dots, q_n)$.

Depending on the manifold M, its tangent bundle can be either trivial or nontrivial. A manifold with a trivial tangent bundle is called *parallelizable*. For example, any Lie group is parallelizable.

Exercise. Among the examples of manifolds considered earlier in this book, find some that are parallelizable and some that are not.

VIII. The *cotangent bundle* $\pi_{T^*}: T^*M \to M$ (Section 9.24). Just as in the previous case, the required trivialization $U \times \mathbb{R}^n \to \pi_{T^*}^{-1}(U)$ can be obtained from the identifications

$$U \times \mathbb{R}^n \longleftrightarrow T^*U \quad \text{and} \quad T^*U \longleftrightarrow \pi_{T^*}^{-1}(U),$$

described in Section 9.24:

$$(z,p) \mapsto \sum_{i=1}^{n} p_i d_z x_i \in T_z^* M \subset T^*U, \quad \text{where } p = (p_1, \dots, p_n).$$

IX. *The bundle of l-jets of functions* $\pi_{J^l}: J^l M \to M$. In the chart (U,x) on M, the trivializing map $U \times \mathbb{R}^N \to \pi_{J^l}^{-1}(U) = J^l(U)$, where N is the total number of different derivatives of order $\le l$,

$$\big(z, \mathbf{p}^l\big) \mapsto [f_{\mathbf{p}^l}]_z^l \in J_z^l M \subset J^l U, \quad \text{where } f_{\mathbf{p}^l} = \sum_{\sigma \le l} \frac{1}{\sigma!} p_\sigma (x-z)^\sigma,$$

and $\mathbf{p}^l = (p_\sigma)$ is the vector with components p_σ, $|\sigma| \le l$, arranged in the lexicographic order of the subscripts. Formula (2.4) shows that l-jets of functions $f_{\mathbf{p}_l}$ at the point z exhaust $J_z M$. Functions x_i, $i = 1, \dots, n$, and p_σ, $|\sigma| \le l$, constitute a local coordinate system in $\pi_{J^l}^{-1}(U)$, and such special charts form an atlas of $J^l M$.

10.12. Sections. A *section* of the bundle $\pi: E \to M$ is a smooth map $s: M \to E$ that assigns to every point $x \in M$ an element of the fiber over this point: $s(x) \in \pi_x$. In other words, the condition is that $\pi \circ s = \mathrm{id}_M$. The set of all sections of the bundle π is denoted by $\Gamma(\pi)$.

In algebraic language, a section of the bundle $i: A \hookrightarrow B$ is represented by an algebra homomorphism $\sigma: B \to A$, left inverse to i, i.e., such that $\sigma \circ i = \mathrm{id}_A$.

Examples. I. The bundle $\pi: \mathbb{R}^1 \to S^1$, described in Example 10.13, I, has no sections. Indeed, suppose that $f: S^1 \to \mathbb{R}^1$ is a smooth map and $\pi \circ f = \mathrm{id}_{S^1}$. The last equality implies that the restriction $\pi\big|_{f(S^1)}: f(S^1) \to S^1$ is a diffeomorphism of the set $f(S^1) \subset \mathbb{R}^1$ onto S^1. However, the image of a continuous map $f: S^1 \to \mathbb{R}^1$ is a certain segment $[a,b] \subset \mathbb{R}$ and thus cannot be homeomorphic to the circle.

II. The bundle $T_1 S^2 \to S^2$ (Example 10.13, IV) has no sections. This fact is sometimes referred to as the *hedgehog theorem.* Indeed, the existence of a section $f: S^2 \to T_1 S^2$ would lead to the following construction of a diffeomorphism $\varphi: S^2 \times S^1 \to T_1 S^2$: Put $\varphi(x, \alpha)$ equal to the vector, obtained from $f(x)$ by a rotation through angle α in a fixed direction (say, counterclockwise, if the sphere is viewed from the outside).

III. The sections of the trivial bundle over M with fiber F are in one-to-one correspondence with smooth maps from M to F.

IV. The sections of the tangent bundle over M are naturally interpreted as vector fields on M (see Section 9.40). In a similar way, the sections of the cotangent bundle over M are naturally associated with the first-order differential forms, or 1-forms. This notion is introduced and discussed below in Sections 11.41–11.44. A similar situation takes place for the jet bundles as well; see Sections 9.65, 11.46–11.47.

Exercise. Describe the sections s_X, s_{df}, and $s_{j_l(f)}$ of the tangent, cotangent, and l-jet bundles (see 9.40, 9.25, and 9.65, respectively) in terms of special local coordinates.

V. This example concerns some remarkable sections of the tautological bundle over the Grassmannian $\Theta_{n,k}\colon E_{n,k} \to G_{n,k}$ (see Example VI in Section 10.11). Let $\mathfrak{M}_{k,n}$ be the space of $(k \times n)$ matrices of rank k. If $J = (j_1, \dots, j_k)$, $1 \le j_1 < \dots < j_k \le n$, and $\mathcal{M} \in \mathfrak{M}_{k,n}$, then $\mathcal{M}_J$ denotes the $(k \times k)$ matrix formed by the columns of $\mathcal{M}$ with numbers $j_1, \dots, j_k$. At least one of the minors $|\mathcal{M}_J|$ in the matrix $\mathcal{M}$ is different from zero; therefore, $\sum_J |\mathcal{M}_J|^2 > 0$.

Fix a multi-index $I = (i_1, \dots, i_k)$ and consider the following function on $\mathfrak{M}_{k,n}$:

$$\nu_I(\mathcal{M}) = \frac{|\mathcal{M}_I|}{\sum_J |\mathcal{M}_J|^2}.$$

Obviously, $\nu_I \in C^\infty(\mathfrak{M}_{k,n})$, and for any $g \in \mathrm{GL}(k, \mathbb{R})$ we have

$$\nu_I(g\mathcal{M}) = |g|^{-1}\nu_I(\mathcal{M}). \tag{10.3}$$

Further, let $\widetilde{\mathcal{M}}_I$ be the adjoint to the matrix $\mathcal{M}_I$, i.e., one formed by the minors of order $k-1$ of $\mathcal{M}_I$. Denote by $\mathrm{Mat}_{k,n}$ the space of all $(k \times n)$ matrices over $\mathbb{R}$. The map

$$m_I\colon \mathfrak{M}_{k,n} \to \mathrm{Mat}_{k,n}, \quad m_I(\mathcal{M}) = \nu_I(\mathcal{M})\widetilde{\mathcal{M}}_I\mathcal{M}, \tag{10.4}$$

is $\mathrm{GL}(k,\mathbb{R})$-*equivariant*, i.e., satisfies

$$m_I(g\mathcal{M}) = m_I(\mathcal{M}), \quad g \in \mathrm{GL}(k, \mathbb{R}). \tag{10.5}$$

It is evidently smooth. Furthermore, $m_I(\mathcal{M}) \in \mathfrak{M}_{k,n}$ if $|\mathcal{M}_I| \ne 0$; otherwise, $m_I(\mathcal{M}) = 0$.

Consider the natural projection

$$\mu\colon \mathfrak{M}_{k,n} \to G_{n,k},$$

where $\mu(\mathcal{M})$ is the subspace of $\mathbb{R}^n$ spanned by the rows of the matrix $\mathcal{M}$. The two conditions: $\mu(\mathcal{M}') = \mu(\mathcal{M})$ and $\mathcal{M}' = g(\mathcal{M})$ for some $g \in \mathrm{GL}(k,\mathbb{R})$ are equivalent. Therefore, $\mu(m_I(\mathcal{M})) = \mu(\mathcal{M})$ if $|\mathcal{M}_I| \ne 0$. On the other hand, the last condition means that $\mu_I(\mathcal{M}) \in U_I$ (see Example VI in Section 10.11).

We are now in a position to define the section

$$s_{I,i}\colon G_{n,k} \to E_{n,k}$$

of the tautological bundle $\Theta_{n,k}$ by setting

$$s_{I,i}(L) = (i\text{th line of the matrix } m_I(\mathcal{M}), L),$$

where $\mathcal{M}$ is any matrix such that $m_I(\mathcal{M}) = L$. By virtue of (10.5), this construction is well defined. Formula (10.4) ensures that the sections $s_{I,i}$ are smooth.

10.13. Subbundles. A bundle $\eta\colon E_\eta \to M$ is said to be a *subbundle* of the bundle $\pi\colon E_\pi \to M$ (notation: $\eta \subset \pi$) if

(i) the total space E_η is a submanifold of E_π;

(ii) the map η is the restriction of π on E_η;

(iii) for any point $x \in M$ the fiber η_x is a submanifold of the fiber π_x.

Exercise. Give an algebraic definition of subbundles.

Examples. I. The tautological bundle over the Grassmannian (Example VI from 10.11) is a subbundle of the trivial bundle $\mathbb{R}^n \times G_{n,k} \to G_{n,k}$.

II. The bundle of unit tangent vectors of the sphere (Example IV from 10.13), which is a subbundle of the tangent bundle of the sphere, has no proper subbundles.

10.14. Whitney sum. Given two bundles η and ζ over one and the same manifold M, one can construct a new bundle π whose fiber over an arbitrary point $x \in M$ is the Cartesian product of the fibers of η and ζ:

$$\pi_x = \eta_x \times \zeta_x.$$

This bundle π is called the *direct sum*, or *Whitney sum,* of the bundles η and ζ and denoted by $\eta \oplus \zeta$.

To give this construction an exact meaning, we must explain how the individual fibers are put together to make a smooth manifold. As always, there are two ways to do this.

The *algebraic* definition of the Whitney sum reads as follows: If the two given bundles correspond to algebra extensions $i : A \hookrightarrow B$ and $j : A \hookrightarrow C$, then their Whitney sum is represented by the homomorphism

$$i \otimes j : A \hookrightarrow \overline{B \otimes_A C}$$

that takes every element a into $a(1 \otimes 1) = i(a) \otimes 1 = 1 \otimes j(a)$.

Note that the tensor product of the algebras B and C is taken over the algebra A, not over the ground ring. This reflects the fact that it is the fibers that get multiplied in this construction, not the total spaces of the bundles.

The *geometric* construction of the Whitney sum consists in the following. The total space of the bundle $\eta \oplus \zeta$ is defined as

$$E_{\eta\oplus\zeta} = \{(y, z) \in E_\eta \times E_\zeta \mid \eta(y) = \zeta(z)\},$$

and the projection as the map that takes the pair (y, z) to the point $\eta(y)$.

Exercise. Check that these definitions of the Whitney sum are equivalent and the fiber of the resulting bundle over an arbitrary point x is in a natural bijection with the manifold $\eta_x \times \zeta_x$.

There is one more useful description of Whitney sum. The map

$$\eta \times \zeta \colon E_\eta \times E_\zeta \to M \times M, \quad (e_1, e_2) \mapsto (\eta(e_1), \zeta(e_2)),$$

is a bundle with fiber $\eta_u \times \zeta_v$ over the point (u, v), where $u, v \in M$. The diagonal $M_\Delta = \{(z, z) \mid z \in M\} \subset M \times M$ is a submanifold in $M \times M$, identified with M via the map $z \mapsto (z, z)$. The total space $E_{\eta\oplus\zeta}$ and the projection $\eta \oplus \zeta$ are identified with the manifold $(\eta \times \zeta)^{-1}(M_\Delta)$ and the map $\eta \times \zeta\big|_{(\eta\times\zeta)^{-1}(M_\Delta)}$, respectively. The restrictions of

$$p_\eta \colon E_\eta \times E_\zeta \to E_\eta \quad \text{and} \quad p_\zeta \colon E_\eta \times E_\zeta \to E_\zeta$$

to the submanifold $(\eta \times \zeta)^{-1}(M_\Delta)$ give rise to smooth surjective maps

$$p_\eta \colon E_{\eta\oplus\zeta} \to E_\eta \quad \text{and} \quad p_\zeta \colon E_{\eta\oplus\zeta} \to E_\zeta.$$

If $s \in \Gamma(\eta \oplus \zeta)$, then $s_\eta \overset{\text{def}}{=} p_\eta \circ s \in \Gamma(\eta)$, $s_\zeta \overset{\text{def}}{=} p_\eta \circ s \in \Gamma(\zeta)$, and $s(z) = (s_\eta(z), s_\zeta(z)) \in \eta_z \times \zeta_z$. This establishes a natural bijection between the sets of sections

$$\Gamma(\eta \oplus \zeta) = \Gamma(\eta) \times \Gamma(\zeta). \tag{10.6}$$

10.15. Examples of direct sums. I. Denote by $\alpha \colon S^1_1 \cup S^1_2 \to S^1$, S^1_i being a copy of S^1, $i = 1, 2$, and $\beta \colon S^1 \to S^1$ the trivial and the nontrivial two-sheeted coverings of the circle. Let A be the algebra of smooth functions on the circle, i.e., the algebra of smooth periodic functions on the line. In algebraic terms, the map α is described by the injection $i \colon A \to A \oplus A$, $f \mapsto (f, f)$, while β corresponds to the map $j \colon A \to A$ taking $f(x)$ to $f(2x)$. Then

1. $\alpha \oplus \alpha$ is a trivial four-sheeted covering of the circle;
2. $\alpha \oplus \beta \cong \beta \oplus \beta$ is a four-sheeted covering of the circle, whose total space consists of two connected components, each of which represents a nontrivial two-sheeted covering.

To understand this fact geometrically, it is sufficient to sketch the behavior of the four points of the fiber after one complete turn of the base circle.

Exercise. Prove these facts algebraically, considering the tensor products of algebras A_1 and A_2 over A, where $A_1 = A_2 = A = C^\infty(S^1)$ and A_1, A_2 are equipped with an A-module structure induced by the inclusions i and j.

II. The direct sum of two Möbius bands, considered as bundles over the circle, is trivial. A visual proof of this fact is shown in Figure 10.3. Represent the Möbius bundle as a subbundle of the trivial bundle over the

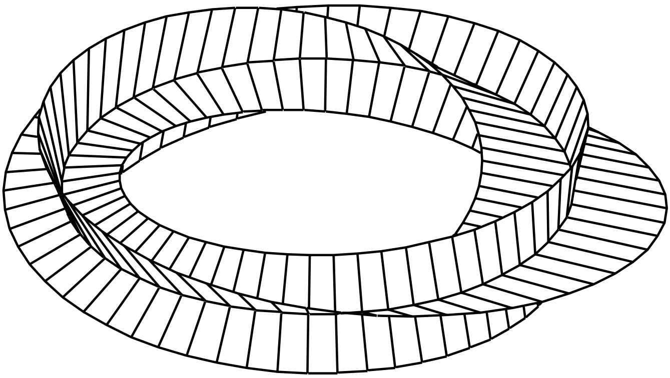


Figure 10.3. Direct sum of two Möbius bands.

circle with fiber $\mathbb{R}^2$. Then the lines perpendicular to its fibers constitute another Möbius bundle.

III. The bundle of 1-jets (see Section 9.28) is a direct sum of the trivial 1-dimensional bundle $M \times \mathbb{R} \to M$ and the cotangent bundle π_{T^*} (see Section 9.24).

10.16. Induced bundle. Given a bundle $\pi\colon E_\pi \to M$ and a smooth map $f\colon N \to M$, we can attach a copy of the fiber $\pi_{f(y)}$ to every point $y \in N$. The union of all these fibers constitutes the total space of the *bundle, induced from* π *by means of the map* f or the *pullback* of π by f.

There are two ways to turn this intuitive picture into a precise definition.

Geometrically, the total space of the induced bundle is defined as

$$E_{f^*(\pi)} \overset{\text{def}}{=} \{(y, z) \mid y \in N, z \in E_\pi, \pi(z) = f(y)\}.$$

The projection $f^*(\pi)$ acts as follows: $f^*(\pi)(y, z) = y$. Let us check the local triviality.

For a point $b \in N$ we set $a = f(b)$ and choose a neighborhood U of the point a in M such that π is trivial over U. Let $\psi\colon \pi^{-1}(U) \to U \times F$ be the trivializing diffeomorphism and let $\overline{\psi}$ be its composition with the projection $U \times F \to F$. Set $\chi(y, z) = \big(y, \overline{\psi}(z)\big)$. Then

$$\chi\colon (f^*(\pi))^{-1}\left(f^{-1}(U)\right) \to f^{-1}(U) \times F$$

is the required diffeomorphism.

The *restriction of the bundle* π to a submanifold $N \subset M$ is a particular case of an induced bundle. It is defined as follows:

$$\pi\big|_N \overset{\text{def}}{=} \pi\big|_{\pi^{-1}(N)} : \pi^{-1}(N) \to N. \tag{10.7}$$

Exercise. Check that $\pi\big|_N = i^*(\pi)$, where $i\colon N \hookrightarrow M$ is the inclusion map.

The algebraic definition of the induced bundle can be stated as follows. Let $i\colon A \hookrightarrow B$ be a bundle, understood as an algebra extension, and let $\varphi\colon A \to A_1$ be the algebra homomorphism corresponding to the smooth map $|\varphi|\colon |A_1| \to |A|$. Consider the algebras A_1 and B as A-modules with multiplication defined via i and φ. Then the *induced bundle* $|\varphi|^*(i)$ is the natural homomorphism $A_1 \to \overline{A_1 \otimes_A B}$.

Remark. There is a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\ \varphi\ } & A_1 \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle |\varphi|^*(i)} \\ B & \longrightarrow & \overline{A_1 \otimes_A B} \end{array}$$

which shows that the notion of induced bundle is a generalization of the Whitney sum.

Exercises. 1. Prove the equivalence of the geometric and the algebraic definitions of the induced bundle.

2. Show that a vector field along a map of manifolds $\varphi\colon N \to M$ (see Section 9.47) can be interpreted as a section of the induced bundle $\varphi^*(\pi_{TM})$ in the same way as an ordinary vector field is interpreted as a section of the tangent bundle (see Section 9.40).

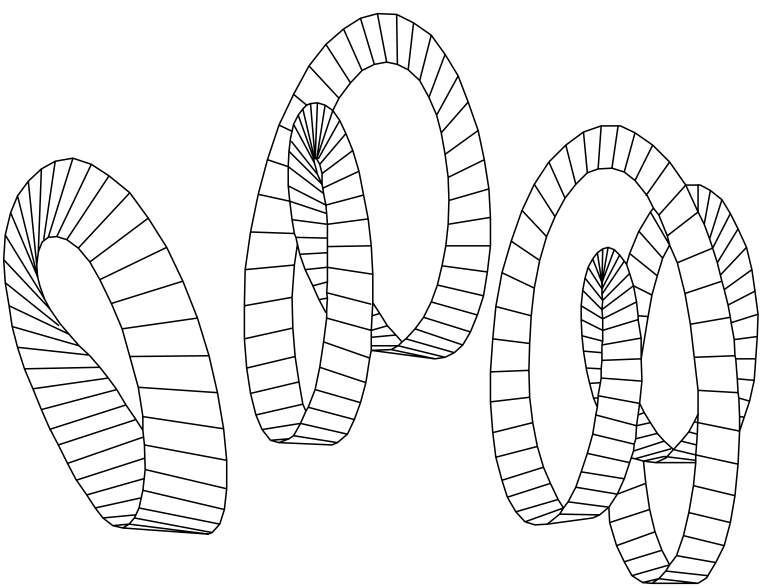


Figure 10.4. Bundles $f_n^*(\mu)$ for $n = 1,\ 2,\ 3$.

10.17. Examples. I. Let μ be the Möbius band bundle over the circle (10.11, III) and $f_n\colon S^1 \to S^1$ the n-sheeted covering of the circle (representing $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$, one can set $f_n(z) = z^n$, $n \in \mathbb{Z}$). Then

$$f_n^*(\mu) = \begin{cases} \mu, & n \text{ odd}, \\ \mathbb{I}_{S^1}, & n \text{ even}. \end{cases}$$

(See figure 10.4; here and below, $\mathbb{I}_M$ denotes the trivial bundle over M with fiber $\mathbb{R}$.)

II. **Triviality criterion in terms of induced bundles.** *A bundle is trivial if and only if it is equivalent to a bundle induced from a bundle over one point.*

Exercise. Prove this fact.

10.18. Define the *canonical morphism* $\varkappa\colon E_{f^*(\pi)} \to E_\pi$ by $\varkappa(y, z) = z$. The map $\varkappa$ is included into the commutative diagram

$$\begin{array}{ccc} E_{f^*(\pi)} & \xrightarrow{\ \varkappa\ } & E_\pi \\ {\scriptstyle f^*(\pi)}\downarrow & & \downarrow{\scriptstyle \pi} \\ N & \xrightarrow[\ f\ ]{} & M \end{array}$$

and therefore is an *f-morphism* from the bundle $f^*(\pi)$ to the bundle π.

More generally, given two bundles π over M and η over N and a smooth map $f\colon N \to M$, then an *f-morphism*, or a *morphism over f*, from η into π is a smooth map $\psi\colon E_\eta \to E_\pi$ that makes commutative the diagram

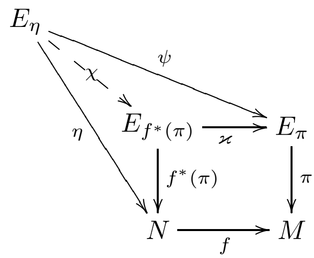
$$\begin{array}{ccc} E_\eta & \xrightarrow{\ \psi\ } & E_\pi \\ {\scriptstyle \eta}\downarrow & & \downarrow{\scriptstyle \pi} \\ N & \xrightarrow[\ f\ ]{} & M \end{array}$$

The notion of f-morphism generalizes the notion of a morphism of bundles over M: The latter is nothing but a morphism over the map id_M.

Example. The map $T\Phi\colon TM \to TN$, arising from a map $\Phi\colon M \to N$ (Section 9.18), is a Φ-morphism from π_{TM} into π_{TN}.

The pair $(f^*(\pi), \eta)$ has the following universal property: For any bundle η over M and any f-morphism $\psi\colon \eta \to \pi$ there exists a unique smooth map

χ that makes commutative the diagram



The proof is easy. For an arbitrary $y \in E_\eta$, both projections of $\chi(y)$ onto N and E_π are uniquely defined due to the commutativity of the diagram. This implies the uniqueness of χ. Existence follows from the explicit formula $\chi(y) = (\eta(y), \psi(y))$.

There is a natural map $\widehat{f}\colon \Gamma(\pi) \to \Gamma(f^*(\pi))$ called the *lifting of sections.* By definition, for any $s \in \Gamma(\pi)$ the value of the section $\widehat{f}(s)$ at the point $y \in N$ is equal to the value of s at $f(y)$. The precise formula is

$$\widehat{f}(s)(y) = (y, s(f(y))) \in E_{f^*(\pi)}.$$

The section $\widehat{f}(s)$ is the *lift* of s along f.

10.19. Regular morphisms. Working with manifolds whose points have one and the same inner structure, it is natural to introduce the class of morphisms that preserve this structure. More specifically, an f-morphism ψ is said to be *regular* if for any point $z \in N$ the map of fibers $\psi_z : \eta_z \to \pi_{f(z)}$ is a diffeomorphism.

Proposition. *Let $\psi\colon \eta \to \pi$ be a regular morphism of bundles over the map $f\colon N \to M$. Then the canonical morphism $\chi\colon \eta \to f^*(\pi)$ defined in the previous section is an equivalence of bundles over N.*

◀ For any $z \in N$ the fiber map $\chi_z : \eta_z \to f^*(\pi)_z$, by the construction of χ, is identified with the map $\psi\colon \eta_z \to \pi_{f(z)}$ and therefore is a diffeomorphism. As a consequence, the map $\chi\colon E_\eta \to E_{f^*(\pi)}$ is a diffeomorphism, too. ▶

The proposition shows that the class of bundles related to a given bundle π by means of regular morphisms is exhausted by the bundles induced from π. This observation leads to the tempting idea of building, for a given type of fibers, a *universal bundle* such that any bundle with this fiber could be induced from the universal bundle via a suitable smooth map. After an appropriate concretization, this idea can be implemented. Here is an example.

10.20. Example (Gauss map). Let M be an n-dimensional manifold. By the Whitney theorem, M can be immersed into $\mathbb{R}^{2n}$, i.e., there exists a map $\varphi\colon M \to \mathbb{R}^{2n}$ such that for any point $z \in M$ the differential $d_z\varphi\colon T_zM \to T_{\varphi(z)}\mathbb{R}^{2n}$ is injective. Denote by r_a, $a \in \mathbb{R}^{2n}$, the linear shift

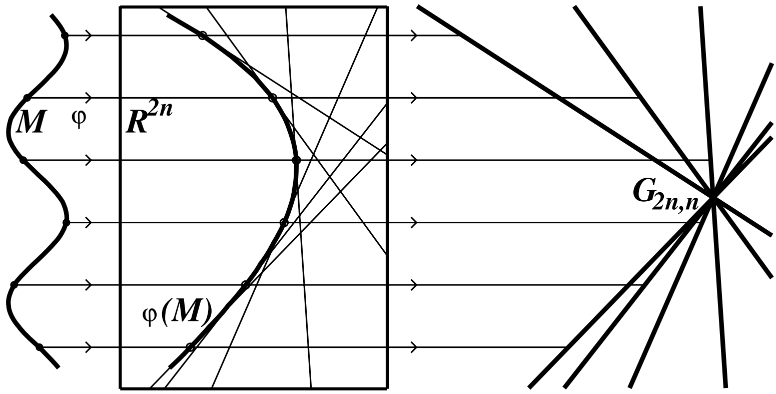


Figure 10.5. The Gauss map.

in $\mathbb{R}^{2n}$ through the vector $-a$:

$$\mathbb{R}^{2n} \ni v \mapsto v - a \in \mathbb{R}^{2n}.$$

Let $G_{2n,n}$ be the Grassmann manifold of n-dimensional linear subspaces in $T_O\mathbb{R}^{2n}$, where $O = (0, \dots, 0) \in \mathbb{R}^{2n}$. The *Gauss map* $g\colon M \to G_{2n,n}$ takes every point $z \in M$ to the image of the corresponding tangent space T_zM under the map

$$d_z(r_{\varphi(z)} \circ \varphi)\colon T_zM \to T_O\mathbb{R}^{2n}$$

(see Figure 10.5). The map g is covered by the morphism of bundles $\gamma\colon \pi_{TM} \to \Theta_{2n,n}$, where $\Theta_{2n,n}\colon E_{2n,n} \to G_{2n,n}$ is the tautological bundle described in Example VI of Section 10.11. Indeed, if $\xi \in T_zM$, then

$$\gamma(\xi) = \big(d_z(r_{\varphi(z)} \circ \varphi)(\xi), g(z)\big) \in E_{2n,n}.$$

Therefore, by Proposition 10.19, the tangent bundles of all n-dimensional manifolds can be induced from one tautological bundle over the Grassmannian $G_{2n,n}$. This fact plays an important role in the study of manifolds. For example, it lies at the foundation of the theory of characteristic classes.

11

Vector Bundles and Projective Modules

11.1. We have seen in the previous chapter that given a bundle, the fiber over a point of the base space describes the inner structure of this point. The fiber may have a certain mathematical structure. For example, the fibers of the tangent bundle have a natural structure of a linear space, and this structure has an evident physical meaning. Indeed, if the manifold M is the configuration space of a mechanical system (see Section 9.22), then for a fixed point $a \in M$ the tangent vector is interpreted as the velocity vector of the system having configuration a. Bundles of this kind, where fibers are vector spaces, are called *vector bundles* (see Section 11.2 for an exact definition). They form an interesting and important class of bundles. In particular, besides tangent bundles, this class contains cotangent bundles and jet bundles, which are fundamental objects of study in the geometric theory of differential equations.

Vector bundles have a simpler algebraic description than bundles of general type. It turns out that under certain natural regularity conditions the extensions of algebras $A \hookrightarrow B$ that correspond to vector bundles are of the form $A \hookrightarrow S(P)$, where $S(P)$ is the symmetric algebra of a certain A-module P, appropriately completed. The study of vector bundles over a manifold M is thus reduced to the study of a certain class of modules over the algebra $A = C^\infty(M)$.

We begin with the geometric definition of a vector bundle. After an investigation of the basic properties of vector bundles, we shall prove the fundamental theorem on the equivalence of the notions of a vector bundle over M and a finitely generated projective module over A, and then explain how symmetric algebras of modules appear in this context.

11.2. Geometric definition of a vector bundle. A fiber bundle $\pi\colon E \to M$ with fiber V is said to be a *vector bundle* if

1. V is a vector space (over $\mathbb{R}$);
2. For any point $x \in M$ the fiber π_x is a vector space;
3. The *vector property of local triviality* holds: for any point $x \in M$ there exists a neighborhood $U \subset M$, $x \in U$, and a trivializing diffeomorphism $\varphi\colon \pi^{-1}(U) \to U \times V$, *linear on every fiber*, i.e., such that all maps $\varphi_y : \pi_y \to V$, $y \in U$, are linear.

The *dimension* of a vector bundle is the dimension of its fiber. Zero-dimensional vector bundles are called *zero bundles* and denoted by $\mathbb{O}_M$. Trivial one-dimensional bundles are called *unit bundles* and denoted by $\mathbb{I}_M$.

11.3. Adapted coordinates in vector bundles. Suppose (U, x) is a chart on M satisfying Condition 3 of Definition 11.2, while $\varphi\colon \pi^{-1}(U) \to U \times V$ is the corresponding trivializing diffeomorphism, and $\xi \in V$. The map

$$s_\xi^\varphi : U \to \pi^{-1}(U), \quad U \ni x \mapsto \varphi^{-1}(x, \xi) \in \pi^{-1}(U),$$

is a section of the fibering $\pi\big|_U$. If $v_1, \dots, v_m$ is a basis of the vector space V, then the sections $e_j = s_{v_j}^\varphi$, $1 \leqslant j \leqslant m$, have the property that at every point $z \in U$ the vectors

$$e_1(z) = \varphi^{-1}(z, v_1), \dots, e_m(z) = \varphi^{-1}(z, v_m)$$

form a basis of the space $\pi_U^{-1}(z) \cong V$. Below (see Section 11.7) we shall show that the totality of all sections of a vector bundle π over M has a natural $C^\infty(M)$-module structure induced by the linear structure in the fibers π_z. In this sense we can say that in the chosen coordinate chart U the module of sections of the bundle $\pi|_U$ is free and $e_j = s_{v_j}^\varphi$, $1 \leqslant j \leqslant m$, is its basis.

Now suppose that $z \in U$ and $(x_1, \dots, x_n)$ are coordinate functions on U. A point $y \in \pi_z \subset \pi^{-1}(U)$ is defined by the set of $n + m$ numbers $(x_1, \dots, x_n, u^1, \dots, u^m)$, where $(x_1, \dots, x_n)$ are coordinates of the point z and $\big(u^1, \dots, u^m\big)$ are coordinates of the point y with respect to the basis $e_1|_z, \dots, e_m|_z$. The functions $\big(x_1, \dots, x_n, u^1, \dots, u^m\big)$ form a coordinate system on $\pi^{-1}(U)$, called the *adapted* coordinates of the bundle π.

Accordingly, the chart $\big(\pi^{-1}(U), x_1, \dots, x_n, u^1, \dots, u^m\big)$ on the manifold E is referred to as an *adapted* chart. Finally, an atlas made up of adapted charts is also called adapted. It is readily verified that if two charts (U, x) and (U', x') on M are compatible, then the corresponding adapted charts on E are compatible, too. (The reader is invited to check this fact as an exercise.) In particular, this means that every atlas on the manifold M gives rise to an adapted atlas on the total space E.

11.4. Morphisms of vector bundles. A *morphism of vector bundles* $\alpha\colon \pi \to \eta$ *over* M is a bundle morphism $\alpha\colon E_\pi \to E_\eta$, which is fiberwise linear (i.e., the map α_z is $\mathbb{R}$-linear for any point of the base space $z \in M$). The set of all morphisms from π to η is denoted by $\mathrm{Mor}(\pi, \eta)$, and the category of vector bundles thus arising will be denoted by VB_M.

For the local study of morphisms, the following point of view is convenient: A morphism of trivial vector bundles over M is the same thing as an operator-valued function on the manifold M. An exact statement of this observation is contained in the obvious lemma that follows.

11.5 Lemma. *Let* $\pi\colon M \times V \to M$ *and* $\eta\colon M \times W \to M$ *be trivial vector bundles. To every fiberwise linear map* $\varphi\colon M \times V \to M \times W$ *one can assign a family of linear operators* $\widetilde{\varphi}\colon M \to \mathrm{Hom}(V, W)$ *by setting the value of the operator* $\widetilde{\varphi}(x)$ *on the vector* $v \in V$ *equal to the* W*-component of the element* $\varphi(x, v) \in M \times W$. *Then the following conditions are equivalent*:

(a) *the map* φ *is smooth* (*i.e.,* φ *is a bundle morphism*);

(b) *the map* $\widetilde{\varphi}$ *is smooth* (*the space* $\mathrm{Hom}(V, W)$ *is endowed with the structure of a manifold, because it is a finite-dimensional real vector space*).

11.6. Examples of vector bundles. I. The Möbius band fibered over the circle (Example 10.11, III) can be viewed as a vector bundle if its fibers are regarded as one-dimensional linear spaces. Representing the algebra of functions on the Möbius band as the subalgebra $B \subset C^\infty(\mathbb{R}^2)$ distinguished by the condition $f(x+1, y) = f(x, -y)$, we can define this bundle by the inclusion of algebras $A \to B$ that takes a function f to the function g, $g(x, y) = f(x)$. Here $A = \{f \in C^\infty(\mathbb{R}) \mid f(x+1) = f(x)\}$ is the algebra of functions on the circle.

II. The tangent bundle $\pi_T\colon TM \to M$ (see Section 9.19). The trivializing diffeomorphisms described in Section 10.11, VII, are fiberwise linear. Therefore, the tangent bundle is a vector bundle.

III. The cotangent bundle $\pi_{T^*}\colon T^*M \to M$ (see Section 9.24). As in the previous example, the trivializing diffeomorphisms of Section 10.11, VIII, are obviously linear.

IV. The trivializations described in Section 10.11, IX, are also fiberwise linear. Therefore, the bundle of l-jets $\pi_{J^l}\colon J^l M \to M$ is a vector bundle.

Note that in the last three examples the special coordinate systems defined in 9.19, 9.24, and 10.11, IX, respectively, are adapted.

Exercise. Check whether the maps

$$\pi_{l,m}\colon J^l M \to J^m M, \quad [f]_z^l \mapsto [f]_z^m, \quad l \geqslant m,$$
$$\tau_l\colon J^l M \to T^* M, \quad [f]_z^l \mapsto d_z(f), \quad l \geqslant 1,$$

are vector bundles.

11.7. Module of sections. A remarkable property of vector bundles is that their sets of sections possess a module structure over the algebra of smooth functions. The resulting interrelation between the bundles and modules is of fundamental importance.

Note first of all that the set of sections of any vector bundle is nonempty: It always contains the *zero section* s_0. By definition, the value of s_0 at any point $z \in M$ is the zero of the vector space π_z.

Using the linear structure in the fibers π_x, one can introduce two operations in the set of sections of a vector bundle: addition and multiplication by a function on the manifold,

$$(s_1 + s_2)(z) = s_1(z) + s_2(z), \qquad (fs)(z) = f(z)s(z),$$

for any sections s, s_1, s_2, any smooth function $f \in C^\infty(M)$, and any point $z \in M$. The definition immediately implies that the sum of two sections and the product of a section and a smooth function are again (smooth) sections. These operations turn the totality of all smooth sections of a vector bundle π into a $C^\infty(M)$-module, denoted by $\Gamma(\pi)$.

The next lemma clarifies the relationship between the global and the pointwise approaches to the sections of a vector bundle. As before, we denote by μ_z the maximal ideal of the algebra $C^\infty(M)$, defined by

$$\mu_z = \{f \in C^\infty(M) \mid f(z) = 0\}$$

and called the *ideal of the point* z.

11.8 Lemma. *Let π be a vector bundle over a manifold M and $z \in M$. Then*

(a) *for any point $y \in \pi_z$ there is a section $s \in \Gamma(\pi)$ such that $s(z) = y$;*

(b) *if $s \in \Gamma(\pi)$ and $s(z) = 0$, then there exist functions $f_i \in \mu_z$ and sections $s_i \in \Gamma(\pi)$ such that s can be written as a finite sum $s = \sum f_i s_i$.*

◀ (a) For a trivial bundle the assertion evidently holds. Therefore, by local triviality there is a neighborhood U of the point z and a section $s\big|_U \in \Gamma(\pi\big|_U)$ satisfying $s\big|_U(z) = y$. In order to obtain a global (i.e., defined over all M) section of the bundle π possessing the same property, it remains to multiply $s\big|_U$ by a smooth function whose support is contained in U and that has value 1 at the point z.

(b) First suppose that the bundle is trivial. In this case (see Section 11.3) there are sections $e_1, \ldots, e_m \in \Gamma(\pi)$ whose values at every point $z \in M$ form a basis of the linear space π_z. A given section s can be expanded over the basis: $s = \sum_{i=1}^m f_i e_i$. The equality $s(z) = 0$ implies that $f_i(z) = 0$ for all $i = 1, \ldots, m$; therefore, $f_i \in \mu_z$, as required.

In the case of an arbitrary bundle, there is a neighborhood U of the point z, sections $e_i \in \Gamma\left(\pi\big|_U\right)$; and functions $f_i \in C^\infty(U)$ such that $s\big|_U = \sum_{i=1}^m f_i e_i$. Choose a smooth function $f \in C^\infty(M)$ such that $\operatorname{supp} f \subset U$

and $f(z) = 1$. Extending the function $ff_i \in C^\infty(U)$ (respectively, the section $fe_i \in \Gamma\left(\pi\big|_U\right)$) as the identical zero outside of U, we shall obtain a smooth function (respectively, section) on the entire manifold M. Keeping the notation ff_i and fe_i for the extensions, we can write

$$f^2 s = \sum_{i=1}^{m} (ff_i)(fe_i),$$

and therefore

$$s = \left(1 - f^2\right) s + \sum_{i=1}^{m} (ff_i)(fe_i).$$

It remains to note that the functions $1 - f^2$, $ff_1, \ldots, ff_m$ belong to μ_z. ▶

The lemma just proven allows a compact reformulation in terms of exact sequences. Recall that a sequence of A-modules

$$\cdots \to P_{i-1} \overset{\alpha_{i-1}}{\to} P_i \overset{\alpha_i}{\to} P_{i+1} \to \cdots$$

is said to be *exact at the term* P_i if $\operatorname{Ker} \alpha_i = \operatorname{Im} \alpha_{i-1}$. The sequence is called *exact* if it is exact at every term.

11.9 Corollary. *For any vector bundle* π *the sequence*

$$0 \to \mu_z \Gamma(\pi) \to \Gamma(\pi) \to \pi_z \to 0,$$

where the first arrow is the inclusion, while the second assigns to every section its value at point $z \in M$*, is exact. Hence* $\Gamma(\pi)/\mu_z\Gamma(\pi) \cong \pi_z$. ▶

Recall that to every element $h \in |A|$ we can assign the ideal $\mu_h = \operatorname{Ker} h \subset A$. The above result justifies the following definition: The *fiber* P_h of an A-module P over a point $h \in |A|$ is the quotient module $P/\mu_h P$. The *value* p_h of an element $p \in P$ at the point h is the image of p under the natural projection $P \to P_h$. For the case in which $A = C^\infty(M)$ and $h = h_z$ for $z \in M$, in the same sense we write $P_z \overset{\text{def}}{=} P/\mu_z P$ and speak of the value p_z of an element $p \in P$ at the point z.

Exercise. Show that for the module of vector fields $P = D(M)$ over the algebra of smooth functions $A = C^\infty(M)$, we have $D(M)_z = T_z M$, and the value of an element $X \in D(M)$ at the point z is just the vector of the field X at this point (see 9.39). (In other words, the notation X_z in both cases has the same meaning.)

11.10. For the analysis of modules of sections, the following fact is important:

Proposition. *Suppose that the sections* $s_1, \ldots, s_l \in \Gamma(\pi)$ *have the property that for every point* $z \in M$ *the vectors* $s_1(z), \ldots, s_l(z)$ *span the fiber* π_z*. Then these sections generate the module* $\Gamma(\pi)$.

◀ Let k be the dimension of the bundle π. For an ordered set of integers $I = (i_1, \ldots, i_k)$, $1 \leqslant i_1 < \cdots < i_k \leqslant l$, put

$$U_I = \{z \in M \mid s_{i_1}(z), \ldots, s_{i_k}(z) \in \pi_z \text{ are linearly independent}\}.$$

Evidently, the set U_I is open, and the sections $s_{i_1}|_{U_I}, \ldots, s_{i_k}|_{U_I}$ generate the $C^\infty(U_I)$-module $\Gamma(\pi|_{U_I})$. Moreover, $\bigcup_I U_I = M$. Indeed, for any point $z \in M$ one can choose a basis $s_{i_1}(z), \ldots, s_{i_k}(z)$ among the vectors $s_1(z), \ldots, s_l(z)$ that span the fiber π_z. This means that $z \in U_I$.

For a section $s \in \Gamma(\pi)$ we have

$$s|_{U_I} = \sum_{\alpha=1}^{k} \lambda_{I,\alpha} s_{i_\alpha}\,|_{U_I}, \qquad \lambda_{I,\alpha} \in C^\infty(U_I).$$

Now let $\mu_I \in C^\infty(M)$ be a function that is strictly positive inside U_I, vanishes outside of this set, and has the property that the functions

$$\nu_{I,i} = \begin{cases} \mu_I \lambda_{I,i} & \text{inside} \quad U_I, \\ 0 & \text{outside} \quad U_I, \end{cases}$$

are smooth on M. Then the function $\mu = \sum_I \mu_I$ is everywhere positive on M and

$$\mu_I s = \sum_i \nu_{I,i} s_i.$$

Therefore,

$$s = \frac{1}{\mu} \sum_I \mu_I s = \sum_{I,i} \frac{\nu_{I,i}}{\mu} s_i. \quad ▶$$

11.11. Geometrization of modules. With every A-module P over a commutative K-algebra A we can associate a geometric object

$$|P| = \bigcup_{h \in |A|} P_h \quad (\text{or } |P| = \bigcup_{z \in M} P_z, \text{ if } A = C^\infty(M)),$$

together with a natural projection onto $|A|$:

$$|P| \supset P_h \ni p_h \overset{\pi_P}{\mapsto} h \in |A|.$$

The A-module $P_h = P/\mu_h P$ can be also viewed as a module over A/μ_h, hence, by virtue of the isomorphism $A/\mu_h = K$, as a K-module.

The projection π_P looks very much like a bundle and, as we shall show below, is equivalent to a vector bundle if $A = C^\infty(M)$ and the module P is projective and finitely generated. We shall refer to such projections as *pseudobundles*.

In the case $P = D(M)$, every element $X \in D(M)$ corresponds to a section $s_X : z \mapsto X_z \in T_z M = D(M)_z$ of the tangent bundle π_T. This construction is of general nature and can be used for arbitrary pseudobundles by assigning the map

$$s_p : |A| \to |P|, \quad h \mapsto p_h,$$

to an element $p \in P$. This allows us to visualize the elements of an arbitrary module P as sections of the pseudobundle $|P|$ much in the same way as the elements of an arbitrary algebra A were viewed as functions on its spectrum $|A|$. One of the main goals of the present chapter is to show that vector bundles are obtained from projective modules just as smooth manifolds are obtained from smooth algebras.

Maps $s_p : |A| \to |P|$ are referred to as *sections* of the pseudobundle π_P. (There is no other way to distinguish a reasonable class among all maps $s : |A| \to |P|$ such that $\pi_P \circ s = \mathrm{id}_{|A|}$.) The set $\Gamma(P)$ of all sections of the pseudobundle π_P forms an A-module with respect to the natural operations

$$(s_{p_1} + s_{p_2}) = s_{p_1+p_2}, \quad p_1, p_2 \in P,$$
$$(as_p) = s_{ap}, \quad a \in A, \quad p \in P.$$

To every A-module P we thus assign the A-module $\Gamma(P)$ of sections of the pseudobundle π_P. Our aim can now be stated more precisely: We want to show that for $A = C^\infty(M)$ and every projective finitely generated A-module P the pseudobundle π_P is a vector bundle and the two modules P and $\Gamma(P)$ are naturally isomorphic.

Exercise. Prove that the assignment $P \mapsto \Gamma(P)$ is a functor in the category of A-modules.

If P is a $C^\infty(M)$-module and its element $p \in P$ belongs to the intersection $\bigcap_{z\in M} \mu_z P$, then the value of p at every point $z \in M$ is zero. Such elements can be called *invisible*, or *unobservable*. Indeed, by the principle of observability, the class $p \bmod \mu_z P$ should be viewed as a certain component of the inner structure of the point $z \in M$, and the fact that p belongs to all subspaces $\mu_z P$ means that this component is unobservable.

A $C^\infty(M)$-module P is said to be *geometric*, if $\bigcap_{z\in M} \mu_z P = 0$, i.e., if all elements of P are observable.

Exercise. Prove that P is geometric if and only if the two modules P and $\Gamma(P)$ are isomorphic.

The algebraic paraphrase of the above discussion is as follows. The map

$$\Gamma = \Gamma_P : P \to P \Big/ \bigcap_{z\in M} \mu_z P = \Gamma(P)$$

kills all unobservable elements of P. Therefore, the quotient module $\Gamma(P)$ defined in this way can be called the *geometrization* of P. The assignment $P \mapsto \Gamma(P)$ defines a functor from the category $\mathrm{Mod}\, C^\infty(M)$ of all $C^\infty(M)$-modules into the category $\mathrm{GMod}\, C^\infty(M)$ of geometric $C^\infty(M)$-modules. In some situations it is sufficient to use the smaller category $\mathrm{GMod}\, C^\infty(M)$ instead of the bigger category $\mathrm{Mod}\, C^\infty(M)$.

Exercise. Show that the subcategory $\operatorname{GMod} C^\infty(M) \subset \operatorname{Mod} C^\infty(M)$ is stable under the operations $\bigotimes$ and Hom: If P and Q are geometric $C^\infty(M)$-modules, then the modules $P \otimes Q$ and $\operatorname{Hom}_{C^\infty(M)}(P, Q)$ are geometric too.

The behavior of the fiber P_h when the point $h \in |A|$ varies provides important information about the module P. For example, one can speak of the support of a module,

$$\operatorname{supp} P = \overline{\{h \in |A| \mid P_h \neq 0\}} \subset |A|,$$

where the bar means closure in the Zariski topology.

Exercises. 1. The tangent space $T_z M$ at a point $z \in M$ can be considered as a $C^\infty(M)$-module with multiplication defined by the rule $(f, \xi) \mapsto f(z)\xi$, $f \in C^\infty(M)$, $\xi \in T_z M$. Show that the support of this module consists of one point z.

2. Prove that the support of the $C^\infty(M)$-module $D(M, N)$ (module of vector fields along a submanifold $N \subset M$, see Section 9.46) coincides with N.

The geometrization of A-modules helps to visualize and thus better understand various algebraic constructions. For example, the structure of an A-module homomorphism $f\colon P \to Q$ is displayed through the family of its values at different points of the spectrum of $|A|$. By the *value* of F at the point $h \in |A|$ we understand the map of quotient modules $F_h\colon P_h \to Q_h$, well defined because $F(\mu_h P) \subset \mu_h Q$. In the geometric situation, when $A = C^\infty(M)$, we can use the notation F_z, P_z, Q_z instead of F_{h_z}, P_{h_z}, Q_{h_z}.

11.12. Topology in $|P|$. The set $|P|$ can be turned into a topological space by using an appropriate generalization of the ideas used in Chapter 9 to prove that the cotangent manifold T^*M is the $\mathbb{R}$-spectrum of the symbol algebra $\mathcal{S}_*$. In the situation under study, a natural candidate to play the role of such an algebra is the symmetric algebra $\mathcal{S}(P^*)$ of the module $P^* = \operatorname{Hom}_A(P, A)$. By definition,

$$\mathcal{S}(P^*) = \bigoplus_{k \geqslant 0} \mathcal{S}_k(P^*),$$

where $\mathcal{S}_k(P^*)$ is the kth symmetric power of the module P^*. An element $f \in P^* = \mathcal{S}_1(P^*)$ can be viewed as a function on $|P|$ by setting

$$f(p_h) \stackrel{\text{def}}{=} f(p) \mod \mu_h \in A/\mu_h = K, \quad h \in |A|,\ p \in P.$$

Since f is a homomorphism, the value $f(p_h)$ does not depend on the choices made. For a general element $f_1 \otimes \ldots \otimes f_k \in (P^*)^{\otimes k}$, where $(P^*)^{\otimes k}$ denotes the kth tensor power of the A-module P^*, we put

$$(f_1 \otimes \cdots \otimes f_k)(p_h) \stackrel{\text{def}}{=} f_1(p_h) \cdots f_k(p_h) \in K.$$

By this formula, elements of $(P^*)^{\otimes k}$ can be understood as functions on $|P|$. For an element $\omega \in (P^*)^{\otimes k}$ of the form

$$\omega = f_1 \otimes \cdots \otimes f_i \otimes \ldots \otimes f_j \otimes \cdots \otimes f_k - f_1 \otimes \cdots \otimes f_j \otimes \cdots \otimes f_i \otimes \cdots \otimes f_k$$

the corresponding function ω is identically zero. The quotient algebra of the complete tensor algebra $(P^*)^{\otimes} = \sum_{k \geqslant 0} (P^*)^{\otimes k}$ over the ideal generated by such elements is the symmetric algebra $\mathcal{S}(P^*)$.

Below, we discuss this idea in detail for modules over the algebra of smooth functions $C^\infty(M)$ and show that for any vector bundle π the two spaces $|\Gamma(\pi)|$ and E_π coincide.

Denote by $\mathcal{F}(|P|)$ the K-algebra of functions on $|P|$ that correspond to elements of the algebra $\mathcal{S}(P^*)$. With the help of this algebra, we can turn the set $|P|$ into a topological space with the Zariski topology, in which the basic closed sets are zero sets of functions belonging to $\mathcal{F}(|P|)$.

Exercise. Show that the maps $\pi_P \colon |P| \to |A|$ and $s_p \colon |A| \to |P|$, $p \in P$, are continuous in this topology.

Using the Zariski topology in $|P|$, one can widen the class of sections of the pseudobundle π_P. Namely, a *continuous section* of π_P is a continuous map $s \colon |A| \to |P|$ such that $\pi_P \circ s = \mathrm{id}_{|A|}$. The set of all continuous sections of π_P will be denoted by $\Gamma_0(P)$.

Exercise. Show that the structure of a K-linear space in each fiber $P_h \subset |P|$ induces the structure of an A-module in $\Gamma_0(P)$.

11.13. The functor of sections. The assignment $\pi \mapsto \Gamma(\pi)$ that associates the module of sections $\Gamma(\pi)$ with a given vector bundle π can be made into a functor as follows. Let $\alpha \in \mathrm{Mor}(\pi, \eta)$. Put

$$\Gamma(\alpha)(s) = \alpha \circ s \quad \text{for any } s \in \Gamma(\pi).$$

Then $\Gamma(a) \colon \Gamma(\pi) \to \Gamma(\eta)$ is a $C^\infty(M)$-module homomorphism, and the assignment $\alpha \mapsto \Gamma(\alpha)$ has all the necessary properties:

(i) $\Gamma(\mathrm{id}_\pi) = \mathrm{id}_{\Gamma(\pi)}$ for any π,

(ii) $\Gamma(\alpha \circ \beta) = \Gamma(\alpha) \circ \Gamma(\beta)$ for any pair of morphisms $\zeta \xrightarrow{\beta} \pi \xrightarrow{\alpha} \eta$.

Exercise. Let $P = \Gamma(\pi)$, $Q = \Gamma(\eta)$, $\alpha \in \mathrm{Mor}(\pi, \eta)$, and $F = \Gamma(\alpha)$. Prove that the map $F_z \colon P_z \to Q_z$ (see Section 11.11) is canonically identified with $\alpha_z \colon \pi_z \to \eta_z$ under the identifications $P_z = \pi_z$, $Q_z = \eta_z$, described in Lemma 11.8.

The study of the functor Γ that relates the geometry of vector bundles with the algebra of rings and modules is the main point of the present chapter. This functor allows one to express the geometric properties of vector bundles and operations with them in algebraic language. Here is a simple example.

Proposition. *A vector bundle π is trivial if and only if the module $\Gamma(\pi)$ is free.*

◀ Indeed, choose a trivializing diffeomorphism $\varphi\colon E_\pi \to M \times V$ and a basis $v_1, \ldots, v_n$ of the linear space V. Let $e_i(x) = \varphi^{-1}(x, v_i)$. Then the set $e_1, \ldots, e_m$ is a free basis of the module $\Gamma(\pi)$.

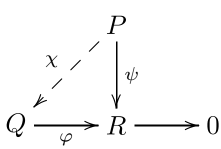
Conversely, supposing that the module $\Gamma(\pi)$ is free with a basis $e_1, \ldots, e_m$, we can define a diffeomorphism $\varphi\colon E_\pi \to M \times \mathbb{R}^n$ by setting

$$\varphi\left(\sum_{i=1}^{m} \lambda_i e_i(x)\right) = (x; \lambda_1, \ldots, \lambda_m). \quad ▶$$

Remark. For any free $C^\infty(M)$-module P of *finite-type*, i.e., with a finite set of generators, there exists a vector bundle whose module of sections is isomorphic to P. Indeed, a free $C^\infty(M)$-module of rank m is isomorphic to $\Gamma(\pi)$, where π is the product bundle $M \times \mathbb{R}^m \to M$.

11.14. Projective modules. It is natural to suppose that section modules of vector bundles must possess certain specific properties originating from the fact that all fibers of a given bundle are equal to each other. These properties serve as a formalization of our doctrine that the inner structures of all points are identical (see Section 10.1). We shall see that an adequate description can be given by using the notion of projectivity.

A module P over a commutative ring A is said to be *projective* if it has the following property: For any epimorphism of A-modules $\varphi\colon Q \to R$ and any homomorphism $\psi\colon P \to R$ there is a homomorphism $\chi\colon P \to Q$ such that $\varphi \circ \chi = \psi$, i.e.; the diagram



commutes. The homomorphism χ is called the *lift* of ψ along φ. Let us give several equivalent definitions of projectivity.

11.15 Proposition. *The following properties of an A-module P are equivalent*:

(a) *P is projective*;

(b) *any epimorphism $\varphi\colon Q \to P$ of an arbitrary A-module onto P splits, i.e., there exists a homomorphism $\chi\colon P \to Q$ such that $\varphi \circ \chi = \mathrm{id}_P$*;

(c) *P is isomorphic to a direct summand of a free A-module*;

(d) *the functor $\mathrm{Hom}_A(P, \cdot)\colon Q \mapsto \mathrm{Hom}_A(P, Q)$ on the category of A-modules is exact, i.e., it preserves the class of exact sequences.*

◀ (a) $\Longrightarrow$ (b). It suffices to set $R = P$ and $\psi = \mathrm{id}_P$ in the definition of projectivity.

(b) $\Longrightarrow$ (c). Let $\varphi\colon Q \to P$ be an epimorphism of a certain free A-module onto P (to construct such an epimorphism, for Q one can take a free module with a basis $\{e_p\}_{p\in P}$, equipotent to the set P and put $\varphi(e_p) = p$ for every $p \in P$). By virtue of (b), there is a homomorphism $\chi \in \mathrm{Hom}(P, Q)$ such that $\chi \circ \varphi = \mathrm{id}_P$. Then $P \cong \mathrm{Im}\,\chi$ and $Q = \mathrm{Im}\,\chi \oplus \mathrm{Ker}\,\varphi$. Indeed, any element $a \in Q$ can be written as $\chi(\varphi(a)) + \big(a - \chi(\varphi(a))\big)$; here the first summand belongs to $\mathrm{Im}\,\chi$ and the second to $\mathrm{Ker}\,\varphi$. On the other hand, if $a \in \mathrm{Im}\,\chi \cap \mathrm{Ker}\,\varphi$, then $a = \chi(p)$, $p \in P$, and $0 = \varphi(a) = \varphi(\chi(p)) = p$. Therefore, $a = 0$.

(c) $\Longrightarrow$ (d). Note that for a free module R the functor $\mathrm{Hom}_A(R, \cdot)$ is exact. This follows from the fact that a homomorphism of a free module R into another module is uniquely determined by its values on the basis elements, and these values can be arbitrary. Now suppose that $R = P \oplus Q$ and

$$\mathcal{S} = \{\cdots \to S_k \xrightarrow{\varphi_k} S_{k+1} \to \cdots\}$$

is an exact sequence of A-modules. Then the sequence $\mathrm{Hom}_A(R, \mathcal{S})$,

$$\cdots \to \mathrm{Hom}_A(R, S_k) \to \mathrm{Hom}_A(R, S_{k+1}) \to \cdots\,,$$

is exact, too. This sequence decomposes into a direct sum of the sequence $\mathrm{Hom}_A(P, \mathcal{S})$ of the form

$$\cdots \to \mathrm{Hom}_A(P, S_k) \to \mathrm{Hom}_A(P, S_{k+1}) \to \cdots\,,$$

and the sequence $\mathrm{Hom}_A(Q, \mathcal{S})$ of the form

$$\cdots \to \mathrm{Hom}_A(Q, S_k) \to \mathrm{Hom}_A(Q, S_{k+1}) \to \cdots\,.$$

In other words, every term of the sequence $\mathrm{Hom}_A(R, \mathcal{S})$ is the direct sum of the corresponding terms of two sequences $\mathrm{Hom}_A(P, \mathcal{S})$ and $\mathrm{Hom}_A(Q, \mathcal{S})$, and every homomorphism of the sequence $\mathrm{Hom}_A(R, \mathcal{S})$ is the direct sum of the corresponding homomorphisms in $\mathrm{Hom}_A(P, \mathcal{S})$ and $\mathrm{Hom}_A(Q, \mathcal{S})$. It remains to apply the following simple observation: The direct sum of two sequences of modules is exact if and only if both summands are exact.

Finally, to prove the implication (d) $\Longrightarrow$ (a) it suffices to apply the property (d) to the exact sequence

$$0 \to \mathrm{Ker}\,\varphi \to Q \xrightarrow{\varphi} R \to 0. \qquad ▶$$

Exercise. Suppose that $P \subset R$, where P is projective and R is free. Is it true that there exists a submodule $Q \subset R$ such that $R = P \oplus Q$?

11.16. Examples of projective modules. I. Over a field A all A-modules are projective, because they are all free.

II. Over the ring of integers $\mathbb{Z}$ all projective modules are free (although in this case not all modules are free). In fact, a $\mathbb{Z}$-module is just an abelian group, and we know that every subgroup of a free abelian group is free.

III. The simplest example of a module that is projective, but not free, is provided by the group $\mathbb{Z}$, considered as a module over the ring $\mathbb{Z} \oplus \mathbb{Z}$ with multiplication $(a, b) \cdot x = ax$.

IV. Modules of sections of vector bundles are projective. This will be proved later, in Theorem 11.32.

Exercise. Describe the projective modules over the ring of residues $\mathbb{Z}/m\mathbb{Z}$ and over the matrix ring.

11.17. Subbundles. We say that a vector bundle $\eta\colon E_\eta \to M$ is a *subbundle* of a vector bundle $\pi\colon E_\pi \to M$ (denoted by $\eta \subset \pi$) if

(i) the total space E_η is a submanifold of E_π;

(ii) the projection η is the restriction of π to E_η;

(iii) for any point $x \in M$ the fiber η_x is a linear subspace of the fiber π_x.

Examples. I. The zero subbundle: Its total space coincides with the image of the zero section.

II. The tangent bundle of the two-sphere does not contain one-dimensional subbundles. To prove this fact, suppose that such a subbundle ξ exists. Then for a smooth oriented closed curve $\Gamma \subset S^2$ we can define an integer invariant $\nu(\Gamma)$ equal to the number of half-turns made by the tangent vector to the curve with respect to the fibers of ξ. The number $\nu(\Gamma)$ does not change under smooth deformations of the curve Γ, and it changes its sign when the orientation of the curve changes. If Γ^+ is a small positively oriented circle and Γ^- the same circle with the negative orientation, then it is evident that $\nu(\Gamma^+) = 2$ and $\nu(\Gamma^-) = -2$. But on the sphere the curve Γ^+ can be smoothly deformed into Γ^-. This contradiction proves our assertion.

11.18. The local structure of subbundles. Suppose that at every point $z \in M$ a linear subspace η_z of the fiber π_z is given. In order that such a distribution should define a subbundle of π, two properties must hold:

(i) The set $\bigcup_{z\in M} \eta_z$ is a submanifold of E_π.

(ii) Local triviality for the family $\{\eta_z\}$.

In the case of a trivial bundle π, these requirements can be stated in the form of the following simple lemma.

11.19 Lemma. *Let $\pi\colon M \times \mathbb{R}^n \to M$ be the projection on the first factor, and at every point $z \in M$ let a k-dimensional linear subspace $\eta_z \subset \pi_z \cong \mathbb{R}^n$ be given. Denote by $\widetilde{\eta}\colon M \to G_{n,k}$ the map $\widetilde{\eta}(z) = \eta_z$, where the plane η_z is considered as a point in the Grassmann manifold $G_{n,k}$. Then the following two conditions are equivalent*:

(a) *The family* $\{\eta_z\}_{z\in M}$ *defines a subbundle* $\eta \subset \pi$.

(b) *The map* $\widetilde{\eta}$ *is smooth.*

◀ (a) $\Longrightarrow$ (b). Pick a point $a \in M$ and a basis $e_1, \ldots, e_k$ of the space η_a. By Lemma 11.8, there exist sections $s_1, \ldots, s_k \in \Gamma(\eta)$ such that $s_i(a) = e_i$. Since the sections s_i are continuous, there is a neighborhood U of a such that the vectors $s_1(z), \ldots, s_k(z)$ are linearly independent for all $z \in U$. Therefore, these vectors form a basis of the space η_z. In the neighborhood U, the map $\widetilde{\eta}$ can be represented as the composition of two maps

$$U \ni z \mapsto \big(s_1(z), \ldots, s_k(z)\big) \mapsto \mathcal{L}\big(s_1(z), \ldots, s_k(z)\big) \in G_{n,k},$$

where $\mathcal{L}\big(s_1(z), \ldots, s_k(z)\big)$ stands for the linear span of the vectors $s_1(z), \ldots, s_k(z)$. Therefore, the map $\widetilde{\eta}$ is smooth.

(b) $\Longrightarrow$ (a). Suppose that the plane $\eta_z \in G_{n,k}$ smoothly depends on the point z. Using the standard coordinate system in $G_{n,k}$ (see 10.11, VI), we can choose a basis $s_1(z), \ldots, s_k(z)$ of η_z that smoothly depends on the point z, if z belongs to a certain neighborhood U of a point $a \in M$.

Define a map $\varphi\colon U \times \mathbb{R}^k \to U \times \mathbb{R}^n$ by

$$\varphi(z, \lambda_1, \ldots \lambda_k) = \left(z, \sum_{i=1}^{k} \lambda_i s_i(z)\right).$$

The linear independence of the vectors $s_1(z), \ldots, s_k(z)$ implies that φ is of maximal rank. By the implicit function theorem (see Theorem 6.23 and Remark 6.24), the image $\operatorname{Im}\varphi$ is a submanifold of the space $U \times \mathbb{R}^n$. According to the construction of the map φ, this means that $\{\eta_z\}_{z\in M}$ defines a subbundle of π. ▶

The lemma describes the structure of subbundles of a trivial vector bundle and thus the local structure of subbundles of any vector bundle. We shall now apply the lemma to investigate the conditions under which the kernel and the image of a bundle morphism $\varphi \in \operatorname{Mor}(\pi, \eta)$ are subbundles in π and η, respectively. By the *kernel* (respectively, *image*) we understand the set $\bigcup_{z\in M} \operatorname{Ker}\varphi_z \subset E_\pi$ together with the restriction to this set of the projection π (respectively, the set $\bigcup_{z\in M} \operatorname{Im}\varphi_z \subset E_\eta$ together with the restriction of η).

11.20 Proposition. *For a vector bundle morphism* $\varphi\colon \pi \to \eta$ *over a manifold* M *the following conditions are equivalent*:

(a) $\dim \operatorname{Ker}\varphi_x$ *does not depend on* x.

(b) $\dim \operatorname{Im}\varphi_x$ *does not depend on* x.

(c) $\operatorname{Ker}\varphi$ *is a subbundle of* π.

(d) $\operatorname{Im}\varphi$ *is a subbundle of* η.

◀ The implications (c) $\Longrightarrow$ (a) and (d) $\Longrightarrow$ (b) are evident. The equivalence (a) $\Longleftrightarrow$ (b) follows from the fact that the sum $\dim \operatorname{Ker} \varphi_x + \dim \operatorname{Im} \varphi_x$ is equal to the dimension of the fiber of π and thus constant.

Let us prove that (b) implies (d). Since assertion (d) is local, it suffices to prove it in a neighborhood of an arbitrary point of the base space. Let $U \subset M$ be a neighborhood of the given point such that the bundles $\pi\big|_U$ and $\eta\big|_U$ are trivial. Therefore, we can assume that we deal with a morphism φ_U of trivial bundles acting from $\pi_U : U \times V \to U$ to $\eta_U : U \times W \to U$. To this morphism there corresponds a smooth map $\widetilde{\varphi}_U : U \to \operatorname{Hom}(V, W)$ sending each point x to the operator $\widetilde{\varphi}(x)$ whose value on the vector $v \in V$ is equal to the W-component of the element $\varphi(x, v) \in M \times W$. By assumption, the rank of $\widetilde{\varphi}_x$ does not depend on the point x; denote it by r. Suppose that at a given point $a \in U$ the vectors $v_1, \ldots, v_r$ have the property that their images under φ_a are linearly independent. Then by continuity there is a neighborhood of a where the vectors $\varphi_x(v_1), \ldots, \varphi_x(v_r)$ form a basis of $\operatorname{Im} \varphi_x$ that smoothly depends on x. Now by Lemma 11.19, $\operatorname{Im} \varphi$ is a subbundle of η.

The previous argument can also be applied to prove the implication (a) $\Longrightarrow$ (c). If a family of operators $\varphi_x \in \operatorname{Hom}(V, W)$ smoothly depends on x and has constant rank r, then $\operatorname{Im} \varphi_x$, as a point of the Grassmannian $G_{W,r}$, is a smooth function of x (by virtue of Lemma 11.19). Note that $\operatorname{Ker} \varphi_x = \operatorname{Ann} \operatorname{Im} \varphi_x^*$ (we recall that $\operatorname{Ann} \operatorname{Im} \varphi_x^*$ denotes the set of mutual zeros of all linear functionals from $\operatorname{Im} \varphi_x^*$). The smooth dependence of φ_x^* on x follows from the fact that the components of this operator in appropriate bases are equal to the components of φ_x. Hence $\operatorname{Im} \varphi_x^*$ is a smooth function of x. It remains to note that the map $\operatorname{Ann} : G_{V^*,r} \to G_{V,\dim V - r}$ sending every subspace into its annihilator is smooth. ▶

11.21. Direct sum of vector bundles. In the case of vector bundles the construction of the direct sum (Section 10.14) must agree with the linear structure in the fibers. We say that a vector bundle π is the *direct sum* of two subbundles η and ζ (notation: $\pi = \eta \oplus \zeta$) if its fiber over every point $x \in M$ is the direct sum of two subspaces: $\pi_x = \eta_x \oplus \zeta_x$.

If two vector bundles η and ζ over one manifold M are given, we can construct a vector bundle π that decomposes into a direct sum of two subbundles isomorphic to η and ζ. Such a vector bundle π is defined uniquely up to isomorphism; it is called the *outer direct sum* or *Whitney sum* of the bundles η and ζ and is also denoted by $\eta \oplus \zeta$. As in the case of arbitrary locally trivial bundles, the total space of $\eta \oplus \zeta$ can be defined as

$$E_{\eta\oplus\zeta} = \{(y, z) \in E_\eta \times E_\zeta \mid \eta(y) = \zeta(z)\},$$

and the projection is the map sending a point (y, z) to $\eta(y)$. The fiber of the Whitney sum over a point $x \in M$ is in a natural bijection with the space $\eta_x \oplus \zeta_x$; this bijection endows $(\eta \oplus \zeta)_x$ with the structure of a vector space.

11.22. Examples of direct sums. I. Let M be a submanifold of a Euclidean space E. Then the trivial bundle $E \times M \to M$ is a direct sum of two subbundles: the tangent subbundle $\pi_T \colon TM \to M$ and the normal subbundle $\nu \colon NM \to M$. The fiber ν_z of the normal subbundle over a point $z \in M$ is by definition the orthogonal complement of the tangent space T_zM in T_zE, the latter being identified with E. It is interesting to note that, for example, for the sphere $S^2 \subset \mathbb{R}^3$ the normal bundle ν is trivial, but the tangent bundle π_T is not (since it does not have nonvanishing sections; see Example 10.11, III). We see that the direct sum of a trivial and a nontrivial bundle can be trivial, a fact that looks quite unexpected at first sight.

II. The Whitney sum of the Möbius band with the trivial one-dimensional (unit) bundle is nontrivial. Indeed, the total space of this sum is the product $[0,1] \times \mathbb{R}^2 \subset \mathbb{R}^3$ with points $(0, y, z)$ and $(1, -y, z)$ identified for any $y, z \in \mathbb{R}$. This manifold is nonorientable and therefore is not diffeomorphic to $S^1 \times \mathbb{R}^2$.

11.23 Proposition. *If* $\pi = \eta \oplus \zeta$, *then* $\Gamma(\pi) = \Gamma(\eta) \oplus \Gamma(\zeta)$ (*the direct sum of submodules*).

◀ The assertion follows from (10.6) by virtue of the natural identifications $\Gamma(\eta) \times \Gamma(\varphi)$ and $\Gamma(\eta) \oplus \Gamma(\varphi)$. ▶

The following proposition is important because of its relationship with the property of projectivity.

11.24 Proposition. *Every subbundle of a vector bundle has a direct complement.*

◀ The proof is based on a standard technical trick: the introduction of a scalar product. A scalar product on a vector bundle π is by definition a scalar product in every fiber $x \in M$, smoothly depending on x. The smoothness requirement can be stated as follows: The scalar product of any two smooth sections is a smooth function.

Remark. After studying the construction of the tensor product of vector bundles (Section 11.35), the reader will see that the scalar product on a vector bundle π is nothing but a smooth function on the manifold $E_{\pi \otimes \pi}$, whose restriction to every fiber is linear and positive definite.

Example. A scalar product on the tangent bundle is the same thing as a Riemannian metric on the given manifold.

11.25 Lemma. *On any vector bundle there exists a scalar product.*

◁ To prove the lemma, note that for a trivial bundle the problem has a trivial solution: It suffices to supply every fiber with one and the same scalar product. Now let $\{U_i\}_{i \in I}$ be a covering of the manifold M with open sets, trivializing for π, and let g_i be a scalar product on the vector bundle $\pi\big|_{U_i}$. Choose a partition of unity $\{e_i\}_{i \in I}$, subjected to the covering $\{U_i\}_{i \in I}$ (see 4.18), and for $y_1, y_2 \in \pi_x$ set

$$g(y_1, y_2) = \sum e_i(x) g_i(y_1, y_2),$$

where the summation ranges over all indices i for which $x \in U_i$. All the necessary properties of the function g can be verified in a straightforward way. ▷

We continue the proof of Proposition 11.24. Let π be a bundle over M and $\eta \subset \pi$. Choose a scalar product g on π. Then in the fiber over any point $x \in M$ we can consider the subspace $\eta_x^\perp$, the orthogonal complement to η in π with respect to the scalar product g. We only have to check the smooth dependence of $\eta_x^\perp$ on x. Since this is a local property, we can assume that the bundle π is trivial, $\pi\colon M \times V \to M$. Then we have smooth maps $\widetilde{\eta}\colon M \to G_{V,k}$ (see 11.19) and $\widetilde{g}\colon M \to (V \otimes V)^*$ (scalar product). Denote by $N \subset (V \otimes V)^*$ the set of all symmetric positive definite bilinear forms. The map $G_{V,k} \times N \to G_{V,n-k}$ sending a pair (L, φ) to the orthogonal complement of L with respect to the scalar product φ is smooth. The map $x \mapsto \eta_x^\perp$ is the composition

$$M \to M \times M \to G_{V,k} \times N \to G_{V,n-k},$$

where the first arrow is the diagonal map $x \mapsto (x, x)$, the second arrow is the direct product of the maps $\widetilde{\eta}$ and $\widetilde{g}$, and the third arrow is the map introduced above. The resulting map is smooth, and by Lemma 11.19, the proposition is proved. ▶

11.26. Induced vector bundles. The geometric construction of the induced vector bundles does not differ from the same construction in the general case (see 10.16).

Example. Let $\pi\colon E \to M$ be an arbitrary bundle and let $\lambda = \pi_{T^*}: T^*M \to M$ be the cotangent bundle of its base space. Then the sections of the induced bundle $\pi^*(\lambda)$ are called *horizontal* 1*-forms* on the total space E (see Section 11.41). Pay attention to the fact that $\Gamma(\pi^*(\lambda))$ is embedded into the module of 1-forms on the manifold E.

The following assertion is one of the key facts in the theory of vector bundles.

11.27 Theorem. *Every vector bundle with connected base can be induced from the tautological bundle over an appropriate Grassmann manifold.*

◀ Let $\eta\colon E_\eta \to M$, $\dim M = n$, $\dim E_\eta = n + k$. Denote by

$$o_z : \eta_z \to T_z(\eta_z) \subset T_z(E_\eta)$$

the canonical identification of the vector space η_z with its tangent space at zero. By Whitney's theorem, there exists an immersion $\phi\colon E_\eta \to \mathbb{R}^{2(n+k)}$ of the manifold E_η into a Euclidean space. This means that all differentials

$$d_y\phi\colon T_y(E_\eta) \to T_{\phi(y)}\mathbb{R}^{2(n+k)}, \qquad y \in E_\eta,$$

are injective.

The *Gauss map* $g\colon M \to G_{2(n+k),k}$ assigns to every point $z \in M$ the k-dimensional subspace of the tangent space $T_O(\mathbb{R}^{2(n+k)}) \cong \mathbb{R}^{2(n+k)}$ that

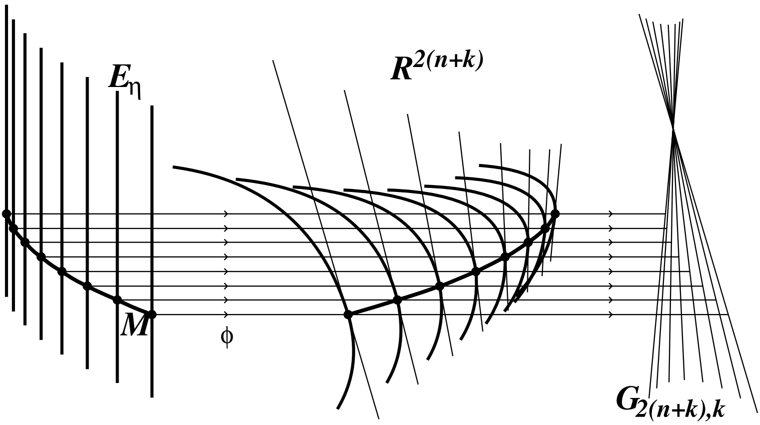


Figure 11.1. The Gauss map.

is the image of the subspace $T_z(\eta_z) \subset T_z(E_\eta)$ under the map

$$d_z(r_{\phi(z)} \circ \phi)\colon T_z(E_\eta) \to T_O\left(\mathbb{R}^{2(n+k)}\right)$$

(see Figure 11.1). Here z is understood as a point of the manifold E_η, the zero element in the fiber η_z, and $r_a\colon v \mapsto v - a$ (for $a, v \in \mathbb{R}^{2(n+k)}$) is the translation of the space $\mathbb{R}^{2(n+k)}$ by the vector $-a$. The map g is covered by the morphism of vector bundles

$$\gamma\colon \eta \to \Theta_{2(n+k),k}, \qquad \gamma_z = d_z(r_{\phi(z)} \circ \phi) \circ o_z.$$

For every point $z \in M$ the map γ_z is an isomorphism of the fiber η_z onto the fiber at $g(z) \in G_{2(n+k),k}$. Now Theorem 10.19 shows that the bundles η and $g^*(\Theta_{2(n+k),k})$ are isomorphic. ▶

11.28 Corollary. *The $C^\infty(M)$-module $\Gamma(\pi)$, $\pi\colon E_\pi \to M$, has a system of generators consisting of no more than N elements, where $N = N(n,k)$ is a natural number that depends only on the dimensions of the base and the fiber (n and k respectively) of the bundle π.*

◀ Note that the sections $s_{I,i}$ of the tautological bundle $\Theta_{m,l}$ described in Example V of Section 10.12 satisfy the assumptions of Proposition 11.10. Therefore, the sections $\widehat{f}(s_{I,i})$ of any induced bundle $f^*(\Theta_{m,l})$ (see Section 10.16) also satisfy these assumptions and thus generate the module $\Gamma(f^*(\Theta_{m,l}))$. Theorem 11.27 implies that any k-dimensional vector bundle over a manifold of dimension n can be induced by a Gauss map g from the tautological bundle $\Theta_{2(n+k),k}$. By Proposition 11.10, the sections $\widehat{g}(s_{I,i})$, where $1 \leqslant i \leqslant k$, $I = \{i_1, \dots, i_k\} \subset \{1, \dots, 2(n+k)\}$, generate the module $\Gamma(\pi)$. The number of these sections is $k\binom{2(n+k)}{k} = N(n,k)$. ▶

We have completed all preparations necessary to state and prove the two main theorems of this chapter (Theorems 11.29 and 11.32), which give an exhaustive description of the section modules of vector bundles.

11.29 Theorem. *For any pair π, η of vector bundles over a manifold M, the section functor Γ determines a one-to-one correspondence*

$$\mathrm{Mor}(\pi, \eta) \cong \mathrm{Hom}_{C^\infty(M)}(\Gamma(\pi), \Gamma(\eta)).$$

◀ We must prove that for any $C^\infty(M)$-homomorphism of modules $F\colon \Gamma(\pi) \to \Gamma(\eta)$ there exists a unique bundle morphism $\varphi\colon \pi \to \eta$ such that $\Gamma(\varphi) = F$.

First we prove the uniqueness. Suppose that $\varphi, \psi \in \mathrm{Mor}(\pi, \eta)$ and $\Gamma(\varphi) = \Gamma(\psi)$. This means that $\varphi, \psi\colon E_\pi \to E_\eta$ and $\varphi \circ s = \psi \circ s$ for any section $s \in \Gamma(\pi)$, i.e., $\varphi(s(x)) = \psi(s(x))$ for all $s \in \Gamma(\pi)$ and all $x \in M$. According to Lemma 11.8(a), every point of E_π can be represented as $s(x)$, which implies that $\varphi = \psi$.

Now suppose that a homomorphism $F\colon \Gamma(\pi) \to \Gamma(\eta)$ is given and we must define the corresponding map $\varphi\colon E_\pi \to E_\eta$, i.e., define its value $\varphi(y) \in E_\eta$ at an arbitrary point $y \in E_\pi$. The point y belongs to a certain fiber: $y \in \pi_x$. By Lemma 11.8(a), we can choose a section $s \in \Gamma(\pi)$ such that $s(x) = y$ and put

$$\varphi(y) = F(s)(x).$$

We must check (a) that φ is well defined, (b) that φ is a bundle morphism, and (c) the equality $\Gamma(\varphi) = F$.

(a) Let s_1 and s_2 be two sections of π such that $s_1(x) = s_2(x)$. Lemma 11.8(b) shows that $s_1 - s_2 \in \mu_x\Gamma(\pi)$. Therefore,

$$F(s_1 - s_2) \in \mu_x\Gamma(\eta) \quad \text{and} \quad F(s_1)(x) = F(s_2)(x).$$

(b) The only thing worth verifying here is the smoothness of the map $\varphi\colon E_\pi \to E_\eta$. It is sufficient to prove that the map $\varphi_U = \varphi\big|_{\pi^{-1}(U)}$ is smooth for an arbitrary open set $U \subset M$, over which both π and η are trivial. Note that

$$\varphi_U(t(x)) = F_U(t)(x), \tag{11.1}$$

where $t \in \Gamma(\pi\big|_U)$ and $F_U\colon \Gamma(\pi\big|_U) \to \Gamma(\eta\big|_U)$ is the localization of the homomorphism F on the subset U (that is, over the multiplicative system of functions that do not vanish at the points of U; see Section 10.6). Choosing some bases of the free $C^\infty(U)$-modules $\Gamma(\pi\big|_U)$ and $\Gamma(\eta\big|_U)$, we can write the homomorphism F_U as a matrix over the ring $C^\infty(U)$. By virtue of equation (11.1), the same matrix gives the coordinate representation of the morphism φ_U (Lemma 11.5). Since the elements of the matrix belong to $C^\infty(U)$, the map φ_U is smooth.

(c) For any $s \in \Gamma_\pi$, we have

$$(\Gamma(\varphi)(s))(x) = (\varphi \circ s)(x) = \varphi(s(x)) = F(s)(x),$$

i.e., $\Gamma(\varphi)(s) = F(s)$. ▶

11.30 Lemma. *If $\varphi\colon \zeta \to \pi$ is a vector bundle morphism and φ_z is an isomorphism of vector spaces $\zeta_z \cong \pi_z$ for any point $z \in M$, then φ is a bundle isomorphism.*

◀ The only thing we must check is that the inverse map $\varphi^{-1}\colon E_\pi \to E_\zeta$ is smooth. By the inverse function theorem (Section 6.21), it suffices to show that the differential $d_y\varphi$ at any point $y \in E_\zeta$ is an isomorphism of the corresponding tangent spaces. Since the dimensions of the manifolds E_ζ and E_π are equal, the dimensions of the tangent spaces T_yE_ζ and $T_{\varphi(y)}E_\pi$ are equal too. Therefore, for the differential $d_y\varphi$ to be an isomorphism, it is sufficient that it be injective. To check the latter, note that $d_{\varphi(y)}\pi \circ d_y\varphi = d_y\zeta$, because $\pi \circ \varphi = \zeta$. Now suppose that $d_y\varphi(v) = 0$. Then $d_y\zeta(v) = 0$, which means that $v \in T_y(\zeta_z)$, where $z = \zeta(y)$. By assumption, $\varphi\big|_{\zeta_z} = \varphi_z$ is an isomorphism between the fibers ζ_z and π_z. This implies $v = 0$. ▶

11.31 Corollary. *If $F \in \mathrm{Hom}_{C^\infty(M)}(\Gamma(\zeta), \Gamma(\pi))$ and for any point $x \in M$ the induced map*

$$F_x\colon \Gamma(\zeta)/\mu_x\Gamma(\zeta) \to \Gamma(\pi)/\mu_x\Gamma(\pi)$$

is an isomorphism of vector spaces, then F is a module isomorphism.

◀ This fact is easily reduced to the lemma just proved. Indeed, we have $F = \Gamma(\varphi)$ for an appropriate morphism $\varphi \in \mathrm{Mor}(\zeta, \pi)$ and $F_x = \varphi_x$ at every point $x \in M$. ▶

11.32 Theorem. *Let M be a connected manifold. A $C^\infty(M)$-module P is isomorphic to the module of sections $\Gamma(\pi)$ of a smooth vector bundle π over M if and only if P is finitely generated and projective.*

◀ (i) Recall, first of all, that *for any vector bundle π over M the module $\Gamma(\pi)$ is finitely generated* (Corollary 11.28).

(ii) Let us prove that *the module $\Gamma(\pi)$ is projective.*

◁ According to (i), there exists a finite system of sections $s_1, \ldots, s_N$ that generate the module $\Gamma(\xi)$. Let Q be a free $C^\infty(M)$-module of rank N with generators $e_1, \ldots, e_N$ and let $F\colon Q \to \Gamma(\xi)$ be the homomorphism such that $F(e_i) = s_i$ for $i = 1, \ldots, N$. By construction, F is epimorphic. Note that $Q = \Gamma(\eta)$ for a certain trivial vector bundle η; hence by Theorem 11.29 there is a bundle morphism $\varphi \in \mathrm{Mor}(\pi, \eta)$ such that $F = \Gamma(\varphi)$.

Lemma 11.8 implies that at every point $z \in M$ the map φ_z is surjective, since F is an epimorphism. Therefore, we can apply Proposition 11.20 and infer that $\mathrm{Ker}\,\varphi$ is a subbundle in η. By Proposition 11.24, there is a direct decomposition $\eta = \mathrm{Ker}\,\varphi \oplus \zeta$ for an appropriate subbundle ζ of π. Proposition 11.23 implies that

$$\Gamma(\eta) = \Gamma(\mathrm{Ker}\,\varphi) \oplus \Gamma(\zeta);$$

i.e., the module $\Gamma(\zeta)$ is a direct summand of a free module. By Proposition 11.15, $\Gamma(\zeta)$ is projective.

Now let us check that the map φ restricted to the total space of the subbundle ζ gives an isomorphism between ζ and π. Indeed, $\varphi_z\colon \zeta_z \to \pi_z$ is a linear isomorphism for any point $z \in M$, and we can use Lemma 11.30. The isomorphism of bundles $\zeta \cong \pi$ implies the isomorphism of modules $\Gamma(\zeta) \cong \Gamma(\pi)$ and thus assertion (ii). $\triangleright$

(iii) We shall now prove that *any projective module of finite-type is isomorphic to the module of sections of a smooth vector bundle.*

$\triangleleft$ Suppose that P is a projective $C^\infty(M)$-module with a finite number of generators. Then (see Proposition 11.15 and the remark in Section 11.13) we can write $\Gamma(\eta) = P' \oplus Q$, where η is the trivial bundle over M, P' and Q are submodules of $\Gamma(\eta)$, and $P' \cong P$. Since we are considering P only up to isomorphism, in the sequel by abuse of notation we suppress the prime and write P instead of P'.

Let $P_z = \{p(z) \mid p \in P\}$. This is an $\mathbb{R}$-linear subspace in η_z. The subspace Q_z is defined similarly. We claim that $\eta_z = P_z \oplus Q_z$.

Indeed, let $y \in \eta_z$. Choose a section $s \in \Gamma(\eta)$ such that $s(z) = y$ and represent it as $p+q$, where $p \in P$, $q \in Q$. Then $y = p(z) + q(z) \in P_z + Q_z$. On the other hand, suppose that $y \in P_z \cap Q_z$, i.e., $y = p(z) = q(z)$, where $p \in P$, $q \in Q$. Then $(p-q)(z) = 0$, hence by Lemma 11.8(b) we have

$$p - q = \sum_i f_i s_i = \sum_i f_i p_i + \sum_i f_i q_i$$

for a certain choice of $f_i \in \mu_z$, $p_i \in P$, $q_i \in Q$. Since $P \cap Q = 0$, the last equation implies $p = \sum_i f_i p_i$. Therefore, $p(z) = 0$, i.e., $y = 0$, and thus $\eta_z = P_z \oplus Q_z$.

We want to verify that the union of all subspaces P_z constitutes a subbundle in η and P that coincides with the module of its (smooth) sections. We shall show first that $\dim P_z$ does not depend on z. Let $\dim P_z = r$ for some point $z \in M$ and let $p_1, \ldots, p_r \in P$ be a set of sections whose values at z span the subspace P_z. The continuity of sections implies the linear independence of vectors $p_1(y), \ldots, p_r(y)$ for all points y in a neighborhood U of the point z. Therefore, $\dim P_y \geqslant \dim P_z$.

A similar argument for the submodule Q shows that we have $\dim Q_y \geqslant \dim Q_z$ in a neighborhood of z. Since the sum $\dim P_y + \dim Q_y$ is constant, we see that $\dim P_y$ is a locally constant function of y. Since M is connected, it is a global constant.

Having in mind Lemma 11.5, we shall now prove that the subspace P_z, viewed as a point of $G_{V,r}$, where V is the fiber of η, smoothly depends on z (here we can assume that η is a trivial bundle, because the problem under consideration is local). Indeed, let $p_1(a), \ldots, p_r(a)$ be a basis of the space P_a at a certain point $a \in M$. Then the vectors $p_1(z), \ldots, p_r(z)$ form a basis of the linear space P_z for all points z belonging to some neighborhood of a.

We see that the family P_z locally has a basis that smoothly depends on z and thus represents a smooth family of points in the Grassmannian $G_{V,r}$.

Denote the bundle with fibers P_z by π. By construction, $P \subset \Gamma(\pi)$. Let us prove the reverse inclusion. If $s \in \Gamma(\pi) \subset \Gamma(\eta)$, then there are elements $p \in P$ and $q \in Q$ such that $s = p + q$. Since $P_z \cap Q_z = 0$, the equation $p(z) + q(z) = s(z)$ implies that $q(z) = 0$ for any $z \in M$. Hence $q = 0$ and $s = p \in P$. ▷

We have shown that $P = \Gamma(\pi)$. This completes the proof of the theorem. ▶

Remark. Our proof shows that the module $\Gamma(\xi)$ is projective also in the case of a disconnected base manifold. On the other hand, any projective module over $C^\infty(M)$ is evidently reduced to the direct sum of modules $\Gamma(\pi_\alpha)$, where π_α is a vector bundle over a connected component M_α of the manifold M. The dimension of the bundle π_α may vary between the connected components.

11.33. Equivalence of the two categories. Theorems 11.29 and 11.32, taken together, establish the equivalence of the category VB_M of vector bundles over the manifold M and the category $\mathrm{Mod_{pf}}\, C^\infty(M)$ of projective finite-type modules over the algebra $C^\infty(M)$. This result is in full parallel with the result of Section 7.19 about the equivalence between the category of smooth manifolds and the category of smooth $\mathbb{R}$-algebras. It can be used in either direction, i.e., by applying algebra to geometry and vice versa.

Here is a simple example: *For any vector bundle π there is a vector bundle η such that the direct sum $\pi \oplus \eta$ is a trivial bundle.* This fact, surprising from the geometrical viewpoint, is reduced to the mere definition of a projective module by the application of Theorem 11.32.

Below (in Section 11.38) we shall give an example of an algebraic statement (*the tensor square of a one-dimensional projective module over $C^\infty(M)$ is isomorphic to $C^\infty(M)$*), which becomes evident after a geometrical trick (introduction of a scalar product on the corresponding vector bundle).

In the next sections we discuss two operations on vector bundles, namely the construction of tensor products and induced bundles, and the corresponding operations on projective modules.

11.34 Proposition. *The tensor product of two projective modules over a commutative ring is a projective module.*

◀ The well-known property (whose proof we leave to the reader as an algebraic exercise) of Hom$-\otimes$-*associativity* reads

$$\mathrm{Hom}_A(P \otimes_A Q, R) \cong \mathrm{Hom}_A(P, \mathrm{Hom}_A(Q, R))$$

for any three A-modules P, Q, R. In other words, the functor $\mathrm{Hom}_A(P \otimes Q, \cdot)$ is the composition of functors $\mathrm{Hom}_A(P, \cdot)$ and $\mathrm{Hom}_A(Q, \cdot)$. It remains to use the equivalence of (a) and (d) from Proposition 11.15. ▶

11.35. Tensor product of vector bundles. Let η and ζ be vector bundles over a manifold M. The fiber of the new bundle $\pi = \eta \otimes \zeta$ (*the tensor product of* η *and* ζ) over a point $z \in M$ is, by definition, the linear space $\pi_z = \eta_z \otimes \zeta_z$. The structure of a smooth manifold on the total space $E_\pi = \bigcup_{z\in M} \pi_z$ is introduced in the following way.

Every point of the base manifold M has a neighborhood U over which both bundles η and ζ are trivial. The sets $\pi^{-1}(U)$ form a covering of the space E_π. Let V be the "outer" fiber of the bundle η, let W be the "outer" fiber of the bundle ζ, while $\varphi\colon \eta^{-1}(U) \to U \times V$ and $\psi\colon \zeta^{-1}(U) \to U \times W$ are the corresponding trivializing diffeomorphisms. The chart

$$\chi\colon \pi^{-1}(U) \to U \times (V \otimes W)$$

is constructed as follows. Let $u \in \pi^{-1}(U)$. Then $u \in \eta_z \otimes \zeta_z$ for some point $z \in M$. The maps φ and ψ, restricted to the fiber over z, give isomorphisms $\eta_z \to V$ and $\zeta_z \to W$ and hence an isomorphism $\eta_z \otimes \zeta_z \to V \otimes W$. Denote the image of an element u under this isomorphism by v and put $\chi(u) = (z, v)$.

Exercise. Prove that these charts form a smooth atlas on E_π.

The bundle $\eta \otimes \zeta$ is thus defined. Note that it satisfies the axiom of local triviality by construction.

For brevity, we denote by $\eta^{\otimes k}$ the kth tensor power of a bundle η.

Example. If π_T and π_{T^*} are the tangent and cotangent bundles of a manifold, then $\pi_T^{\otimes k} \otimes \pi_{T^*}^{\otimes l}$ is the bundle of tensors of type (k, l) (k times contravariant and l times covariant).

11.36 Proposition. *Let P and Q be projective modules over a commutative ring A. Then the module* $\mathrm{Hom}_A(P, Q)$ *is also projective. If both P and Q are finitely generated, then the module* $\mathrm{Hom}_A(P, Q)$ *is also of finite-type.*

◀ If R and S are (finitely generated) free modules, then $\mathrm{Hom}_A(R, S)$ is also (finitely generated and) free. Indeed, let $\{r_i\}$ and $\{s_j\}$ be the free generators of R and S, respectively. Then the A-homomorphisms $h_{i,j} \in \mathrm{Hom}_A(R, S)$, defined by the rule $h_{i,j}(r_i) = \delta_{i,j}s_j$, where $\delta_{i,j}$ is the Kronecker delta, constitute a basis of $\mathrm{Hom}_A(R, S)$.

Now suppose that R and S are free A-modules that contain the modules P and Q, respectively, as direct summands. Let

$$\alpha_P\colon P \hookrightarrow R \quad \text{and} \quad \beta_P\colon R \to P, \quad \alpha_P \circ \beta_P = \mathrm{id}_P,$$

and similarly with α_Q and β_Q for Q, be the injections and projections that realize the corresponding decompositions of R and S into direct sums. Then the A-homomorphism

$$\mathrm{Hom}_A(P, Q) \ni h \mapsto \alpha_Q \circ h \circ \beta_P \in \mathrm{Hom}_A(R, S)$$

is the injection of $\mathrm{Hom}_A(P, Q)$ into the free module $\mathrm{Hom}_A(R, S)$, whose image, together with the kernel of the projection

$$\mathrm{Hom}_A(R, S) \ni H \mapsto \beta_Q \circ H \circ \alpha_P \in \mathrm{Hom}_A(P, Q),$$

turns the module $\mathrm{Hom}_A(P, Q)$ into a direct summand in the free A-module $\mathrm{Hom}_A(R, S)$. ▶

There is a natural map

$$\iota\colon P^* \otimes_A Q \to \mathrm{Hom}_A(P, Q), \quad \text{where } P^* = \mathrm{Hom}_A(P, A), \tag{11.2}$$

defined by the formula $\iota(p^* \otimes q)(p) = p^*(p)q$. If the modules P and Q are projective and finitely generated, then ι is an isomorphism. This fact has an evident proof for finitely generated free modules and can be generalized to arbitrary projective modules of finite-type by an argument similar to the proof of Proposition 11.36.

11.37. To every vector bundle ζ we can associate the dual bundle ζ^*, whose fiber ζ_z^* over a point $z \in M$ is the vector space dual to ζ_z. The precise construction of a smooth atlas on the set $E_{\zeta^*} = \bigcup_{z\in M} \zeta_z^*$ repeats the corresponding construction for the case of the cotangent bundle from the tangent bundle given in Section 9.24.

Example. $\pi_T^* = \pi_{T^*}$.

For any vector bundle ζ there is a natural pairing

$$\Gamma(\zeta) \times \Gamma(\zeta^*) \to C^\infty(M), \quad (s, s^*)(z) \stackrel{\text{def}}{=} (s(z), s^*(z)),$$

where $s \in \Gamma(\zeta),\ s^* \in \Gamma(\zeta^*),\ z \in M$.

Exercise. Using local triviality, verify that $(s, s^*)(z)$ is a smooth function for any smooth sections s and s^*.

For the bundle $\zeta = \pi_{TM}$ this pairing, by the equality $(\pi_{TM})^* = \pi_{T^*M}$, turns into the pairing

$$\Gamma(\pi_{TM}) \times \Gamma(\pi_{T^*M}) \to C^\infty(M).$$

A metric on ζ allows us to identify ζ_z and ζ_z^* for any point $z \in M$. Lemma 11.30 implies that vector bundles ζ and ζ^* are isomorphic.

Given a pair of vector bundles η and ζ, one can define the bundle $\mathrm{Hom}(\eta, \zeta)$ with the fiber $\mathrm{Hom}_{\mathbb{R}}(\eta_z, \zeta_z)$ over every point $z \in M$ in the same way. The construction follows that of the tensor product of two vector bundles (see Section 11.35).

Another possibility to define the bundle $\mathrm{Hom}(\eta, \zeta)$ is to explicitly reduce it to the constructions of the tensor product and the dual bundle, using the natural isomorphism of vector spaces $\mathrm{Hom}_{\mathbb{R}}(\eta_z, \zeta_z) = \eta_z^* \otimes \zeta_z$.

11.38. Example. The tensor square of the Möbius band, viewed as a one-dimensional vector bundle over the circle (Example 11.6, III), is $\mathbb{I}_{S^1}$, the trivial one-dimensional bundle. This result can either be understood directly from the construction of the Möbius band and the definition of the tensor product (we recommend to the reader to carry it out) or be deduced from a more general fact: *The tensor square of any one-dimensional bundle over an arbitrary manifold is isomorphic to the unit bundle.* To prove the latter, we use the above-mentioned isomorphism

$$\zeta \otimes \zeta \cong \zeta^* \otimes \zeta \cong \mathrm{Hom}(\zeta, \zeta).$$

If $\dim \zeta = 1$, then the bundle $\mathrm{Hom}(\zeta, \zeta)$ is trivial: The trivializing diffeomorphism $\varphi\colon M \times \mathbb{R} \to E_{\mathrm{Hom}(\zeta,\zeta)}$ can be defined by the formula $\varphi(a, \lambda)y = \lambda y$ for all $a \in M$, $\lambda \in \mathbb{R}$, $y \in \pi^{-1}(a)$.

Note, finally, that tensor multiplication introduces a group structure into the set $V^1(M)$ of equivalence classes of one-dimensional vector bundles over the manifold M. The order of any nonunital element of this group is 2. Thus, $V^1(S^1) = \mathbb{Z}_2$, the two elements being represented by the trivial bundle and the Möbius band.

In the next theorem we establish the relationship between the functors Γ, $\otimes$, and Hom.

11.39 Theorem. *The functor* Γ *preserves tensor products and homomorphisms, i.e.,*

$$\Gamma(\pi \otimes \eta) \cong \Gamma(\pi) \otimes \Gamma(\eta),$$
$$\Gamma(\mathrm{Hom}(\pi, \eta)) \cong \mathrm{Hom}_{C^\infty(M)}(\Gamma(\pi), \Gamma(\eta)),$$

for any two vector bundles π *and* η *over the manifold* M *(here and below, tensor products of* $C^\infty(M)$*-modules are computed over the ring* $C^\infty(M)$*).*

◀ The natural isomorphisms

$$\mathrm{Hom}(\pi, \eta) \cong \pi^* \otimes \eta, \quad \mathrm{Hom}_{C^\infty(M)}(\Gamma(\pi), \Gamma(\eta)) \cong \Gamma(\pi^*) \otimes \Gamma(\eta)$$

(see Section 11.36) together with the natural identifications $\pi^{**} = \pi$, $\Gamma(\pi)^{**} = \Gamma(\pi)$ show that it suffices to prove the assertion only for tensor products.

We shall construct a map from $\Gamma(\pi) \otimes \Gamma(\eta)$ to $\Gamma(\pi \otimes \eta)$ and verify that it is an isomorphism.

Let $s \in \Gamma(\pi)$, $t \in \Gamma(\eta)$. Define the section $s \otimes t \in \Gamma(\pi \otimes \eta)$ by the formula $(s \otimes t)(x) = s(x) \otimes t(x)$. From the construction of the bundle $\pi \otimes \eta$, we see that $s \otimes t$ is a smooth section of $\pi \otimes \eta$. The assignment of $s \otimes t$ to the pair (s, t) is homomorphic with respect to both arguments and thus defines a $C^\infty(M)$-module homomorphism $\iota\colon \Gamma(\pi) \otimes \Gamma(\eta) \to \Gamma(\pi \otimes \eta)$. Let us show that the value of this homomorphism at a point $x \in M$,

$$\iota_x\colon \Gamma(\pi) \otimes \Gamma(\eta)/\mu_x\bigl(\Gamma(\pi) \otimes \Gamma(\eta)\bigr) \to \pi_x \otimes \eta_x,$$

is an isomorphism of vector spaces. Note that

$$\iota_x \left[\sum s_i \otimes t_i\right] = \sum s_i(x) \otimes t_i(x),$$

where the square brackets denote the equivalence class of an element in the quotient space.

(i) Surjectivity of ι_x. An arbitrary element of the space $\pi_x \otimes \eta_x$ has the form $\sum y_i \otimes z_i$ with $y_i \in \pi_x$, $z_i \in \eta_x$. By Lemma 11.8(a), there are sections $s_i \in \Gamma(\pi)$, $t_i \in \Gamma(\eta)$ that take values y_i and z_i, respectively, at the point x. Then $\iota_x[\sum s_i \otimes t_i] = \sum y_i \otimes z_i$.

(ii) Injectivity of ι_x. We must show that if $s_i \in \Gamma(\pi)$, $t_i \in \Gamma(\eta)$, and $\sum s_i(x) \otimes t_i(x) = 0$ in the space $\pi_x \otimes \eta_x$, then there exist sections $p_i \in \Gamma(\pi)$, $q_i \in \Gamma(\eta)$ and functions $f_i \in \mu_x$ such that $\sum s_i \otimes t_i = \sum f_i p_i \otimes q_i$ (equality in $\Gamma(\pi) \otimes \Gamma(\eta)$).

The following lemma clarifies the structure of zero elements in the tensor product of vector spaces.

11.40 Lemma. *Let V and W be vector spaces over a certain field. Suppose that $v_i \in V$, $w_i \in W$ are nonzero vectors and*

$$\sum_{i=1}^{m} v_i \otimes w_i = 0 \quad \textit{in } V \otimes W.$$

Then there exist a natural number k, $1 \leqslant k \leqslant m$, and elements of the ground field a_{ij}, $1 \leqslant i \leqslant k$, $k < j \leqslant m$, such that after an appropriate renumeration

$$\{1, \dots, m\} \to \{1, \dots, m\},$$

and the same for both $\{v_i\}$ and $\{w_i\}$, the following relations hold:

$$v_j = \sum_{i=1}^{k} a_{ij} v_i, \quad j = k+1, \dots, m;$$

$$w_i = -\sum_{j=k+1}^{m} a_{ij} w_j, \quad i = 1, \dots, k.$$

◁ Indeed, if the elements $v_1, \dots, v_m$ are linearly independent, then the equality $\sum v_i \otimes w_i = 0$ implies that all vectors w_i are zero. If not, choose from $v_1, \dots, v_m$ a maximal linearly independent subset. Let it be $v_1, \dots, v_k$. Expand v_j for $j = k+1, \dots, m$ in terms of this basis: $v_j = \sum_{i=1}^{k} a_{ij} v_i$. Then

$$\begin{aligned}\sum_{i=1}^{m} v_i \otimes w_i &= \sum_{i=1}^{k} v_i \otimes w_i + \sum_{j=k+1}^{m} \Big(\sum_{i=1}^{k} a_{ij} v_i\Big) \otimes w_j \\ &= \sum_{i=1}^{k} v_i \otimes \Big(w_i + \sum_{j=k+1}^{m} a_{ij} w_j\Big),\end{aligned}$$

whence $w_i = -\sum_{j=k+1}^{m} a_{ij} w_i$. The lemma is proved. ▷

Applying the lemma in the current situation, we obtain

$$s_j = \sum_{i=1}^{k} a_{ij} s_i + s'_j, \quad j = k+1, \dots, m; \quad s'_j \in \mu_x \Gamma(\pi);$$

$$t_i = - \sum_{j=k+1}^{m} a_{ij} t_i + t'_i, \quad i = 1, \dots, k; \quad t'_i \in \mu_x \Gamma(\eta).$$

Therefore,

$$\sum_{i=1}^{m} s_i \otimes t_i = \sum_{i=1}^{k} s_i \otimes \left(-\sum_{j=k+1}^{m} a_{ij} t_j + t'_i \right) + \sum_{j=k+1}^{m} \left(\sum_{i=1}^{k} a_{ij} s_i + s'_j \right) \otimes t_j$$

$$= \sum_{i=1}^{k} s_i \otimes t'_i + \sum_{j=k+1}^{m} s'_j \otimes t_j \in \mu_x \big(\Gamma(\pi) \otimes \Gamma(\eta) \big),$$

as desired.

We see that the morphism ι is an isomorphism for every point $x \in M$. Using the corollary of Lemma 11.30, we want to conclude that ι is a module isomorphism. For this corollary to apply, both modules must be modules of sections of smooth vector bundles. In our case, only the module $\Gamma(\pi) \otimes \Gamma(\eta)$ is to be checked in this respect. But by Theorem 11.32 and Proposition 11.34 it is projective. Hence, by Theorem 11.32, this module is isomorphic to a certain module of sections, and thus the corollary can be used. This completes the proof of the theorem. ▶

11.41. Differential 1-forms. Fix a manifold M. The module of sections $\Gamma(\pi_{T^*})$ of the bundle $\pi_{T^*} = \pi_{T^*M}$ is called the *module of differential 1-forms* of the manifold M and denoted by $\Lambda^1(M)$. The elements of this module, i.e., smooth sections of the bundle π_{T^*}, are referred to as *differential 1-forms* on the manifold M.

According to Section 9.25, any function $f \in C^\infty(M)$ gives rise to a section of π_{T^*} defined by

$$s_{df} \colon M \to T^*M, \quad s_{df}(z) = d_z(f).$$

Sections of this kind are called *differentials* of smooth functions. The differential of a function f, viewed as a purely algebraic object, will be denoted by df. The same thing, viewed geometrically, as a map from M to T^*M, will be denoted by s_{df} (the "graph" of df).

Exercise 9.22 implies that the map

$$d \colon C^\infty(M) \to \Lambda^1(M), \quad f \mapsto df,$$

is a derivation of the algebra $C^\infty(M)$ with values in the $C^\infty(M)$-module $\Lambda^1(M)$. It is called *universal derivation*. The origin of this term will be clarified later.

Now let (U, x) be a chart on M and $\big(\pi_{T^*}^{-1}(U), T^*x\big)$ the corresponding chart on T^*M. In Section 9.24 we denoted by T^*x the system of coordinate functions $\{x_i, p_j\}$, where the value of x_i at $(z, \theta) \in T^*U$ is the ith coordinate of the point z, while p_j is the jth component in the expansion of the covector θ over the basis dx_i. Within this chart every smooth section s has the coordinate representation

$$\begin{aligned} x_i &= x_i, \quad i = 1, \dots, n, \\ p_j &= p_j(x), \quad j = 1, \dots, n, \end{aligned}$$

where $p_j(x) \in C^\infty(U)$. As before, the section $s_{dx_i} \in \Gamma(T^*U) = \Lambda^1(U)$ will be denoted by dx_i. It follows that the sections dx_i, $i = 1, \dots, n$, form a basis of the free $C^\infty(U)$-module $\Lambda^1(U)$. In particular, the restriction of a differential form $\omega \in \Lambda^1(M)$ to U belongs to $\Lambda^1(U)$ and can therefore be written as

$$\omega = \sum_i p_i(x) dx_i.$$

For the differential of a function df we can write

$$df = \sum \frac{\partial f}{\partial x_i} dx_i$$

(see the exercise in Section 9.25).

Vector fields on M can be viewed as sections of the tangent bundle (see Section 9.40), and the pairing

$$\Gamma(\pi_{TM}) \times \Gamma(\pi_{T^*M}) \to C^\infty(M)$$

becomes

$$D(M) \times \Lambda^1(M) \to C^\infty(M). \tag{11.3}$$

Returning to local coordinates, we recall that for any point $z \in U \subset M$ the basis $d_z x_1, \dots, d_z x_n$ of the linear space $T_z^* M$ is by definition dual to the basis

$$\left.\frac{\partial}{\partial x_1}\right|_z, \dots, \left.\frac{\partial}{\partial x_n}\right|_z$$

of the space $T_z M$. If X is a vector field and ω is a 1-form represented in special local coordinates by

$$X = \sum_i \alpha_i(x) \frac{\partial}{\partial x_i}, \quad \omega = \sum_i p_i(x) dx_i,$$

then the result of the pairing (X, ω) restricted to U is the function $(X, \omega)\big|_U = \sum_i \alpha_i p_i(x) \in C^\infty(U)$. In particular, for $\omega = df$ we get

$$(X, df)\big|_U = \sum_i \alpha_i \frac{\partial f}{\partial x_i} = X(f)\big|_U.$$

11.42. Universal derivation. The definition of the module of differential 1-forms $\Lambda^1(M)$ given above was descriptive. Here we want to give a conceptual definition of the same thing.

Let $\mathfrak{M}$ be a certain category of modules over an algebra A. A pair (δ, Λ), where Λ is an object of $\mathfrak{M}$ and $\delta \in D(\Lambda)$ a derivation from A to Λ, is called the *universal derivation* in the category $\mathfrak{M}$ if for any module P from $\mathfrak{M}$ the correspondence

$$\mathrm{Hom}_A(\Lambda, P) \ni h \mapsto h \circ \delta \in D(P)$$

is an isomorphism of A-modules $\mathrm{Hom}_A(\Lambda, P)$ and $D(P)$.

Proposition. *The universal derivation is unique up to isomorphism; i.e., if (δ', Λ') is another universal derivation, then there exists an isomorphism of A-modules $\gamma\colon \Lambda \to \Lambda'$ such that $\delta' = \gamma \circ \delta$.*

◀ The universality of δ implies that there is a homomorphism $\gamma\colon \Lambda \to \Lambda'$ such that $\delta' = \gamma \circ \delta$. Similarly, $\delta = \gamma' \circ \delta'$ for an appropriate $\gamma' \in \mathrm{Hom}_A(\Lambda', \Lambda)$. Therefore, $\delta = \gamma' \circ \gamma \circ \delta$ and $\mathrm{Im}\,\delta \subset \Lambda_0$, where

$$\Lambda_0 = \{\omega \in \Lambda \mid \gamma'(\gamma(\omega)) = \omega\} \subset \Lambda.$$

Let $\alpha\colon \Lambda \to \Lambda/\Lambda_0$ be the natural projection. Then $0 = \alpha \circ \delta \in D(\Lambda/\Lambda_0)$, hence $\alpha = 0$ by the universality of δ. Since α is surjective, it follows that $\Lambda = \Lambda_0$ and $\gamma' \circ \gamma = \mathrm{id}_\Lambda$.

Symmetrically, $\gamma \circ \gamma' = \mathrm{id}_{\Lambda'}$; hence γ and γ' are mutually inverse. ▶

11.43 Theorem. *The pair $(d, \Lambda^1(C^\infty(M)))$ is the universal derivation in the category of geometric $C^\infty(M)$-modules.*

◀ Let us prove that the natural homomorphism

$$\eta_P\colon \mathrm{Hom}_{C^\infty(M)}\left(\Lambda^1(M), P\right) \to D(P), \quad h \mapsto h \circ d,$$

is an isomorphism if the module P is geometric. To this end, we shall construct the inverse homomorphism

$$\nu_P\colon D(P) \to \mathrm{Hom}_{C^\infty(M)}\left(\Lambda^1(M), P\right), \quad X \mapsto h_X.$$

We use the fact that any 1-form $\omega \in \Lambda^1(M)$ can be written as $\omega = \sum_i f_i dg_i$ (this will be independently proved below; see Corollary 11.49). Put

$$h_X(\omega) \stackrel{\text{def}}{=} \sum_i f_i X(g_i)$$

and check that h_X is well defined.

Let $z \in M$ and $f = \sum_i c_i g_i$, where $c_i = f_i(z) \in \mathbb{R}$. Then we can write $\omega_z = \sum_i c_i d_z g_i = d_z f$. Moreover,

$$h_X(\omega)(z) = \sum_i f_i(z) X_z(g_i) = X_z\left(\sum_i c_i g_i\right) = X_z(f).$$

Here X_z denotes the composition $C^\infty(M) \xrightarrow{X} P \to P_z$ (see the end of Section 11.11); therefore, the value of $h_X(\omega)$ at an arbitrary point $z \in M$ is well defined, i.e., does not depend on the choice of the representation $\omega = \sum_i f_i dg_i$. Since P is geometric, this implies that $h_X(\omega)$ is well defined.

In the case $\omega = dg$ we have by definition $h_X(dg) = X(g)$, which means that $X = h_X \circ d \Leftrightarrow \eta_P \circ \nu_P = \mathrm{id}_{D(P)}$. If $X = h \circ d$, then

$$h_X(\omega) = \sum_i f_i h(dg_i) = h\left(\sum_i f_i dg_i\right) = h(\omega),$$

and thus $h_X = h \Leftrightarrow \nu_P \circ \eta_P = \mathrm{id}_{D(P)}$. ▶

The theorem implies a pairing

$$\Lambda^1(M) \times D(P) \to P, \quad (\omega, X) \mapsto h_X(\omega),$$

whose result can be written as $\omega(X) = h_X(\omega)$.

Exercise. Show that for $P = C^\infty(M)$ this pairing coincides with the one defined in Section 11.41.

11.44. Conceptual definition of differential forms. The theorem proved in the previous section suggests a conceptual approach to the theory of differential forms over an arbitrary algebra A. Namely, differential 1-forms should be understood as elements of the A-module Λ, the target of the universal derivation $\delta\colon A \to \Lambda$. It is important that this module depends on the choice of the category of A-modules M (see Section 11.42) and is referred to as the *representing object* for the functor D in this category.

Exercise. Indicate a category of $C^\infty(M)$-modules in which the functor D is not representable, i.e., does not determine a universal derivation.

In the category of all A-modules over an arbitrary commutative K-algebra A the functor D is representable. We denote the corresponding universal derivation by

$$d_{\mathrm{alg}}\colon A \to \Lambda_{\mathrm{alg}}(A).$$

To prove its existence consider the free A-module $\widetilde{\Lambda}$, generated by the symbols $\widetilde{da}$ for all $a \in A$. Let $\widetilde{\Lambda}_0$ be the submodule spanned by all relations of the form

$$\widetilde{d}(ka) - k\widetilde{d}a, \quad \widetilde{d}(ab) - a\widetilde{d}b - b\widetilde{d}a, \quad k \in K,\ a, b \in A.$$

Then

$$\Lambda_{\mathrm{alg}}(A) = \widetilde{\Lambda}/\widetilde{\Lambda}_0 \quad \text{and} \quad d_{\mathrm{alg}}a = \widetilde{d}a \mod \widetilde{\Lambda}_0, \quad a \in A.$$

If $A = C^\infty(M)$, then $\Lambda_{\mathrm{alg}}(A) \neq \Lambda^1(M)$. For example, if $M = \mathbb{R}$, then we have $d_{\mathrm{alg}}e^x - e^x d_{\mathrm{alg}}x \neq 0$. One can, however, prove that $\Lambda^1(M)$ is the geometrization of the module $\Lambda_{\mathrm{alg}}(A)$; see Section 11.11.

Other functors of differential calculus have similar properties. For the functor Diff_l they are treated below.

11.45. Behavior of differential forms under morphisms of manifolds. Let $F\colon M \to N$ be a smooth map of manifolds. Note that the composition

$$d \circ F^*\colon C^\infty(N) \to \Lambda^1(M)$$

is a derivation of the algebra $C^\infty(N)$ with values in the $C^\infty(N)$-module $\Lambda^1(M)$. We recall that the $C^\infty(N)$-module structure in $\Lambda^1(M)$ is defined by

$$(f, \omega) \mapsto F^*(f)\omega, \quad f \in C^\infty(N),\ \omega \in \Lambda^1(M).$$

Exercise. Prove that $\Lambda^1(M)$ is a geometric $C^\infty(N)$-module.

By Theorem 11.43 there exists a $C^\infty(N)$-homomorphism

$$h_{d\circ F^*}\colon \Lambda^1(N) \to \Lambda^1(M)$$

such that $h_{d\circ F^*} \circ d = d \circ F^*$. For the sake of brevity we shall write F^* instead of $h_{d\circ F^*}$. We thus have an $\mathbb{R}$-linear map

$$F^*\colon \Lambda^1(N) \to \Lambda^1(M),$$

with the following properties:

(i) $F^* \circ d = d \circ F^*$;

(ii) $F^*(f\omega) = F^*(f)F^*(\omega)$, if $f \in C^\infty(N)$, $\omega \in \Lambda^1(N)$.

In view of (i) and (ii), we see that

$$F^*(\omega) = \sum_i F^*(f_i)dF^*(g_i), \quad \text{if} \quad \omega = \sum_i f_i dg_i.$$

Exercise. Show that

1. $(F \circ G)^* = G^* \circ F^*$, if $L \xrightarrow{G} M \xrightarrow{F} N$;
2. $(F^*)^{-1} = (F^{-1})^*$;
3. $F^*(\omega)_z(\xi) = \omega_{F(z)}(d_zF(\xi))$, or, equivalently,

$$F^*(\omega)_z = (d_zF)^*(\omega_{F(z)}), \quad \text{where } z \in M \text{ and } \omega \in \Lambda^1(N).$$

11.46. Jet algebras $\mathcal{J}^l(M)$. We return to the case $A = C^\infty(M)$ and for every natural l define the $C^\infty(M)$-module $\mathcal{J}^l(M)$ as the module of sections of the vector bundle $\pi_{J^l}\colon J^lM \to M$ (see Example IX in Section 10.11). The elements of this module are referred to as *l-jets* on the manifold M.

According to Section 9.65, every function $f \in C^\infty(M)$ gives rise to the section

$$s_{j_l(f)}\colon M \to J^lM, \quad z \mapsto [f]^l_z.$$

Sections of this kind are called *l-jets* of smooth functions.

Note that the multiplication in the algebra $C^\infty(M)$ induces an algebra structure in each fiber of the bundle π_{J^l}. In fact, let $z \in M$, $f, g \in C^\infty(M)$, and $h \in \mu_z^{l+1}$. Then $fg = f(g+h) \mod \mu_z^{l+1}$. Therefore, the formula

$$[f]_z^l \cdot [g]_z^l \stackrel{\text{def}}{=} [fg]_z^l$$

gives a well-defined product in the fiber $J_z^l M$. This multiplication induces a $C^\infty(M)$-algebra structure in the module $\mathcal{J}^l(M) = \Gamma(\pi_{J^l})$.

Using this multiplication, we can give a more transparent coordinate expression of jets. Let $\delta x_i = j_l(x_i) - x_i j_l(1)$ and $\delta^\sigma = \delta x_{i_1} \cdots \delta x_{i_k}$ if $\sigma = (i_1, \dots, i_k)$, $0 < |\sigma| \leqslant l$. We also put $\delta x^\varnothing = j_l(1)$. Then the sections δ^σ, $|\sigma| \leqslant l$, form a basis of the vector bundle π_{J^l} over U. This follows from the fact that the l-jets of polynomials $(x-z)^\sigma$, $|\sigma| \leqslant l$, are a basis of $J_z^l M$. We see that a jet of order l on the manifold M in the special coordinate system corresponding to a local chart (U, x) can be written as $\sum_{|\sigma| \leqslant l} \alpha_\sigma \delta x^\sigma$.

11.47. Jet algebras $\mathcal{J}^l(M)$ as representing objects. With respect to differential operators in the algebra $C^\infty(M)$, the jet algebras $\mathcal{J}^l(M)$ play a role similar to the role of the module of differential forms $\Lambda^1(M)$ with respect to derivations. This fact can be proved by an argument very close to the one that we used in Section 11.41.

If we forget the geometric meaning of the module $\mathcal{J}^l(M)$ as the module of sections of the bundle $J^l M$ and view its elements from a purely algebraic standpoint, we denote the jet of the function $f \in C^\infty(M)$ by $j_l(f)$.

11.48 Theorem. *There is a finite set of functions $f_1, \dots, f_m \in C^\infty(M)$ whose l-jets $j_l(f_1), \dots, j_l(f_m)$ generate the $C^\infty(M)$-module $\mathcal{J}^l(M)$.*

◀ If $M = \mathbb{R}^k$, then as such we can take the set of all monomials x^σ with $|\sigma| \leqslant l$, where $x = (x_1, \dots, x_k)$ are the usual coordinates in $\mathbb{R}^k$. In the general case, choose an appropriate k and consider an immersion $F \colon M \to \mathbb{R}^k$ (Whitney's theorem). Then the family $F^*(x^\sigma)$, $|\sigma| \leqslant l$, will have the required property. Indeed, by Corollary 11.28 it suffices to show that for every point $z \in M$ the l-jets $[F^*(x^\sigma)]_z^l$ generate the vector space $J_z^l M$. Since F is an immersion, the differential $d_z F$ is injective; hence $(d_z F)^*$ is surjective. Therefore, among the differentials $d_z F^*(x_i)$, $i = 1, \dots, k$, there are $n = \dim M$ linearly independent ones, say $d_z F^*(x_1), \dots, d_z F^*(x_n)$. The functions $F^*(x_1), \dots, F^*(x_n)$ form a local system of coordinates near the point z. As Corollary 2.9 to the generalized Hadamard's lemma shows, the monomials of degree $\leqslant l$ in these variables generate the space $J_z^l M$. ▶

11.49 Corollary. *There is a finite set of functions $f_1, \dots, f_m$ from $C^\infty(M)$ whose differentials $df_1, \dots, df_m$ generate the $C^\infty(M)$-module $\Lambda^1(M)$.*

◀ Let $f_1, \dots, f_m$ be the functions whose jets generate $\mathcal{J}^1(M)$. Then their differentials generate $\Lambda^1(M)$. Indeed, the canonical direct decomposition $J_z^1 M = \mathbb{R} \oplus T_z^* M$ (see the proof of Corollary 9.27) shows that the bundle

π_{J^1} is the direct sum of the two bundles $\mathbb{I}_M : M \times \mathbb{R} \to M$ and π_{T^*}. Passing to the modules of sections and using Proposition 11.23, we infer

$$\mathcal{J}^1(M) = C^\infty(M) \oplus \Lambda^1(M).$$

Therefore, the images of the elements that generate $\mathcal{J}^1(M)$ under the projection $\mathcal{J}^1(M) \to \Lambda^1(M)$ generate $\Lambda^1(M)$. It remains to notice that the image of $j_1(f)$ is df. ▶

11.50 Proposition. *The* $\mathbb{R}$*-homomorphism* $j_l \colon C^\infty(M) \to \mathcal{J}^l(M)$, $f \mapsto j_l(f)$ *is a differential operator of order* l, *i.e., satisfies* Definition 9.57.

◀ We must prove that we have $\big(\delta_{g_0} \circ \cdots \circ \delta_{g_l}\big)(j_l) = 0$ for any functions $g_0, \ldots, g_l$. Let $\theta \in \mathcal{J}^l(M)$ and

$$\Delta = \theta \cdot j_l \colon C^\infty(M) \to \mathcal{J}^l(M), \quad \Delta(f) = \theta \cdot j_l(f).$$

Then

$$\begin{aligned}\delta_g(\Delta)(f) &= \theta \cdot j_l(gf) - g\theta \cdot j_l(f) = \theta \cdot j_l(g) \cdot j_l(f) - \theta \cdot g j_l(f)\\ &= \big(j_l(g) - g j_l(1)\big) \cdot \theta \cdot j_l(f) = \delta_l(g) \cdot \theta \cdot j_l(f) = \delta_l(g) \cdot \Delta(f),\end{aligned}$$

where $\delta_l(g) = j_l(g) - g j_l(1)$. Therefore,

$$\big[\big(\delta_{g_0} \circ \cdots \circ \delta_{g_l}\big)(j_l)\big](f) = \delta_l(g_0) \cdots \delta_l(g_l) \cdot j_l(f).$$

The required fact follows from the equality $\delta_l(g_0) \cdots \delta_l(g_l) = 0$. To prove the latter, note that the image of the element $\delta_l(g)$ under the natural projection

$$\mathcal{J}^l(M) \to \mathcal{J}^0(M) = C^\infty(M), \quad j_l(f) \mapsto j_0(f),$$

is equal to $\delta_0(g) = 0$. This means that the value of the section $\delta_l(g)$ at any point $z \in M$ belongs to $\mu_z J^l_z M = \mu_z / \mu_z^{l+1}$. Hence

$$\delta_l(g_0) \cdots \delta_l(g_l) \in \mu_z^{l+1} J^l_z M = \mu_z^{l+1} / \mu_z^{l+1} = 0.$$

The element $\delta_l(g_0) \cdots \delta_l(g_l)$ is thus the zero section of the vector bundle π_{J^l}, i.e., the zero element of the module $\mathcal{J}^l(M)$. ▶

The operator $j_l \colon C^\infty(M) \to \mathcal{J}^l(M)$ is referred to as the *universal differential operator of order* $\leqslant l$ on the manifold M. The origin of the word "universal" will become clear in a little while.

If P is a geometric $C^\infty(M)$-module, then there is a natural pairing

$$\operatorname{Diff}_l P \times \mathcal{J}^l(M) \to P.$$

Indeed, suppose that $\Delta \in \operatorname{Diff}_l P$, $\Theta \in \mathcal{J}^l(M)$, $z \in M$, and let $f \in C^\infty(M)$ be a smooth function such that $\Theta(z) = [f]^l_z$. Put

$$(\Delta, \Theta)(z) = \Delta(f)(z) \in P_z.$$

By virtue of Corollary 9.64 the value $(\Delta, \Theta)(z)$ does not depend on the choice of f.

Exercise. As in the proof of Theorem 11.43, show that the totality of all values $\Delta(f)(z) \in P_z$ uniquely determines the element $(\Delta, \Theta) \in P$.

To an arbitrary operator $\Delta \in \operatorname{Diff}_l P$ we can assign the homomorphism of $C^\infty(M)$-modules $h_\Delta\colon \mathcal{J}^l(M) \to P$ by putting $h_\Delta(\Theta) = (\Delta, \Theta)$. It follows from the definition of the pairing that we have $(\Delta, j_l(f)) = \Delta(f)$. Therefore, $j_l \circ h_\Delta = \Delta$. On the other hand, if $h\colon \mathcal{J}^l(M) \to P$ is an arbitrary $C^\infty(M)$-homomorphism, then the composition

$$\Delta_h = h \circ j_l\colon C^\infty(M) \to P,$$

according to 9.67, 9.59, is a differential operator of order $\leqslant l$ that satisfies $h_{\Delta_h} = h$. We thus arrive at the following important result:

11.51 Proposition. *For any geometric $C^\infty(M)$-module P the assignment*

$$\operatorname{Hom}_{C^\infty(M)}(\mathcal{J}^l(M), P) \ni h \mapsto h \circ j_l \in \operatorname{Diff}_l P$$

defines a natural isomorphism of $C^\infty(M)$-modules

$$\operatorname{Hom}_{C^\infty(M)}(\mathcal{J}^l(M), P) \cong \operatorname{Diff}_l P.$$

In other words, the functor Diff_l *in the category of geometrical $C^\infty(M)$-modules is representable, with representing object $\mathcal{J}^l(M)$.* ▶

This proposition explains why the differential operator $j_l\colon C^\infty(M) \to \mathcal{J}^l(M)$ is called universal. As in the case of differential forms, the assumption that P is geometrical is essential: Without it, Proposition 11.51 is not valid.

The significance of Proposition 11.51 is also explained by the fact that it shows how to introduce correctly the notion of jet in the differential calculus over any commutative K-algebra A. To do this, one must, first of all, choose the corresponding category of A-modules, say $\mathfrak{M}$ (see Section 11.42) and then define the module of l-jets $\mathcal{J}^l_{\mathfrak{M}}(A)$ as the range of values of the universal differential operator $j_l^{\mathfrak{M}}\colon A \to \mathcal{J}^l_{\mathfrak{M}}(A)$. The universality of the operator $j_l^{\mathfrak{M}}$ means that for any module P in the category $\mathfrak{M}$ the correspondence

$$\operatorname{Hom}_A(\mathcal{J}^l_{\mathfrak{M}}(A), P) \ni h \mapsto h \circ j_l \in \operatorname{Diff}_l P$$

determines a natural A-module isomorphism $\operatorname{Hom}(\mathcal{J}^l_{\mathfrak{M}}(A), P) \cong \operatorname{Diff}_l P$.

Exercise. Describe the module of l-jets $\mathcal{J}^l_{\mathfrak{M}}(A)$, where A is the algebra of smooth functions on the cross $\mathbf{K}$, while $\mathfrak{M}$ is the category of geometrical modules over this algebra.

11.52. Change of rings. To translate the construction of the induced bundle into algebraic language, we must understand what relations between modules over different rings arise when a homomorphism from one ring to another is given.

Let $\varphi\colon A \to B$ be a ring homomorphism, and P a module over A. The homomorphism φ allows us to view B as an A-module with multiplication $a \cdot b = \varphi(a)b$ and hence define an A-module $\varphi_*(P) = B \otimes_A P$. Setting $b_1(b_2 \otimes p) = b_1 b_2 \otimes p$, we convert $\varphi_*(P)$ into a B-module. The assignment

$P \mapsto \varphi_*(P)$ extends to a functor from $\operatorname{Mod} A$ to $\operatorname{Mod} B$, called the functor of *change of rings*.

Proposition. *The change of rings functor preserves projectivity.*

◀ We shall show that the projectivity of an A-module P implies the projectivity of the B-module $\varphi_*(P)$, using property (d) from Proposition 11.15. For an arbitrary B-module Q there is an isomorphism of abelian groups

$$\operatorname{Hom}_A(P \otimes_A B, Q) \cong \operatorname{Hom}_A(P, \operatorname{Hom}_A(B, Q)). \tag{11.4}$$

More exactly, the elements

$$\gamma \in \operatorname{Hom}_A(P \otimes_A B, Q) \quad \text{and} \quad \delta \in \operatorname{Hom}_A(P, \operatorname{Hom}_A(B, Q))$$

that correspond to each other under this isomorphism are related by the equations

$$\gamma(p \otimes b) = \delta(p)(b), \quad p \in P, \ b \in B.$$

In particular, if $\gamma \in \operatorname{Hom}_B(P \otimes_A B, Q)$, then for any $p \in P$, $b_1, b_2 \in B$ we have

$$\gamma(p \otimes b_1 b_2) = b_1 \gamma(p \otimes b_2),$$

and so

$$\delta(p)(b_1 b_2) = b_1 \cdot \delta(p)(b_2),$$

i.e., $\delta(p) \in \operatorname{Hom}_B(B, Q) \cong Q$. The converse argument is also valid. Therefore, isomorphism (11.4) induces an isomorphism

$$\operatorname{Hom}_B(P \otimes_A B, Q) \cong \operatorname{Hom}_A(P, Q).$$

This isomorphism is natural with respect to Q, i.e., it extends to an isomorphism of functors on the category of B-modules with values in the category of abelian groups,

$$\operatorname{Hom}_B(P \otimes_A B, \cdot) \cong \operatorname{Hom}_A(P, \cdot),$$

which, by Proposition 11.15, implies that the B-module $P \otimes_A B$ is projective. ▶

11.53. Algebraic formulation of induced bundles. We now establish the algebraic meaning of the procedure of inducing vector bundles. Let $\varphi\colon N \to M$ be a smooth map of manifolds,

$$\Phi = \varphi^*\colon C^\infty(M) \to C^\infty(N),$$

the corresponding homomorphism of function rings, and let Φ_* be the functor of change of rings. According to Proposition 11.52, the functor Φ_* preserves projectivity; besides, it preserves the finite-type property.

Therefore, the functor Φ_* can be restricted to the subcategory of finitely generated projective modules:

$$\Phi_*\colon \operatorname{Mod}_{\mathrm{pf}} C^\infty(M) \to \operatorname{Mod}_{\mathrm{pf}} C^\infty(N).$$

11.54 Theorem. *For any vector bundle π over M there is an isomorphism of $C^\infty(N)$-modules*

$$\Gamma(\varphi^*(\pi)) \cong \Phi_*(\Gamma(\pi)).$$

This isomorphism can be chosen to be natural with respect to π, so that the functors $\Gamma \circ \varphi^*$ and $\Phi_* \circ \Gamma$ are isomorphic, and the functor diagram

$$\begin{array}{ccc} \mathrm{VB}_M & \xrightarrow{\ \varphi^*\ } & \mathrm{VB}_N \\ \Big\downarrow{\scriptstyle\Gamma} & & \Big\downarrow{\scriptstyle\Gamma} \\ \operatorname{Mod}_{\mathrm{pf}} C^\infty(M) & \xrightarrow[\ \Phi_*\ ]{} & \operatorname{Mod}_{\mathrm{pf}} C^\infty(N) \end{array}$$

is commutative.

◀ Below we will refer to the lifting of sections $\hat{\varphi}$ defined in Section 10.18. Let $A = C^\infty(M)$, $B = C^\infty(N)$. The map

$$B \times \Gamma(\pi) \to \Gamma(\varphi^*(\pi)),$$

which sends the pair (f, s) to the section $f \cdot \hat{\varphi}(s)$, is homomorphic over A with respect to either argument (here the B-module $\Gamma(\varphi^*(\pi))$ is viewed as an A-module with multiplication introduced via the ring homomorphism Φ). Therefore, this map defines an A-homomorphism

$$\nu\colon B \otimes_A \Gamma(\pi) \to \Gamma(\varphi^*(\pi)).$$

Note that in fact, ν is a homomorphism not only over A, but also over B. Indeed, for $f, g \in B$ and $s \in \Gamma(\pi)$ we have

$$\nu(fg \otimes s) = fg\hat{\varphi}(s) = f\nu(g \otimes s).$$

Let us prove that ν is an isomorphism. The module $B \otimes_A \Gamma(\pi)$ is finitely generated and projective; hence by Theorem 11.32 it is isomorphic to the module of sections of a bundle over N. Using Lemma 11.30, we can consider the value of the homomorphism ν at a point $w \in N$:

$$\nu_w\colon B \otimes_A \Gamma(\pi)/\mu_w \otimes_A \Gamma(\pi) \to (\varphi^*(\pi))_w \cong \pi_{\varphi(w)}.$$

Using the identification of $(\varphi^*(\pi))_w$ with $\pi_{\varphi(w)}$, we obtain

$$\nu_w([g \otimes s]) = g(w)s(\varphi(w)).$$

The map ν_w is epimorphic, because for any $z \in \pi_{\varphi(w)}$ we can, by Lemma 11.8(a), find a section s such that $s(\varphi(w)) = z$ and hence $\nu_w([1 \otimes s]) = z$.

Now let us check that ν_w is monomorphic. Let $\sum_i g_i \otimes s_i$ be an element of $B \otimes_A \Gamma(\pi)$ such that $\sum_i g_i(w)s_i(\varphi(w)) = 0$. Set $g_i(w) = \beta_i \in \mathbb{R}$ and put $\overline{g}_i = g_i - \beta_i$. The previous equation can be rewritten as $\sum_i \beta_i s_i \in \mu_{\varphi(w)}\Gamma(\pi)$, i.e.,

$$\sum_i \beta_i s_i = \sum_j f_j t_j,$$

where $f_j \in \mu_{\varphi(w)}$, $t_i \in \Gamma(\pi)$. By the definition of the A-module structure in B,

$$g \otimes ft = \varphi^*(f)g \otimes t \quad \text{for all} \quad g \in B,\ f \in A,\ t \in \Gamma(\pi).$$

Therefore, the following transformations are valid:

$$\begin{aligned}\sum_i g_i \otimes s_i &= \sum_i \overline{g}_i \otimes s_i + \sum_i \beta_i \otimes s_i = \sum_i \overline{g}_i \otimes s_i + 1 \otimes \sum_i \beta_i s_i \\ &= \sum_i \overline{g}_i \otimes s_i + 1 \otimes \sum_j f_j t_j \\ &= \sum_i \overline{g}_i \otimes s_i + \sum_j \varphi^*(f_j) \otimes t_j \in \mu_w \otimes \Gamma(\pi)\end{aligned}$$

(the last inclusion holds because $\varphi^*(f_j) \in \varphi^*(\mu_{\varphi(w)}) \subset \mu_w$).

We see that ν_w is an isomorphism at any point $w \in N$; hence ν is an isomorphism of B-modules. Its naturality with respect to π is evident. The theorem is proved. ▶

Exercise. Show that the $C^\infty(N)$-module $D_\varphi(M)$ consisting of vector fields along the map $\varphi\colon N \to M$ (see Section 9.47) is naturally isomorphic to the module $\Gamma(\varphi^*(\pi_T))$.

What is the algebraic meaning of section lifting; i.e., what map $\Gamma(\pi) \to B \otimes_A \Gamma(\pi)$ is it described by? It is easy to see from the definitions that this is the map that takes each element $s \in \Gamma(\pi)$ to $1 \otimes s$.

11.55. Pseudobundles and geometric modules. Let M be a smooth manifold and $\mathcal{F} = C^\infty(M)$. Theorem 11.32 gives a geometric meaning to the notion of finitely generated projective $\mathcal{F}$-module. We want to find out, in the spirit of Sections 11.11–11.12, to what extent arbitrary modules over $\mathcal{F}$ possess a geometrical interpretation.

In Section 11.11, to an arbitrary module P over an arbitrary commutative K-algebra A we assigned a pseudobundle π_P. For the function algebra $\mathcal{F}(|P|)$ on the total space of the bundle, we took the symmetric algebra $\mathcal{S}(P^*)$ of the module P. In the case of the algebra of smooth functions $A = C^\infty(M)$, it is natural to take the smooth envelope $\overline{\mathcal{F}(|P|)} = \overline{\mathcal{S}(P^*)}$ instead of just $\mathcal{S}(P^*)$, as we did in Section 9.80 for the cotangent bundle and the algebra of symbols.

11.56 Exercises. 1. Show that maps

$$\pi_P\colon |P| \to |A|, \quad s_p\colon |A| \to |P| \quad (p \in P),$$

defined in Section 11.11, are continuous in the Zariski topology defined by the function algebra $\overline{\mathcal{S}(P^*)}$.

2. Let $P = D(C^\infty(\mathbf{K}))$ be the module of vector fields on the cross (see Exercises 7.14, 9.35, 9.45, 9.78). Find $|P|$.

From now on, by continuous sections of a pseudobundle π_P we shall understand sections, continuous in the Zariski topology, corresponding to the smooth envelope of the symmetric algebra $\overline{\mathcal{F}(|P|)} = \overline{\mathcal{S}(P^*)}$. The set of all such sections forms a module over the ring $C^\infty(M)$, which we denote by $\Gamma_c(\pi)$. The assignment $p \mapsto s_p$ defines a $C^\infty(M)$-module homomorphism

$$\mathbb{S}\colon P \to \Gamma_c(\pi).$$

By Theorem 11.32, for a projective finitely generated module P, this homomorphism is a monomorphism, and its image coincides with the submodule of smooth sections in $\Gamma_c(\pi)$.

We pass to examples of geometric and nongeometric modules over the algebras of smooth functions.

11.57. Examples. *A. Geometric nonprojective modules.*

I. A smooth map of manifolds $\varphi\colon M \to N$ gives rise to a homomorphism of the corresponding smooth function rings $\varphi^*: B \to A$ and thereby turns A into a B-module. An easy check shows that this module is always geometric. However, it is projective only in exceptional cases (for instance, if φ is a diffeomorphism). The simplest example of a geometric nonprojective module of this kind is obtained if M is the manifold consisting of one point.

Exercise. Describe all smooth maps φ for which the B-module A is projective.

II. The ideal μ_a of any point $a \in M$, viewed as a $C^\infty(M)$-module, is obviously geometric. It turns out that this module is projective if and only if $\dim M = 1$.

Indeed, the value of the module μ_a at a point $b \in M$ is the quotient space $\mu_a/\mu_a\mu_b$. Its dimension is

$$\dim \mu_a/\mu_a\mu_b = \begin{cases} \dim M, & \text{if } b = a, \\ 1, & \text{if } b \neq a. \end{cases}$$

The first equality follows from the fact that μ_a/μ_a^2 is the cotangent space of the manifold M at the point a (see Section 9.27). The second one is a consequence of Lemma 2.11.

The fiber dimension of the vector bundle corresponding to μ_a is thus constant in the case $\dim M = 1$ and nonconstant in the case $\dim M > 1$.

There exist two different connected one-dimensional manifolds: the line $\mathbb{R}^1$ and the circle S^1. What vector bundles correspond to μ_a in each case? The answer, at first sight unexpected, is that for the line it is the trivial one-dimensional bundle $\mathbb{I}_{\mathbb{R}^1}$, while for the circle it is the Möbius band bundle described in Example 11.6, I. Here is the proof.

◀ Let $\mathcal{F} = C^\infty(\mathbb{R}^1)$ and let $a \in \mathbb{R}^1$ be an arbitrary point. Hadamard's Lemma 2.10 implies that the map $\mathcal{F} \to \mu_a$, sending every function f into the product $(x-a)f$, establishes the module isomorphism $\mathcal{F} \to \mu_a$. Therefore, $\mu_a \cong \mathcal{F} \cong \Gamma(\mathbb{I}_{\mathbb{R}^1})$.

This argument does not apply to $\mathcal{F} = C^\infty(S^1)$ and $a \in S^1$, because in this case μ_a is not a principal ideal: There is no smooth function on the circle that vanishes only at one point and has a nonzero derivative at this point. The isomorphism between μ_a and the module of sections of the nontrivial vector bundle π, whose total space E_π is the Möbius band, can be defined as follows. We know that the tensor square of π is isomorphic to $\mathbb{I}_{\mathbb{R}^1}$ (see Example 11.38); hence there is a well-defined multiplication $\Gamma(\pi) \times \Gamma(\pi) \to \mathcal{F}$. Fix a section $f_0 \in \Gamma(\pi)$ with a single simple zero at the point a (i.e., $f_0(a) = 0$, $f_0'(a) \neq 0$, $f_0(b) \neq 0$ for $b \neq a$). Then the map that sends every section $f \in \Gamma(\pi)$ to the product $f_0 f \in \mathcal{F}$ establishes the required isomorphism $\Gamma(\pi) \to \mu_a$. ▶

B. Nongeometric modules.

III. The $C^\infty(M)$-module of lth order jets $J^l_z M = C^\infty(M)/\mu_z^{l+1}$ (see Section 9.64) is not geometric if $l \geqslant 1$. This is due to the facts that:

(i) $\mu_{z'} \cdot J^l_z M = J^l_z M$, if $z' \neq z$;

(ii) $\mu_z \cdot J^l_z M = \mu_z/\mu_z^{l+1}$.

Leaving the proof to the reader we deduce that

$$\bigcap_{z' \in M} \mu_{z'} \cdot J^l_z M = \mu_z/\mu_z^{l+1}.$$

The last module is composed of all "invisible" elements of $J^l_z M$ (see 11.11) and is nontrivial for $l \geqslant 1$.

However, $C^\infty(M)$-modules $T_z M = D(M)/\mu_z D(M)$ (see Lemma 9.75) and $T^*_z M = \Lambda^1(M)/\mu_z \Lambda^1(M)$ are geometric. Indeed, if P is one of them, then $\mu_{z'} \cdot P = P$ for $z' \neq z$ and $\mu_z \cdot P = 0$ (prove that) and hence $\bigcap_{z' \in M} \mu_{z'} \cdot P = 0$.

IV. Let $A = C^\infty(\mathbb{R})$ and let $I \subset A$ be the ideal that consists of all functions with compact support. The reader can prove, by way of exercise, that the quotient module $P = A/I$ has the property $P = \bigcap_{x \in \mathbb{R}} \mu_x P$, so that the corresponding map $\mathbb{S}$ (see Section 11.56) is identically zero. The module P in this example consists entirely of invisible (unobservable) elements.

11.58. Vector bundles as quasi-bundles. We are now in a position to keep the promise given previously and explain the relationship between the algebraic treatment of a vector bundle as a module and the treatment of a quasi-bundle as an embedding of smooth algebras.

Proposition. *The algebra of functions on the total space of a vector bundle π is isomorphic to the smooth envelope of the complete symmetric algebra of the module of sections $\Gamma(\pi)$.*

◀ Since the modules of sections of a given bundle π and its conjugate π^* are isomorphic (see the remark in Section 11.38), it suffices to construct an isomorphism of the algebra $C^\infty(E_\pi)$ with the smooth envelope of the symmetric algebra of $\Gamma(\pi^*)$. Such an isomorphism can be built in a natural way. Indeed, every section $s \in \Gamma(\pi^*) = S^1(\Gamma(\pi^*))$ defines a function on E_π that is linear on the fibers. Elements of $S^2(\Gamma(\pi^*))$ correspond to functions on E_π that are quadratic on the fibers; elements of $S^3(\Gamma(\pi^*))$ correspond to functions on E_π that are cubic on the fibers; etc. Such functions are obviously smooth. The whole symmetric algebra $S(\Gamma(\pi^*))$ can be considered as the algebra of all functions on E_π polynomial on every fiber of π. The construction of the smooth envelope (Section 3.36) extends the set of polynomial functions to the set of all smooth functions. ▶

We shall complete this chapter by proving the equivalence of two definitions of differential operator, the conventional one and the algebraic one (see Section 9.67) in the class of projective $C^\infty(M)$-modules, and by constructing a representing object in the category of geometric $C^\infty(M)$-modules for the functor $Q \mapsto \mathrm{Diff}_l(P, Q)$, where P is a projective module.

11.59. Jet bundles. Suppose that P is a projective $C^\infty(M)$-module, i.e., the module of sections of a vector bundle $\pi_P\colon E \to M$, and $\mu_z \in C^\infty(M)$ is the maximal ideal corresponding to the point $z \in M$. Note that $\mu_z^{l+1}P$ is a submodule of P, and let $J_z^l P \stackrel{\mathrm{def}}{=} P/\mu_z^{l+1}P$ be the quotient module. The image of the element $p \in P$ under the natural projection will be denoted by $[p]_z^l \in J_z^l P$.

The vector space $J_z^l P$ is a module over the algebra $J_z^l M$ with respect to the multiplication

$$[f]_z^l [p]_z^l \stackrel{\mathrm{def}}{=} [fp]_z^l, \quad f \in C^\infty(M),\ p \in P.$$

Exercise. Prove that this multiplication is well defined.

Put $J^l P \stackrel{\mathrm{def}}{=} \bigcup_{z\in M} J_z^l P$. Our nearest aim is to equip the set $J^l P$ with the structure of a smooth manifold in such a way that the natural projection

$$\pi_{J^l P}\colon J^l P \to M, \quad J_z^l P \mapsto z \in M,$$

will define a vector bundle structure over M on the smooth manifold $J^l P$. This vector bundle will be called the bundle of *jets of order l* (or *l-jets*) of the bundle π_P.

On the total space E of the bundle π_P there is an adapted atlas (see Section 11.3). Its charts are of the form $(\pi^{-1}(U), x, u)$, where (U, x) is a chart of the corresponding atlas on M and $u = u^1, \dots, u^m$, $m = \dim \pi_P$, are the fiber coordinates. Then, according to Proposition 11.13, the localization $P_U = \Gamma(\pi_P\big|_U)$ is a free $C^\infty(U)$-module. Let $e_1, \dots, e_m$ be its basis. The restriction of an element $q \in P$ to U can be written as

$$q\big|_U = \sum_{i=1}^{m} f^i e_i, \quad \text{where } f^i \in C^\infty(U).$$

In other words, in the adapted coordinates a section of the vector bundle is represented by a vector function $(f^1, \dots, f^m)$ in the variables $(x_1, \dots, x_n)$. This implies that $[q]_z^l$ is uniquely determined by the m-uple $([f^1]_z^l, \dots, [f^m]_z^l)$, and therefore, the collection of numbers

$$\left(x_1, \dots, x_n, u^1, \dots, u^m, \dots, p_\sigma^i, \dots\right), \quad |\sigma| \leqslant l, \quad p_\sigma^j = \frac{\partial^{|\sigma|} f^j}{\partial x^\sigma},$$

uniquely determines the point $[q]_z^l \in \pi_{J^l P}^{-1}(U)$, where $\left(u^j, \dots, p_\sigma^i, \dots\right)$ are the special local coordinates of the l-jet of the function f^j in $J^l M$ (see Section 10.11, IX). The functions

$$x_i,\ u^j,\ p_\sigma^j, \quad 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m,\ 0 < |\sigma| \leqslant l,$$

form a coordinate system in the domain $\pi_{J^l P}^{-1}(U) \subset J^l P$. Charts of this type will be referred to as *special charts* on $J^l P$.

Exercise. Show that special charts on $J^l P$ that correspond to compatible charts on M are compatible as well. In other words, these charts form an atlas, thus defining the structure of a smooth manifold in $J^l P$.

The projection

$$\pi_{J^l P} \colon J^l P \to M, \quad J_z^l P \mapsto z \in M,$$

is evidently smooth. The special charts described above are direct products of the form $U \times \mathbb{R}^{mN}$, where N is the number of all derivatives of order $\leqslant l$ (see Example IX in Section 10.11). Therefore, $\pi_{J^l P}$ is a bundle. Moreover, trivializing diffeomorphisms $\pi_{J^l P}^{-1}(U) \to U \times \mathbb{R}^{mN}$ are linear on each fiber, so that $\pi_{J^l P}$ is a vector bundle over M.

11.60. Suppose that $A = C^\infty(M)$, $z \in M$, and P and Q are $C^\infty(M)$-modules corresponding to vector bundles π_P and π_Q over M, respectively. We have the following generalization of Corollary 9.61.

11.61 Proposition. *Suppose* $p_1,\ p_2 \in P$, $\Delta \in \mathrm{Diff}_l(P, Q)$, *and* $p_1 - p_2 \in \mu_z^{l+1} P$. *Then* $\Delta(p_1)(z) = \Delta(p_2)(z)$. *In particular, if the elements* $p_1,\ p_2 \in P$ *coincide in a neighborhood* $U \ni z$, *then for any differential operator* $\Delta \in \mathrm{Diff}(P, Q)$ *we have* $\Delta(p_1)(z) = \Delta(p_2)(z)$. *In other words, differential operators that act on sections of vector bundles are local.*

◀ Indeed, if $p_1 - p_2 \in \mu_z^{l+1} P$, then $\Delta(p_1 - p_2) \in \mu_z Q$ by Proposition 9.67. Now, if the two sections p_1 and $p_2 \in P$ coincide in a neighborhood of $U \ni z$, then $p_1 - p_2 \in \mu_z^{l+1} P$ for any l. ▶

This proposition allows us to correctly define the restriction

$$\Delta\big|_U : P\big|_U \to Q\big|_U, \quad \Delta\big|_U(\bar{p})(z) = \Delta(p)(z), \quad \bar{p} \in P\big|_U,\ p \in P,\ z \in U,$$

for any differential operator $\Delta \in \mathrm{Diff}(P, Q)$ and any open set $U \subset M$, where p is an arbitrary element of the module P coinciding with $\bar{p}$ in a certain neighborhood of the point z. According to this definition $\Delta_U(p\big|_U)(z) = \Delta(p)\big|_U(z)$ if $p \in P$. An operator Δ is uniquely determined by its restrictions to the charts of an arbitrary atlas of the manifold M.

Now fix a system of local coordinates $x_1, \ldots, x_n$ in a neighborhood $U \subset M$ so that both vector bundles $\pi_P\big|_U$ and $\pi_Q\big|_U$ are trivial. Let $e_1, \ldots, e_m$ and $\varepsilon_1, \ldots, \varepsilon_k$ be bases of modules $P\big|_U$ and $Q\big|_U$, respectively. Then the restriction of the elements $p \in P$ and $q \in Q$ to U is represented as

$$p\big|_U = \sum_{i=1}^{m} f^i e_i, \quad q\big|_U = \sum_{r=1}^{k} g^r \varepsilon_r, \quad \text{where } f^i,\ g^r \in C^\infty(U).$$

Fixing the bases $e_1, \ldots, e_m$ and $\varepsilon_1, \ldots, \varepsilon_r$, we can define the $C^\infty(U)$-linear maps

$$\alpha_i : C^\infty(U) \to P\big|_U, \quad f \mapsto fe_i, \quad 1 \leqslant i \leqslant m,$$

$$\beta_j : Q\big|_U \to C^\infty(U), \quad \sum_{r=1}^{k} g^r \varepsilon_r \mapsto g^j, \quad 1 \leqslant j \leqslant r.$$

The composition $\Delta_{i,j} \stackrel{\text{def}}{=} \beta_j \circ \Delta \circ \alpha_i : C^\infty(U) \to C^\infty(U)$ is, according to Sections 9.67 and 9.59, a differential operator of order $\leqslant l$. For scalar differential operators we have already proved that the algebraic definition 9.57 coincides with the conventional one. Now we see that the action of the operator $\Delta\big|_U$ on $p\big|_U$, i.e., on a vector function $\big(f^1, \ldots, f^m\big)$, is given by

$$\begin{pmatrix} \Delta_{1,1} & \ldots & \Delta_{1,m} \\ \vdots & \ddots & \vdots \\ \Delta_{k,1} & \ldots & \Delta_{k,m} \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} = \begin{pmatrix} \Delta_{1,1}(f_1) + \ldots + \Delta_{1,m}(f_m) \\ \vdots \\ \Delta_{k,1}(f_1) + \ldots + \Delta_{k,m}(f_m) \end{pmatrix}.$$

It follows that the standard notion of a matrix differential operator is a particular case of the general algebraic definition 9.67, since for the scalar differential operators, like $\Delta_{i,j}$, this fact has already been established (see Section 9.62). Matrix differential operators are the coordinate description of differential operators (in the sense of Definition 9.67) that send the sections of one vector bundle to the sections of another over the ground algebra $A = C^\infty(M)$.

11.62. Jet modules. The module of smooth sections of the vector bundle π_{J^lP} is called the *module of l-jets of the bundle* π_P and denoted by $\mathcal{J}^l(P)$. The elements of this module are called *geometric l-jets of the module P* or simply *l-jets*. It is worth noticing that the $C^\infty(M)$-module $\mathcal{J}^l(P)$ is also a

$\mathcal{J}^l(M)$-module with respect to multiplication,

$$(\theta \cdot \Theta)(z) \stackrel{\text{def}}{=} \theta(z)\Theta(z), \quad \theta \in \mathcal{J}^l(M),\ \Theta \in \mathcal{J}^l(P),$$

where the multiplication $J^l_z M \times J^l_z P \to J^l_z P$ that appears on the right-hand side was defined in Section 11.59.

As in the scalar case, any element of the module P gives rise to a section $j_l(p)$ of the bundle $\pi_{J^l P}$ defined by $j_l(p)(z) = [p]^l_z$. Suppose that in the adapted coordinates p is represented by the vector function $\left(f^1, \dots, f^m\right)$. Then in the corresponding special coordinates on $J^l P$, the section $j_l(p)$ takes the form of the vector function

$$\left(f^1, \dots, f^m, \dots, \frac{\partial^{|\sigma|} f^i}{\partial x^\sigma}, \dots\right), \quad |\sigma| \leqslant l.$$

The coordinate expression of $j_l(p)$ shows, first, that this section is smooth, i.e., that $j_l(p) \in \mathcal{J}^l(P)$, and, second, that the $\mathbb{R}$-linear map

$$j_l \colon P \to \mathcal{J}^l(P), \quad p \mapsto j_l(p),$$

is a differential operator of order $\leqslant l$.

Exercise. Give a coordinate-free proof of these facts.

The operator j_l is referred to as the *universal differential operator* of order l in the bundle π_P.

11.63 Proposition. *Let P be a projective $C^\infty(M)$-module. There exists a finite set of elements $p_1, \dots, p_m \in P$ such that the corresponding l-jets $j_l(p_1)$, ..., $j_l(p_m)$ generate the $C^\infty(M)$-module $\mathcal{J}^l(P)$.*

◀ Let $\bar{p}_1, \dots, \bar{p}_k \in P$ be a finite system of generators of the module P (see Corollary 11.28). Let $f_1, \dots, f_s$ be a finite set of functions whose l-jets generate $\mathcal{J}^l(M)$ (Proposition 11.48). Then the l-jets

$$j_l(f_i \bar{p}_j), \quad 1 \leqslant i \leqslant s,\ 1 \leqslant j \leqslant k,$$

generate $\mathcal{J}^l(P)$. To establish this fact, it is sufficient, by Proposition 11.10, to show that the l-jets $[f_i \bar{p}_j]^l_z$ generate the fiber $J^l_z P$ for any $z \in M$. We know that any element of this fiber has the form $[p]^l_z$ for a certain $p \in P$. Let

$$p = \sum_j g_j \bar{p}_j, \quad g_j \in C^\infty(M), \quad \text{and} \quad [g_j]^l_z = \sum_i \lambda_{ji} [f_j]^l_z, \quad \lambda_{ji} \in \mathbb{R}.$$

Then

$$[p]^l_z = \sum_j [g_j \bar{p}_j]^l_z = \sum_j [g_j]^l_z [\bar{p}_j]^l_z = \sum_{i,j} \lambda_{ji} [f_i]^l_z [\bar{p}_j]^l_z = \sum_{i,j} \lambda_{ji} [f_i \bar{p}_j]^l_z. \quad ▶$$

Suppose that Q is a geometric $C^\infty(M)$-module. Following the approach used in Section 11.50, we define the pairing

$$\mathrm{Diff}_l(P, Q) \times \mathcal{J}^l(P) \to Q.$$

For a point $z \in M$ and a jet $\Theta \in \mathcal{J}^l(M)$, we can choose an element $p \in P$ such that $\Theta(z) = [p]_z^l$. For an arbitrary differential operator $\Delta \in \mathrm{Diff}_l(P, Q)$ put $(\Delta, \Theta)(z) = \Delta(p)(z) \in Q_z$. By virtue of Proposition 11.61, the value $(\Delta, \Theta)(z)$ does not depend on the choice of the representative in the class $[p]_z^l$.

If $\Theta = \sum_i h_i j_l(p_i)$ (see Proposition 11.63), then for p we can take the element $\sum_i \lambda_i p_i$, where $\lambda_i = h_i(z) \in \mathbb{R}$. Therefore,

$$\Delta(p)(z) = \sum_i \lambda_i \Delta(p_i)(z) = \Big[\sum_i h_i \Delta(p_i)\Big](z).$$

Since the module Q is geometric, it follows that

$$(\Delta, \Theta) = \sum_i h_i \Delta(p_i) \in Q.$$

This proves the existence of the pairing.

Proceeding as at the end of Section 11.50, assign to an operator $\Delta \in \mathrm{Diff}_l(P, Q)$ a homomorphism of $C^\infty(M)$-modules

$$h_\Delta \colon \mathcal{J}^l(P) \to Q, \quad h_\Delta(\Theta) = (\Delta, \Theta).$$

It follows from the definition of the pairing that $(\Delta, j_l(p)) = \Delta(p)$. Therefore, $h_\Delta \circ j_l = \Delta$. On the other hand, if $h \colon \mathcal{J}^l(P) \to Q$ is an arbitrary $C^\infty(M)$-homomorphism, then the composition

$$\Delta_h = h \circ j_l \colon P \to Q$$

is, according to Sections 9.59 and 9.67, a differential operator of order $\leqslant l$. Also, evidently, $h_{\Delta_h} = h$. We have thus established the following important fact.

11.64 Theorem. *Let a projective $C^\infty(M)$-module P be given. For any geometric $C^\infty(M)$-module Q, the correspondence*

$$\mathrm{Hom}_{C^\infty(M)}(\mathcal{J}^l(P), Q) \ni h \mapsto h \circ j_l \in \mathrm{Diff}_l(P, Q)$$

defines a natural isomorphism of $C^\infty(M)$-modules

$$\mathrm{Hom}_{C^\infty(M)}(\mathcal{J}^l(P), Q) \cong \mathrm{Diff}_l(P, Q).$$

In other words, the functor $Q \mapsto \mathrm{Diff}_l(P, Q)$ is representable in the category of geometric $C^\infty(M)$-modules, and the $C^\infty(M)$-module $\mathcal{J}^l(P)$ is its representing object. ▶

The last theorem makes it possible to change our point of view and define the module $\mathcal{J}^l(P)$ (together with the operator $j_l \colon P \to \mathcal{J}^l(P)$) as the representing object of the functor $Q \mapsto \mathrm{Diff}_l(P, Q)$ in the category of geometric $C^\infty(M)$-modules. This approach is conceptual and thereby immediately extends to arbitrary algebras and categories of modules. Of course, the question of existence must be answered separately in each particular case.

Exercise. Prove that the A-module $\mathrm{Diff}_l(P, Q)$ is geometric, provided that the module Q is geometric.

11.65. In this book we have dealt with smooth manifolds, smooth functions, smooth vector fields, smooth sections of vector bundles, etc. What can we say about similar objects that are not infinitely smooth, but for instance of class C^m? These notions of the standard calculus can be treated in the algebraic framework, using different functional algebras and using the procedure of the change of rings.

Exercises. 1. Let P be the module of smooth (of class C^∞) sections of a vector bundle π_P. For an arbitrary $m \geqslant 0$, give an algebraic definition of the module of sections of this bundle belonging to the class C^m (e.g., to C^0, i.e., continuous sections).

2. Give an algebraic definition of vector fields (differential operators) of class C^m on a smooth manifold M.

Afterword

If we continue on the path traced out by this book and analyze to what extent contemporary mathematics corresponds to the observability principle, we see that many things in our science are simply conceptually unfounded. This unavoidably leads to serious difficulties, which are usually ignored from force of habit even when they contradict our experience. If, for example, measure theory is the correct theory of integration, then why is it that all attempts to construct the continual integral on its basis have failed, although the existence of such integrals is experimentally verified?

As the result of this, physicists are forced to use "unobservable" mathematics in their theories, which leads to serious difficulties, say, in quantum field theory, which some even regard as an inherent aspect of the theory. It is generally believed that the mathematical basis of quantum mechanics is the theory of self-adjoint operators in Hilbert space. But then why does Dirac write that "physically significant interactions in quantum field theory are so strong that they throw any Schrödinger state vector out of Hilbert space in the shortest possible time interval"?

Having noted this, one must either avoid writing the Schrödinger equation in the context of quantum fields theory or refuse to consider Hilbert spaces as the foundation of quantum mechanics. Dirac reluctantly chose the first alternative, and this refusal was forced, since the mathematics of that time allowed him to talk about solutions of differential equations only in a very limited language (see the quotation at the beginning of the Introduction). On the other hand, since the Hilbert space formalism contains no procedure for distinguishing one vector from another, the observability principle is not followed here. Thus the second alternative seems more appropriate, but it requires specifying many other points, e.g., finding out how one can observe solutions of partial differential equations; this question, however, is outside the sphere of interests of the PDE experts: To them even setting the question would seem strange, to say the least.

Thus the systematic mathematical formalization of the observability principle requires rethinking many branches of mathematics that seemed established once and for all. The main difficult step that must be taken in this direction is to find solutions in the framework of the differential calculus, avoiding the appeal of functional analysis, measure theory, and other purely set-theoretical constructions. In particular, we must refuse measure theory as integration theory in favor of the purely cohomological approach. One page suffices to write out the main rules of measure theory. The number of pages needed to explain de Rham cohomology is much larger. The conceptual distance between the two approaches shows what serious difficulties must be overcome on this road.

The author intends to explain, in the next issues of his infinite series of books, how this road leads to the secondary differential calculus (already mentioned in the Introduction) and its main applications, e.g., cohomological physics. The reader may obtain an idea of what has already been done, and what remains to be done in this direction, by consulting the references appearing below.

Appendix

A. M. Vinogradov

Observability Principle, Set Theory and the "Foundations of Mathematics"

The following general remarks are meant to place the questions discussed in this book in the perspective of observable mathematics.

Propositional and Boolean algebras. While the physicist describes nature by means of measuring devices with $\mathbb{R}$-valued scales, the ordinary man or woman does so by means of statements. Using the elementary operations of conjunction, disjunction, and negation, new statements may be constructed from given ones. A system of statements (propositions) closed with respect to these operations is said to be a *propositional algebra.* Thus, the means of observation of an individual not possessing any measuring devices is formalized by the notion of propositional algebra. Let us explain this in more detail.

Let us note, first of all, that the individual observing the world without measuring devices was considered above only as an example of the main, initial mechanism of information processing, which in the sequel we shall call *primitive.* Thus, we identify propositional algebras with primitive means of observation.

Further, let us recall that any propositional algebra A may be transformed into a unital commutative algebra over the field $\mathbb{Z}_2$ of residues modulo 2 by introducing the operations of multiplication and addition as follows:

$$pq \stackrel{\text{def}}{=} p \wedge q,$$
$$p + q \stackrel{\text{def}}{=} (p \wedge \bar{q}) \vee (\bar{p} \wedge q),$$

where $\vee$ and $\wedge$ are the propositional connectives conjunction and disjunction, respectively, while the bar over a letter denotes negation. All elements of the algebra thus obtained are idempotent, i.e., $a^2 = a$. Let us call any unital commutative $\mathbb{Z}_2$-algebra *Boolean* if all its elements are idempotent. Conversely, any Boolean algebra may be regarded as a propositional algebra with respect to the operations

$$\begin{aligned} p \wedge q &\stackrel{\text{def}}{=} pq, \\ p \vee q &\stackrel{\text{def}}{=} p + q + pq, \\ \bar{p} &\stackrel{\text{def}}{=} 1 + p. \end{aligned}$$

This shows that there is no essential difference between propositional and Boolean algebras, and the use of one or the other only specifies what operations are involved in the given context. Thus we can restate the previous remarks about means of observation as follows: *Boolean algebras are primitive means of observation.*

Boolean spectra. The advantage of the previous formulation is that it immediately allows us to discern the remarkable analogy with the observation mechanism in classical physics as interpreted in this book. Namely, in this mechanism one must merely replace the $\mathbb{R}$-valued measurement scales by $\mathbb{Z}_2$-valued ones (i.e., those that say either "yes" or "no") and add the idempotence condition. This analogy shows that *what we can observe by means of a Boolean algebra A is its $\mathbb{Z}_2$-spectrum, i.e., the set of all its homomorphisms as a unital $\mathbb{Z}_2$-algebra to the unital $\mathbb{Z}_2$-algebra $\mathbb{Z}_2$.*

Let us denote this spectrum by $\mathrm{Spec}_{\mathbb{Z}_2}$ and endow it with the natural topology, namely the Zariski one. Then we can say, more precisely, that Boolean algebras allow us to observe topological spaces of the form $\mathrm{Spec}_{\mathbb{Z}_2}$, which we shall call, for this reason, *Boolean spaces.*

In connection with the above, one may naturally ask whether the spectra of Boolean algebras possess any structure besides the topological one, say, a smooth structure, as was the case for spectra of $\mathbb{R}$-algebras. The reader who managed to do Exercise 4 from Section 9.45 already knows that the differential calculus over Boolean algebras is trivial in the sense that any differential operator on such an algebra is of order zero, i.e., is a homomorphism of modules over this algebra. This means, in particular, that *the phenomenon of motion cannot be adequately described and studied in mathematical terms by using only logical notions* or, to put it simply, by using everyday language (recall the classical logical paradoxes on this topic).

The Stone theorem stated below, which plays a central role in the theory of Boolean algebras, shows that the spectra of Boolean algebras possess only one independent structure: the topological one. In the statement of the theorem it is assumed that the field $\mathbb{Z}_2$ is supplied with the discrete topology.

Stone's theorem. *Any Boolean space is an absolutely disconnected compact Hausdorff space and, conversely, any Boolean algebra coincides with the algebra of open-and-closed sets of its spectrum with respect to the set-theoretic operations of symmetric difference and intersection.*

Recall that the absolute disconnectedness of a topological space means that the open-and-closed sets form a base of its topology. The appearance of these simultaneously open and closed sets in Boolean spaces is explained by the fact that any propositional algebra possesses a natural duality. Namely, the negation operation maps it onto itself and interchanges conjunction and disjunction. Note also that Stone's theorem is an identical twin of the Spectrum theorem (see Sections 7.2 and 7.7). Their proofs are based on the same idea, and differ only in technical details reflecting the specifics of the different classes of algebras under consideration. The reader may try to prove this theorem as an exercise, having in mind that the elements of the given Boolean algebra can be naturally identified with the open-and-closed subsets of its spectrum, while the operations of conjunction, disjunction, and negation then become the set-theoretical operations of intersection, union, and complement, respectively. It is easy to see that the spectrum of a finite Boolean algebra is a finite set supplied with the discrete topology. Thus any finite Boolean algebra turns out to be isomorphic to the algebra of all subsets of a certain set.

"Eyes" and "ears". After all these preliminaries, the role of "eyes" and "ears" in the process of observation may be described as follows. First of all, the "crude" data absorbed by our senses are written down by the brain and sent to the corresponding part of our memory. One may think that in the process of writing down, the crude data are split up into elementary blocks, "macros," and so on, which are marked by appropriate expressions of everyday language. These marks are needed for further processing of the stored data. The system of statements constituting some description generates an ideal of the controlling Boolean algebra, thus distinguishing the corresponding closed subset in its spectrum. Supposing that to each point of the spectrum an elementary block is assigned, and this block is marked by the associated maximal ideal, we come to the conclusion that to each closed subset of the spectrum one can associate a certain image, just as a criminalist creates an identikit from individual details described by witnesses. Thus, if we forget about the "material" content of the elementary blocks (they may be "photographs" of an atomic fragment of a visual or an audio image, etc.) that corresponds (according to the above scheme) to points of the spectrum of the controlling Boolean algebra, we may assume that everything that can be observed on the primitive level is tautologically expressed by the points of this spectrum.

Boolean algebras corresponding to the primitive level. It is clear that any rigorous mathematical notion of observability must come from some notion of observer, understood as a kind of mechanism for gathering and processing information. In other words, the notion of observability must be formalized approximately in the same way as Turing machines formalize the notion of algorithm. So as not to turn out to be an a priori formalized metaphysical scheme, such a formalization must take into account "experimental data." The latter may be found in the construction and evolution of computer hardware and in the underlying theoretical ideas. Therefore it is useful to regard the individual mathematician, or better still, the mathematical community, in the spirit of the "noosphere" of Vernadskii, as a kind of computer. Then, having in mind that the operational system of any modern computer is a program written in the language of binary codes, we can say that there is no alternative to Boolean algebras as the mechanism describing information on the primitive level. For practical reasons, as well as for considerations of theoretical simplicity, it would be inconvenient to limit the size of this algebra by some concrete number, say the number of elementary particles in the universe. Hence it is natural to choose the free algebra in a countable number of generators. The notion of level of observability is apparently important for the mathematical analysis of the notion of observability itself, and we shall return to it below.

How set theory appeared in the foundations of mathematics. As we saw above, any propositional algebra is canonically isomorphic to the algebra of all subsets of the spectrum of the associated Boolean algebra. If this spectrum is finite, then its topology is discrete. So we can forget about the topology without losing anything. Moreover, any concrete individual, especially if he/she is not familiar with Boolean algebras, feels sure that what she/he is observing are just subsets or, more precisely, the identikits which he defined. Therefore, such an immediate "material" feeling leads us to the idea that the initial building blocks of precise abstract thinking are "points" ("elements") grouped together in "families," i.e., sets. Having accepted or rather having experienced this feeling of primitivity of the notion of set under the pressure of our immediate feelings, we are forced to place set theory at the foundation of exact knowledge, i.e., of mathematics. On the primitive level of finite sets, this choice, in view of what was explained above, does not contradict the observability principle, since any finite set can be naturally and uniquely interpreted as the spectrum of some Boolean algebra.

However, if we go beyond the class of finite sets, the situation changes radically: The notion of observable set, i.e., of Boolean space, ceases to coincide with the general notion of a set without any additional structure. Therefore, our respect for the observability principle leads us to abandon the notion of a set as the formal-logical foundation of mathematics and leave the paradise so favored by Hilbert. One of the advantages of such a step, among others, is that it allows us to avoid many of the paradoxes inherent to set theory. For example, the analogue of the "set of all sets" in observable mathematics is the "Boolean space of all Boolean spaces." But this last construction is clearly meaningless, because it defines no topology in the "Boolean space of all Boolean spaces." Or the "observable" version of the "set (not) containing itself as an element," i.e., the "Boolean space (not) containing itself as an element" is so striking that no comment is needed. In this connection we should additionally note that in order to observe Boolean spaces (on the primitive level!) as individual objects, a separate Boolean space that distinguishes them is required.

Observable mathematical structures (Boole groups). Now is the time to ask what observable mathematical structures are. If we are talking about groups observable in the "Boolean" sense, then we mean topological groups whose set of elements constitutes a Boolean space. Such a group should be called Boolean. In other words, a *Boole group* is a group structure on the spectrum of some Boolean algebra. If we replace in this definition the notion of Boolean observability by that of classical observability, we come to the notion of Lie group, i.e., of a group structure on the spectrum of the classical algebra of observables.

Observing observables: different levels of observability. Just as the operating system in a computer manipulates programs of the next level, one can imagine a Boolean algebra of the primitive level (see above) with the points of its spectrum marking other Boolean algebras. In other words, this is a Boolean algebra observing other Boolean algebras. Iterating this procedure, we come to "observed objects," which, if one forgets the multistep observation scheme, can naively be understood as sets of cardinality higher than finite or countable. For instance, starting from the primitive level, we can introduce into observable mathematics things that in "nonobservable" mathematics are related to sets of continual cardinality. In this direction, one may hope that there is a constructive formalization of the observability of smooth $\mathbb{R}$-algebras, which, in turn, formalize the observation procedure in classical physics.

Down with set theory? The numerous failed attempts to construct mathematics on the formal-logical foundations of set theory, together with the considerations related to observability developed above, lead us to refuse this idea altogether. We can note that it also contradicts the physiological basis of human thought, which ideally consists in the harmonious interaction of the left and right hemispheres of the brain. It is known that the left hemisphere is responsible for rational reasoning, computations, logical analysis, and pragmatic decision-making. Dually, the right hemisphere answers for "irrational" thought, i.e., intuition, premonitions, emotions, imagination, and geometry. If the problem under consideration is too hard for direct logical analysis, we ask our intuition what to do. We also know that in order to obtain a satisfactory result, the intuitive solution must be controlled by logical analysis and, possibly, corrected on its basis. Thus, in the process of decision-making, in the search for the solution of a problem, etc., the switching of control from one hemisphere to the other takes place, and such iterations can be numerous.

All this, of course, is entirely relevant to the solution of mathematical problems. The left hemisphere, i.e., the algebro-analytical part of our brain, is incapable of finding the solution to a problem whose complexity is higher than, say, the possibilities of human memory. Indeed, from any assumption one can deduce numerous logically correct consequences. Therefore, in the purely logical approach, the number of chains of inference grows at least exponentially with their length, while those that lead to a correct solution constitute a vanishingly small part of that number. Thus if the correct consequence is chosen haphazardly at each step, and the left hemisphere knows no better, then the propagation of this "logical wave" in all directions will overfill our memory before it reaches the desired haven.

The only way out of this situation is to direct this wave along an appropriate path, i.e., to choose at each step the consequences that can lead in a more or less straight line to the solution. But what do we mean by a "straight line"? This means that an overall picture of the problem must be sketched, a picture on which possible ways of solution could be drawn. The construction of such an overall picture, in other words, of the geometric image of the problem, takes place in the right hemisphere, which was created by nature precisely for such constructions. The basic building blocks for them, at least when we are dealing with mathematics, are sets. These are sets in the naive sense, since they live in the right hemisphere. Hence any attempt to formalize them, moving them from the right hemisphere to the left one, is just an outrage against nature. So let us leave set theory in the right hemisphere in its naive form, thanks to which it is has been so useful.

Infinitesimal observability. Above we considered Boolean algebras as analogue of smooth algebras. But we can interchange our priorities and do things the other way around. From this point of view, the operations or, better, the functors of the differential calculus, will appear as the analogue of logical operations, and the calculus itself as a mechanism for manipulating infinitesimal descriptions. In this way we would like to stress the infinitesimal aspect related to observability.

Some of the "primary" functors were described in this book. Their complete list should be understood as the logic algebra of the differential calculus. The work related to the complete formalization of this idea is still to be completed.

In conclusion let us note, expressing ourselves informally, that in our imaginary computer, working with stored knowledge, the program called "differential calculus" is not part of its operating system, and so is located at a higher level than the primitive one (see above). This means that the geometric images built on its basis cannot be interpreted in a material way. They should retain their naive status in the sense explained above.

The constructive differential calculus, developed in the framework of "constructive mathematical logic," illustrates what can happen if this warning is ignored.

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